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The Riemann Integral in Weak Systems of Analysis

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Abstract: Taking as a starting point (a modification of) a weak theory of arithmetic of Jan Johannsen and Chris Pollett (connected with the *hierarchy of counting functions*), we introduce successively stronger theories of *bounded arithmetic* in order to set up a system for analysis (TCA^2). The extended theories preserve the connection with the *counting hierarchy* in the sense that the algorithms which the systems prove to halt are exactly the ones in the hierarchy. We show that TCA^2 has the exact strength to develop Riemannian integration for functions with a modulus of uniform continuity.
Key Words: Weak analysis, Riemann integral, counting hierarchy
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1 Introduction

The formalization of mathematics in systems of second-order arithmetic has attracted eminent mathematicians, including Richard Dedekind, Hermann Weyl and David Hilbert, among others. More recently, the field has been revitalized with contributions of Harvey Friedman, Stephen Simpson and their co-workers. Their program of *reverse mathematics* seeks to find the exact correspondence between theorems of ordinary mathematics and axioms, calibrating the strength of specific mathematical results. A survey of the state of the art on this subject can be found in [Simpson 1999].

In [Fernandes and Ferreira 2002], the efforts to formalize mathematics within second-order arithmetic are extended to weaker (sub-exponential) theories, in an area known as *weak analysis*. More precisely, some very basic notions of analysis are developed within BTFA, a second-order system of 0-1 strings related with *polynomial time computability* (see [Ferreira 1994]). On this regard, see also [Yamazaki 2005] and [Fernandes and Ferreira 2005]. The following quote is from [Fernandes and Ferreira 2002]: "BTFA is [...] insufficient for developing Riemannian integration for (general) continuous functions with a modulus of uniform continuity." In the present paper, we pursue this line of research and present a system of weak analysis with the exact strength to develop the Riemann integral.

We know from [Ferreira and Ferreira 2006] that *counting* is a consequence of *integration* over the theory BTFA. Therefore, if we take BTFA as our base theory, the formalization of the Riemann integral has to be carried out in a system which allows counting. We show in this paper that counting is exactly what is needed for developing Riemannian integration. This explains our interest in the *hierarchy of counting functions* (FCH), a computational complexity class lying between PTIME and PSPACE.

The links concerning bounded theories of arithmetic and computational complexity classes are a good example of the rich interaction between *mathematical logic* and *computer science*. There are several examples of systems introduced because of their connection to some complexity classes. We already mentioned BTFA characterizing PTIME, but could also refer to (for instance) Buss' theories S^1_2, U^1_2, and V^1_2 [Buss 1985] characterizing precisely PTIME, PSPACE and $\mathsf{EXPTIME}$ as the class of functions provably total in these systems (with appropriate graphs). For related work in the area, see also [Krajíček 1993], [Krajíček and Pudlák 1991], [Johannsen 1996], [Johannsen and Pollett 1998] and [Clote and Takeuti 1995].

After the introduction of FCH, by Wagner in 1986 [Wagner 1986], some machine independent characterizations of the class were developed (for instance in [Vollmer and Wagner 1996]). Johannsen and Pollett introduced in [Johannsen and Pollett 1998] a second-order bounded theory of arithmetic D^0_2 and proved that it is related to FCH. This is our departure point. Based on D^0_2, we introduce TCA a second-order theory in binary notation, compatible with Ferreira's theories of feasible analysis, and still connected with FCH. The second-order variables of this theory (as well as those of the original theory D^0_2) are intended to range over bounded sets. We expand this theory to a *bona fide* second-order system for analysis TCA^2, with second-order variables intended to range over arbitrary subsets of the first-order domain. If we compare TCA and TCA^2 with the theories Σ^{b}_1-NIA (see [Ferreira 1994]) and BTFA, we are led to the following informal correspondence:

$$\frac{\mathsf{BTFA}}{\Sigma^{\mathsf{b}}_1\text{-}\mathsf{NIA}} \sim \frac{\mathsf{TCA}^2}{\mathsf{TCA}}$$

Since $\mathsf{BTFA} \subseteq \mathsf{TCA}^2$, the portion of real analysis already developed in BTFA (see [Fernandes and Ferreira 2002]) can also be developed within TCA^2. We pursue these efforts in order to formalize the Riemann integral for functions with a modulus of uniform continuity in TCA^2. We verify some properties of the integral; in particular, we prove the *fundamental theorem of calculus*.

2 Theory for counting arithmetic (TCA)

The purpose of this section is to present a second-order theory of bounded arithmetic in *binary notation*, called TCA, whose class of provably total functions is FCH. The intent is facilitated because Johannsen and Pollett have already developed a number system D_2^0 and proved its relation to the counting hierarchy. Apropos bounded arithmetic theories related with counting see also the systems C_2^0 and C_3^0 [Johannsen and Pollett 1998] by Johannsen and Pollett and the system VTC^0 [Nguyen and Cook 2004] by Stephen Cook and Phuong Nguyen. However, we do not work in D_2^0 and work instead with a reformulation of this theory in the binary string setting. This has the advantage of building upon the work in weak analysis already done in such setting. Following [Ferreira 1994], we start by quickly recalling the basic concepts concerning second-order systems in binary notation.

Consider $\mathcal{L}_2^b$ a second-order language with equality which has three constant symbols, viz ϵ (for the empty word), 0 and 1; two binary function symbols $\hat{}$ (for concatenation – the symbol "$\hat{}$" is usually omitted) and $\times$ (where $x \times y$ refers to the word x concatenated with itself length of y times) and three binary relation symbols $=$, $\subseteq$ (for initial subwordness, i.e., string prefix) and $\in$ that infixes between a first-order term and a second-order variable. The language has first-order variables denoted by $x, y, z, \ldots$ and second-order variables denoted by F^t, G^q, $\ldots$ with t, q first-order terms. The *standard structure* for this language has domain $(2^{<\omega}, \mathcal{P}_f(2^{<\omega}))$, i.e., the first-order variables are interpreted as finite sequences of zeros and ones, and the second-order variables are subsets X^t of $2^{<\omega}$ satisfying $x \in X^t \to x \preceq t$, where $x \preceq t$ abbreviates $1 \times x \subseteq 1 \times t$ (this means that the length of x is less than or equal to the length of t; we also use the notation, $x \prec t \vee x \equiv t$).

The *class of formulas* in $\mathcal{L}_2^b$ is defined as the smallest class of expressions containing the atomic formulas $t_1 \subseteq t_2, t_1 = t_2, t_1 \in F^t$, with t_1, t_2 terms and F^t a second-order variable, and closed under the boolean operations $\neg, \wedge, \vee, \to$, the first-order quantifications $\forall x, \exists x$, the bounded first-order quantifications $\forall x \preceq t, \exists x \preceq t$ and the second-order quantifications $\forall F^t, \exists F^t$.

The theory that we present below is a reformulation in binary notation of the theory D_2^0, introduced in [Johannsen and Pollett 1998].

Definition 2.1 TCA *(*Theory for Counting Arithmetic*) is the second-order theory in the language $\mathcal{L}_2^b$, which has the following axioms:*

- Basic axioms*: $x\epsilon = x$; $x(y0) = (xy)0$; $x(y1) = (xy)1$; $x \times \epsilon = \epsilon$; $x \times y0 = (x \times y)x$; $x \times y1 = (x \times y)x$; $x \subseteq \epsilon \leftrightarrow x = \epsilon$; $x \subseteq y0 \leftrightarrow x \subseteq y \vee x = y0$; $x \subseteq y1 \leftrightarrow x \subseteq y \vee x = y1$; $x0 = y0 \to x = y$; $x1 = y1 \to x = y$; $x0 \neq y1$; $x0 \neq \epsilon$; $x1 \neq \epsilon$;*

– $\forall y \forall F^t (y \in F^t \to y \preceq t)$*;*

– Induction on notation for $\Sigma_0^{1,b}$-formulas*:*

$$A(\epsilon) \wedge \forall x (A(x) \to A(x0) \wedge A(x1)) \to \forall x A(x),$$

with A *a* $\Sigma_0^{1,b}$-formula *(i.e., with no second-order quantifications and where all first-order quantifications are bounded). In the standard model, if the second-order parameters are in the Polynomial Hierarchy (a.k.a. Meyer-Stockmeyer Hierarchy) then these formulas define predicates in this hierarchy;*

– Bounded comprehension*:* $\exists F^t \forall y \preceq t (y \in F^t \leftrightarrow A(y))$*, where* t *is a term in which* y *does not occur, and* A *is a* $\Sigma_0^{1,b}$*-formula that may have other free variables other than* y *and where the variable* F^t *does not occur;*

– Substitution for $\Sigma_0^{1,b}$-formulas*:* $\forall x \preceq t \exists F^q\ \varphi(x, F^q) \to \exists G^r \forall x \preceq t \bar{\varphi}(x, G^r)$*, with* φ *a* $\Sigma_0^{1,b}$*-formula,* t *a term where* x *does not occur, and* $\bar{\varphi}$ *results from* φ *by replacing all the occurrences of "*$s \in F^q$*" by "*$\langle x, s\rangle \in G^r$*" (where* $\langle\, ,\, \rangle$ *is a pairing function and* r *is a certain term depending on* t *and* q*). We are omitting the exact term* r *in order to facilitate reading (the term depends on the particular definition of the pairing function – see [Ferreira 2006] for a concrete implementation of these matters). This is a technical axiom that permits a kind of "permutation" between bounded first-order universal quantifications and second-order existential quantifications;*

– Counting axiom*:* $\forall F^t \exists C^v Count(C^v, F^t)$*, where* v *is a certain term (we omit it) which depends on* t*, and* $Count(C^v, F^t)$ *abbreviates the conjunction of* $\forall x \preceq t \exists^1 j \preceq v \langle x, j\rangle \in C^v$ *– a clause which states the functionality of* C^v *– together with*

$$(\epsilon \notin F^t \to\ \langle \epsilon, \epsilon\rangle \in C^v) \wedge (\epsilon \in F^t \to\ \langle \epsilon, 0\rangle \in C^v),$$

and

$$\begin{aligned}\forall x <_l 1 \times t \big((S(x) \notin F^t \to \forall j \preceq v(\langle x, j\rangle \in C^v \to \langle S(x), j\rangle \in C^v))\wedge \\ (S(x) \in F^t \to \forall j \preceq v(\langle x, j\rangle \in C^v \to \langle S(x), S(j)\rangle \in C^v))\big),\end{aligned}$$

where $<_l$ *is linearly ordered according to increasing length and, within the same length, lexicographically (0 before 1),* S *is the successor function induced by* $<_l$*, and* t *is a term in which* x *does not occur. (Again, the term* v *depends on the particular definition of the pairing function; [Ferreira 2006] presents a concrete implementation.)*

In the last scheme, the idea behind the formula $Count$ is that C^v counts the number of elements in F^t: given $x \preceq t$, we have $\langle x, j\rangle \in C^v$ if and only if there are j elements less than or equal to x (with respect to the order $\leq_l$) in F^t.

A $\Sigma_1^{1,b}$-*formula* (resp. $\Pi_1^{1,b}$-*formula*) is a formula in $\mathcal{L}_2^b$ of the form: $\exists F_1^{t_1}...\exists F_k^{t_k}\ \varphi(F_1^{t_1},...,F_k^{t_k},\bar{x},\bar{G}^r)$ (resp. $\forall F_1^{t_1}...\forall F_k^{t_k}\ \varphi(F_1^{t_1},...,F_k^{t_k},\bar{x},\bar{G}^r)$), where φ is a $\Sigma_0^{1,b}$-formula. A $\Sigma_1^{1,b}$-*extended formula* (respectively $\Pi_1^{1,b}$-*extended formula*) is a formula that can be built in a finite number of steps, starting with $\Sigma_0^{1,b}$-formulas and allowing conjunctions, disjunctions, bounded first-order quantifications and second-order existential (respectively universal) quantifications. A formula is $\Delta_1^{1,b}$ (respectively $\Delta_1^{1,b}$-*extended*) if it is equivalent in TCA to both a $\Sigma_1^{1,b}$-formula (respectively a $\Sigma_1^{1,b}$-extended formula) and a $\Pi_1^{1,b}$-formula (respectively a $\Pi_1^{1,b}$-extended formula).

Proposition 2.1 *The following is provable in* TCA *([Ferreira 2006] or [Johannsen and Pollett 1998]):*

- *substitution for* $\Sigma_1^{1,b}$*-extended formulas*
- *bounded comprehension for* $\Delta_1^{1,b}$*-extended formulas*
- *induction on notation for* $\Delta_1^{1,b}$*-extended formulas*
- *minimization scheme for* $\Delta_1^{1,b}$*-extended formulas, i.e.,*

  $$\exists x A(x) \rightarrow \exists x(A(x) \wedge \forall y <_l x \neg A(y)),$$

  with A *a* $\Delta_1^{1,b}$*-extended formula.*

Similarly to [Johannsen and Pollett 1998], it can be proved that FCH is the class of functions provably total in TCA with $\Sigma_1^{1,b}$-graphs. The proof uses the free cut elimination theorem, after formulating the theory TCA into Gentzen's sequent calculus (the details can be found in [Ferreira 2006]). Here we just present that formulation, denoted by $\mathsf{LK_{FCH}}$.

Besides the initial sequents of the form $A \Rightarrow A$, with A an atomic formula, and the sequents for equality, $\mathsf{LK_{FCH}}$ has also the following axioms:

1) $\Rightarrow A(\bar{s})$, with A a basic axiom of TCA and $\bar{s}$ terms;
2) $s \in F^t \Rightarrow s \preceq t$;
3) $\Rightarrow A(\epsilon) \wedge \forall x \prec s\,(A(x) \rightarrow A(x0) \wedge A(x1)) \rightarrow \forall x \preceq s A(x)$, with A a $\Sigma_0^{1,b}$-formula, and x does not occur in s;
4) $\Rightarrow \exists F^s \forall y \preceq s\,(y \in F^s \leftrightarrow A(y))$, with A a $\Sigma_0^{1,b}$-formula where the variable F^s does not occur;
5) $\Rightarrow \exists C^v Count(C^v, F^t)$, the term v as in the counting axiom,

and all the second-order inference rules (like the ones presented in [Buss 1985] with the obvious modification to our language), complemented with the following *substitution rule:*

$$\frac{\Gamma, a \preceq t \Rightarrow \exists F^q \varphi(a, F^q)}{\Gamma \Rightarrow \exists G^r \forall x \preceq t \bar{\varphi}(x, G^r)},$$

where a is a proper variable, φ a $\Sigma_0^{1,b}$-formula , and r and $\bar{\varphi}$ are as in the substitution scheme.

Of course, the point of this sequent calculus formulation of TCA is that all of its axioms are given by sequents consisting only of $\Sigma_1^{1,b}$-formulas.

3 Enriching TCA

In this section, we enrich TCA and set up a system for analysis (with second-order variables intended to range over arbitrary subsets of 0-1-strings), still characterizing FCH. Firstly, we need to add to TCA a suitable collection scheme.

A $\Sigma_\infty^{1,b}$-*formula* is a formula that belongs to the smallest class of expressions in $\mathcal{L}_2^b$ that contains the $\Sigma_0^{1,b}$-formulas and is closed under $\neg, \wedge, \vee, \exists X^t, \forall X^t, \exists x \preceq t$ and $\forall x \preceq t$. Consider the following scheme, denoted by $\mathsf{B}^1\Sigma_\infty^{1,b}$:

$$\forall X^t \exists y\, \varphi(y, X^t) \rightarrow \exists z \forall X^t \exists y \preceq z\, \varphi(y, X^t)$$

with φ a $\Sigma_\infty^{1,b}$-formula which can have free variables other than y and X^t.

This *bounded collection* scheme, although somewhat technical, is of paramount importance for the introduction of some principles of recursive comprehension. Observe that this is a true scheme in the standard model because (for a fixed 0-1-string t) there are only finitely many values for X^t. Even though we may be collecting exponentially many elements, adjoning the collection scheme does not entail the totality of exponentiation. In point of fact, one obtains a theory which is suitably conservative over the original theory (see the next proposition). This conservativity is argued below via an analysis within a sequent style Gentzen calculus. For a recent new angle on this issue see [Ferreira and Oliva 2007].

The theory $\mathsf{TCA} + \mathsf{B}^1\Sigma_\infty^{1,b}$ can be formulated in Gentzen's sequent calculus by $\mathsf{LK}'_{\mathsf{FCH}}$, which results from $\mathsf{LK}_{\mathsf{FCH}}$ adding the following inference rule:

$$\frac{\Gamma \Rightarrow \exists y \varphi(y, C^t)}{\Gamma \Rightarrow \exists z \forall X^t \exists y \preceq z \varphi(y, X^t)},$$

where C^t is a proper variable, t is a term where y does not occur and φ is a $\Sigma_\infty^{1,b}$-formula which can have other free variables.

The characterization of the provably total functions in the enlarged theory $\mathsf{TCA} + \mathsf{B}^1\Sigma_\infty^{1,b}$ is obtained via the following conservation result:

Proposition 3.1 *The theory* $\mathsf{TCA} + \mathsf{B}^1\Sigma_\infty^{1,b}$ *is* $\forall\exists\Sigma_\infty^{1,b}$*-conservative over* TCA*, i.e., whenever* $\mathsf{TCA} + \mathsf{B}^1\Sigma_\infty^{1,b}$ *proves a sentence of the form* $\forall \bar{x} \exists y \varphi(\bar{x}, y)$*, with* φ *a* $\Sigma_\infty^{1,b}$*-formula, then* TCA *already proves it.*

Proof. The proof uses the cut elimination theorem and is based on a proof, in a different context, presented by Buss (see [Buss 1998]).

Suppose $\mathsf{TCA} + \mathsf{B}^1\Sigma^{1,b}_\infty \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$, with φ a $\Sigma^{1,b}_\infty$-formula. So, there is a $\mathsf{LK}'_{\mathsf{FCH}}$-proof of the sequent $\Rightarrow \exists y\varphi(\bar{x}, y)$. The free cut elimination theorem ensures that there is a $\mathsf{LK}'_{\mathsf{FCH}}$-proof P of $\Rightarrow \exists y\varphi(\bar{x}, y)$ without free cuts, which means that we can assume that every formula in P is a $\Sigma^{1,b}_\infty$-formula or is a formula of the form $\exists x\theta(x, \bar{x}, \bar{X}^p)$, with θ a $\Sigma^{1,b}_\infty$-formula. We prove, by induction on the number of lines of the proof P, that for every sequent $\Gamma \Rightarrow \Delta$ in P (consider $\Gamma := \exists x_1\varphi_1, ..., \exists x_n\varphi_n$ and $\Delta := \exists y_1\psi_1, ..., \exists y_k\psi_k$, with φ_i, ψ_i $\Sigma^{1,b}_\infty$-formulas), we have $\mathsf{TCA} \vdash \forall u\exists v\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\Gamma^{\preceq u} \to \Delta^{\preceq v})$, where $\Gamma^{\preceq u}$ abbreviates $\exists x_1 \preceq u\varphi_1 \wedge ... \wedge \exists x_n \preceq u\varphi_n$ and $\Delta^{\preceq v}$ abbreviates $\exists y_1 \preceq v\psi_1 \vee ... \vee \exists y_k \preceq v\psi_k$. Once we apply it to the last sequent of P, the result follows immediately. We illustrate the proof by induction on the number of lines of P with the cut rule. Suppose $\Gamma \Rightarrow \Delta$ is obtained by cut, i.e., the line immediately above is formed by the sequents $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$. If A is a $\Sigma^{1,b}_\infty$-formula it is enough to choose v as being the concatenation of the v's that exist by induction hypotheses for the two sequents in the line above. Suppose $A := \exists z\theta(z, \bar{x}, \bar{X}^{p(\bar{x})})$, with θ a $\Sigma^{1,b}_\infty$-formula. By induction hypothesis we have that (1) $\mathsf{TCA} \vdash \forall u\exists v\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\Gamma^{\preceq u} \to \Delta^{\preceq v} \vee \exists z \preceq v\theta(z, \bar{x}, \bar{X}^{p(\bar{x})}))$ and (2) $\mathsf{TCA} \vdash \forall u\exists v\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\exists z \preceq u\theta(z, \bar{x}, \bar{X}^{p(\bar{x})}) \wedge \Gamma^{\preceq u} \to \Delta^{\preceq v})$. We want to prove that $\mathsf{TCA} \vdash \forall u\exists v\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\Gamma^{\preceq u} \to \Delta^{\preceq v})$. Let us work in TCA. Given u, by (1) there is v_1 such that $\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\Gamma^{\preceq u} \to \Delta^{\preceq v_1} \vee \exists z \preceq v_1\theta(z, \bar{x}, \bar{X}^{p(\bar{x})}))$. We can suppose that $u \preceq v_1$, because if it is not the case we replace v_1 by $v_1\hat{\ }u$. By (2), there is v_2 such that $\forall\bar{x} \preceq v_1\forall\bar{X}^{p(\bar{x})}(\exists z \preceq v_1\theta(z, \bar{x}, \bar{X}^{p(\bar{x})}) \wedge \Gamma^{\preceq v_1} \to \Delta^{\preceq v_2})$. So $\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\exists z \preceq v_1\theta(z, \bar{x}, \bar{X}^{p(\bar{x})}) \wedge \Gamma^{\preceq u} \to \Delta^{\preceq v_2})$. With $v = v_1\hat{\ }v_2$ we have $\forall\bar{x} \preceq u\forall\bar{X}^{p(\bar{x})}(\Gamma^{\preceq u} \to \Delta^{\preceq v})$. The other rules follow in a similar way. $\Box$

It follows immediately that the provably total functions in $\mathsf{TCA} + \mathsf{B}^1\Sigma^{1,b}_\infty$, with graphs $\Sigma^{1,b}_1$, are exactly the functions of FCH.

We are now ready to define the theory TCA^2, which includes the previous systems and is stated in a language that permits variables ranging over infinite sets. Let $\mathcal{L}_2$ be a second-order language with equality which differs from $\mathcal{L}^b_2$ only by the presence of second-order variables, denoted by $F, G, \ldots, X, Y$, instead of the previous second-order "bounded" variables $F^t, G^q, \ldots$ The formulas of $\mathcal{L}_2$ are defined as in $\mathcal{L}^b_2$, replacing X^t by X. The definitions of $\Sigma^{1,b}_0$ (respectively $\Sigma^{1,b}_1$, $\Pi^{1,b}_1$, $\Delta^{1,b}_1$, $\Sigma^{1,b}_\infty$)-formulas in $\mathcal{L}_2$ (and its extended versions) are given by obvious modifications, namely replacing $\forall X^t$, $\exists X^t$ by the second-order quantifications $\forall X \preceq t$, $\exists X \preceq t$ where $X \preceq t$ abbreviates $\forall z(z \in X \to z \preceq t)$.

A structure for $\mathcal{L}_2$ has domain (M, S), with the first-order variables taking values in M and the second-order variables varying over S, a given subset of $\mathcal{P}(M)$. The standard model is $(2^{<\omega}, \mathcal{P}(2^{<\omega}))$. Note that although we work in a second-order language, our logic is of first-order kind (first-order logic in a two-sorted language): our semantics only specifies S to be a subset of $\mathcal{P}(M)$, not necessarily all of $\mathcal{P}(M)$.

Consider the following axiom, known as the *recursive comprehension scheme*:

$$\forall x(\exists y\varphi(x,y) \leftrightarrow \forall y\psi(x,y)) \rightarrow \exists X\forall x(x \in X \leftrightarrow \exists y\varphi(x,y)),$$

with φ a $\Sigma_1^{1,b}$-formula and ψ a $\Pi_1^{1,b}$-formula, possibly with other free variables. In the standard model, this scheme ensures that all recursive sets exist. Therefore, it may seem that adding this scheme to weak theories like ours would increase its computational power. We will prove that this is not the case. NB the existence of a set X is guaranteed only in the case the theories have enough resources to prove the equivalence in the antecedent of the scheme, i.e., enough "power" to prove that some algorithms do halt.

Definition 3.1 TCA^2 *is the second-order theory, in the language $\mathcal{L}_2$, with the following axioms: basic axioms, induction on notation for $\Sigma_0^{1,b}$-formulas, substitution for $\Sigma_0^{1,b}$-formulas, counting axiom, bounded collection and the recursive comprehension scheme mentioned above.*

Note that $\mathcal{L}_2^b \subseteq \mathcal{L}_2$, in the sense that every expression in $\mathcal{L}_2^b$ can be formulated in $\mathcal{L}_2$. Note also that every model of TCA^2 satisfies the axioms of $\mathsf{TCA} + \mathsf{B}^1\Sigma_\infty^{1,b}$ by definition. In order to proceed, we need in some sense the inverse, i.e., to get models of TCA^2 from models of $\mathsf{TCA} + \mathsf{B}^1\Sigma_\infty^{1,b}$.

Lemma 3.1 *Let $\mathcal{M}$ be a model of the theory $\mathsf{TCA}+\mathsf{B}^1\Sigma_\infty^{1,b}$ with domain (M, S_b). Then there is $S \subseteq \mathcal{P}(M)$ such that $\mathcal{M}^*$, with domain (M, S), is a model of TCA^2 and $S_b = \{X^a : X \in S \wedge a \in M\}$, where X^a collects the elements of X with length less than or equal to a.*

Proof. In order to get $\mathcal{M}^*$ from $\mathcal{M}$ we have in some sense to "close" $\mathcal{M}$ for recursive comprehension. Let S be formed by the subsets $X \subseteq M$ for which there is a $\Sigma_1^{1,b}$-formula φ, a $\Pi_1^{1,b}$-formula ψ and elements $\bar{a}, \bar{b}$ in M and $\bar{A}^p, \bar{B}^u$ in S_b such that $X = \{x \in M : \mathcal{M} \vDash \exists y\, \varphi(x, y, \bar{a}, \bar{A}^{p(x,y,\bar{a})})\} = \{x \in M : \mathcal{M} \vDash \forall y\, \psi(x, y, \bar{b}, \bar{B}^{u(x,y,\bar{b})})\}$. The proof that $S_b \subseteq \{X^a : X \in S \wedge a \in M\}$ follows immediately because whenever $C^c \in S_b$, we have $C^c \in S$. For the other inclusion consider $C \in S$ and $c \in M$. We want to prove that $C^c \in S_b$. By definition of S there are formulas φ, ψ and elements $\bar{a}, \bar{b}, \bar{A}, \bar{B}$ (to simplify notation we omit the bounded term in the second-order parameters) such that $\mathcal{M} \vDash \forall x(\exists y\varphi(x, y, \bar{a}, \bar{A}) \leftrightarrow \forall y\psi(x, y, \bar{b}, \bar{B}))$. We claim that there is $d \in M$ such that $\forall x \preceq c(\exists y \preceq d\varphi(x, y, \bar{a}, \bar{A}) \leftrightarrow \forall y \preceq d\psi(x, y, \bar{b}, \bar{B}))$. The existence of this d uses the fact that from $\forall x \preceq c\exists y\, \theta(x, y)$, with θ a $\Sigma_\infty^{1,b}$-formula, we have $\exists d\forall x \preceq c\exists y \preceq d\, \theta(x, y)$, which is a consequence of $\mathcal{M}$ being a model of $\mathsf{B}^1\Sigma_\infty^{1,b}$.

Therefore, by bounded comprehension in $\mathsf{TCA} + \mathsf{B}^1\Sigma^{1,b}_\infty$ for $\Delta^{1,b}_1$-extended formulas we have that $F^c = \{x \preceq c : \exists y \preceq d\varphi(x, y, \bar{a}, \bar{A})\}$ is an element in S_b. Since $F^c = C^c$, we have $C^c \in S_b$.

The proof that $\mathcal{M}^*$ is a model of TCA^2 is done axiom by axiom. The case of the basic axioms is trivial, because the interpretation of the constants, the function and relation symbols is the same in $\mathcal{M}$ and $\mathcal{M}^*$ and both models have the same first-order domain. The study of the other axioms follows more or less in a straightforward manner from the following technical fact (which can be proved by induction on the complexity of φ): *Given $\varphi(\bar{u}, \bar{U})$ a $\Sigma^{1,b}_\infty$-formula, there is a term $q(\bar{u})$ with the following property: given $\bar{c}$ elements in M and $\bar{C}$ subsets in S then $\mathcal{M}^* \vDash \varphi(\bar{c}, \bar{C}) \Leftrightarrow \mathcal{M} \vDash \varphi(\bar{c}, \bar{C}^b)$ whenever $q(\bar{c}) \preceq b$.* We illustrate the application of the fact studying the counting axiom and the recursive comprehension scheme. For the counting axiom, given $\bar{a} \in M$, fix $F \in S$ such that $F \preceq t(\bar{a})$. We want to prove that $\mathcal{M}^* \vDash \varphi(\bar{a}, F)$ with $\varphi(\bar{u}, U)$ the $\Sigma^{1,b}_\infty$-formula defined by $\exists C \preceq vCount(C, U)$. Since $F^{t(\bar{a})} \in S_b$ and $\mathcal{M}$ is a model of TCA, we know that $\mathcal{M} \vDash \varphi(\bar{a}, F^{t(\bar{a})})$. Applying the technical fact above, there is a term $q(\bar{u})$ such that $\mathcal{M}^* \vDash \varphi(\bar{a}, F) \Leftrightarrow \mathcal{M} \vDash \varphi(\bar{a}, F^b)$ whenever $q(\bar{a}) \preceq b$. If $q(\bar{a}) \preceq t(\bar{a})$ the result is immediate. If $t(\bar{a}) \preceq q(\bar{a})$, then $F^{t(\bar{a})} = F^{q(\bar{a})}$ because $F \preceq t(\bar{a})$. Therefore, $\mathcal{M}^* \vDash \varphi(\bar{a}, F)$.

For the recursive comprehension scheme, suppose that $\mathcal{M}^* \vDash \forall x(\exists y\varphi(x, y, \bar{a}, \bar{A}) \leftrightarrow \forall y\psi(x, y, \bar{b}, \bar{B}))$ with φ a $\Sigma^{1,b}_1$-formula and ψ a $\Pi^{1,b}_1$-formula. We want to prove that $\mathcal{M}^* \vDash \exists X\forall x(x \in X \leftrightarrow \exists y\varphi(x, y, \bar{a}, \bar{A}))$. Note that the formulas $\varphi(x, y, \bar{u}, \bar{U})$ and $\psi(x, y, \bar{u}, \bar{U})$ are, in particular, $\Sigma^{1,b}_\infty$-formulas. Applying the fact above to the formula φ, there exists a term $q(x, y, \bar{u})$ such that given $s, r, \bar{a} \in M$ and $\bar{A} \in S$ we have $(\star)$ $\mathcal{M}^* \vDash \varphi(s, r, \bar{a}, \bar{A}) \Leftrightarrow \mathcal{M} \vDash \varphi(s, r, \bar{a}, \bar{A}^b)$ whenever $q(s, r, \bar{a}) \preceq b$. Applying the same fact to ψ, there exists a term $p(x, y, \bar{u})$ such that given $s, r, \bar{b} \in M$ and $\bar{B} \in S$ we have $\mathcal{M}^* \vDash \varphi(s, r, \bar{b}, \bar{B}) \Leftrightarrow \mathcal{M} \vDash \varphi(s, r, \bar{b}, \bar{B}^{b'})$ whenever $p(s, r, \bar{b}) \preceq b'$. Since $\mathcal{M}^* \vDash \forall x(\exists y\varphi(x, y, \bar{a}, \bar{A}) \leftrightarrow \forall y\psi(x, y, \bar{b}, \bar{B}))$ we know that $\mathcal{M} \vDash \forall x(\exists y\varphi(x, y, \bar{a}, \bar{A}^{q(x,y,\bar{a})}) \leftrightarrow \forall y\psi(x, y, \bar{b}, \bar{B}^{p(x,y,\bar{b})}))$. Take $X = \{x \in M : \mathcal{M} \vDash \exists y\varphi(x, y, \bar{a}, \bar{A}^{q(x,y,\bar{a})})\} = \{x \in M : \mathcal{M} \vDash \forall y\psi(x, y, \bar{b}, \bar{B}^{p(x,y,\bar{b})})\}$. By the definition of S, $X \in S$. From $(\star)$ we know that $\mathcal{M}^* \vDash \exists y\varphi(s, y, \bar{a}, \bar{A}) \Leftrightarrow \mathcal{M} \vDash \exists y\varphi(s, y, \bar{a}, \bar{A}^{q(s,y,\bar{a})})$, so $X = \{x \in M : \mathcal{M}^* \vDash \exists y\varphi(x, y, \bar{a}, \bar{A})\}$. Thus $\mathcal{M}^* \vDash \exists X\forall x(x \in X \leftrightarrow \exists y\varphi(x, y, \bar{a}, \bar{A}))$. □

Theorem 3.1 *If* $\mathsf{TCA}^2 \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$*, with* φ *a* $\Sigma^{1,b}_\infty$*-formula, then* $\mathsf{TCA} \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$.

Proof. Suppose that $\mathsf{TCA} \nvdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$. Therefore, by Proposition 3.1, we also have $\mathsf{TCA}+\mathsf{B}^1\Sigma^{1,b}_\infty \nvdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$. By the completeness theorem, there is a model (M, S_b) of $\mathsf{TCA} + \mathsf{B}^1\Sigma^{1,b}_\infty$ and $\bar{a} \in M$ such that $(M, S_b) \models \forall y\neg\varphi(\bar{a}, y)$. Using the previous lemma, take $S \subseteq \mathcal{P}(M)$ such that $S_b = \{X^a : X \in S \wedge a \in M\}$.

Clearly, $\Sigma^{1,b}_{\infty}$-formulas are absolute between (M, S_b) and (M, S). Therefore, we also have $(M, S) \models \forall y \neg\varphi(\bar{a}, y)$. Since (M, S) is a model of TCA^2, by soundness we conclude that $\mathsf{TCA}^2 \nvdash \forall \bar{x} \exists y \varphi(\bar{x}, y)$. □

As a consequence, the provably total functions of TCA^2, with $\Sigma^{1,b}_1$-graphs, are still the functions of FCH.

4 Analysis in weak systems – background

In this section, we briefly review the formalization of the basic analytic concepts given in the paper *Groundwork for Weak Analysis* [Fernandes and Ferreira 2002]. The formalization was worked out within BTFA and, consequently, it also applies to the stronger TCA^2. All the elementary notions that we will need, such as that of a real number or of a continuous real function are briefly introduced. Some basic concepts not dealt with in [Fernandes and Ferreira 2002] are also stressed. For a more detailed study see [Fernandes and Ferreira 2002], [Yamazaki 2005] and [Ferreira 2006].

In TCA^2 we consider two sorts of natural numbers:

- The *tally numbers*, $\mathbb{N}_1$, which are the sequences satisfying $x = \epsilon \vee x = 1 \times x$. The idea is that a natural number y corresponds to the *tally* $\underbrace{1^\frown \ldots {}^\frown 1}_{y\ times}$.

- The *dyadic natural numbers*, $\mathbb{N}_2$, which are the sequences satisfying $x = \epsilon \vee x = 1y$ (with y a word). The idea is that if z is $1x_1x_2 \cdots x_{n-1}$ where each x_i is 0 or 1, then we should view z as the number $\sum_{i=0}^{n-1} x_i 2^{n-i-1}$, with $x_0 = 1$. The empty string ϵ represents the number zero.

We define $0_{\mathbb{N}_1}$, $1_{\mathbb{N}_1}$, $\leq_{\mathbb{N}_1}$, $+_{\mathbb{N}_1}$ and $\cdot_{\mathbb{N}_1}$ as ϵ, 1, $\subseteq$, $^\frown$ and $\times$ (resp.), and obtain a structure of ordered semi-ring in $\mathbb{N}_1$. It is also possible to define $0_{\mathbb{N}_2}$, $1_{\mathbb{N}_2}$, $\leq_{\mathbb{N}_2}$, $+_{\mathbb{N}_2}$ and $\cdot_{\mathbb{N}_2}$ in order to reproduce the usual operations of the natural numbers and verify that $\mathbb{N}_2$ is an ordered semi-ring (see [Ferreira and Oitavem 2006]). The indexes $\mathbb{N}_1$ and $\mathbb{N}_2$ are omitted whenever it is clear from the context which operations are being used.

Remark 4.1 - *By the* counting axiom *it is possible to do the counting in* $\mathbb{N}_2$ *as well.*

- *The induction scheme along* $\mathbb{N}_2$ *is valid in* TCA^2*:*

$$A(0_{\mathbb{N}_2}) \wedge \forall x \in \mathbb{N}_2 (A(x) \rightarrow A(x +_{\mathbb{N}_2} 1)) \rightarrow \forall x \in \mathbb{N}_2 A(x),$$

with A *a* $\Delta^{1,b}_1$*-extended formula.*

We follow [Fernandes and Ferreira 2002] and let the *dyadic rational numbers*, $\mathbb{D}$, be triples $(0, x, y)$ and $(1, x, y)$ (coded as strings in a smooth way), with $x \in \mathbb{N}_2$ and $y = \epsilon \vee y = z1$ (with z a word). The idea is that the triple $(s, x_0 \dots x_{n-1}, y_0 \dots y_{m-1})$ represents the rational number $(-1)^s(\sum_{i=0}^{n-1} x_i 2^{n-i-1} + \sum_{j=0}^{m-1} \frac{y_j}{2^{j+1}})$. Usually we denote such dyadic rational number by $\pm x_0 x_1 \dots x_{n-1} \cdot y_0 \dots y_{m-1}$. The middle dot "$\cdot$" is the *radix point* of the bitwise representation of the dyadic number. It is possible to introduce $0_{\mathbb{D}}$, $1_{\mathbb{D}}$, $\leq_{\mathbb{D}}$, $+_{\mathbb{D}}$ and $\cdot_{\mathbb{D}}$ extending, to the dyadic rational numbers, the operations already mentioned in the dyadic natural numbers. Such operations reproduce the usual operations in the rational numbers and with them $\mathbb{D}$ becomes an ordered ring. We can also introduce: $-_{\mathbb{D}}$, $-_{\mathbb{N}_1}$ (considering positive and *negative* tally) and $|x|$ in the expected way.

Given a *tally* n, we use the notation 2^n to abbreviate the representation of the dyadic rational number $+1\underbrace{00\dots0}_{n\ zeros}\cdot\epsilon$. Note that it is just notation, since both the dyadic and unary exponential functions are not total in TCA^2.

Notation 4.1 *The dyadic rational numbers of the form $+m \cdot \epsilon$ are sometimes used when we want to refer to the dyadic natural numbers m.*

Functions are suitable sets of codes of ordered pairs. A function $\alpha : \mathbb{N}_1 \to \mathbb{D}$ is a *real number* if $|\alpha(n) - \alpha(m)| \leq 2^{-n}$ for all $n \leq m$. Two real numbers α and β are *equal* and we write $\alpha = \beta$ if $\forall n \in \mathbb{N}_1 |\alpha(n) - \beta(n)| \leq 2^{-n+1}$. We identify each dyadic rational number x with the real number α_x defined by the constant function $\alpha_x(n) = x$, for all $n \in \mathbb{N}_1$. Therefore, we have a natural embedding of $\mathbb{D}$ into $\mathbb{R}$. The basic arithmetical operations, with the usual properties, can be defined on the real numbers in the following way:

- $\alpha + \beta$ is the real number $n \rightsquigarrow \alpha(n+1) + \beta(n+1)$
- $\alpha - \beta$ is the real number $n \rightsquigarrow \alpha(n+1) - \beta(n+1)$
- $\alpha \cdot \beta$ is the real number $n \rightsquigarrow \alpha(n+k) \cdot \beta(n+k)$, where k is the least *tally* such that $|\alpha(0)| + |\beta(0)| + 2 \leq 2^k$ (the symbol $\cdot$ is usually omitted)
- $\alpha \leq \beta$ is defined by $\forall n(\alpha(n) \leq \beta(n) + 2^{-n+1})$
- $\alpha < \beta$ is defined by $\alpha \leq \beta \wedge \alpha \neq \beta$
- $|\alpha|$ is the real number $n \rightsquigarrow |\alpha(n)|$,

and, with these operations, the real numbers form an ordered field.

In the above, the only operation not introduced in [Fernandes and Ferreira 2002] is the absolute value $|\alpha|$, which is obviously a real number because $||\alpha|(n) - |\alpha|(m)| = ||\alpha(n)| - |\alpha(m)|| \leq |\alpha(n) - \alpha(m)| \leq 2^{-n}$, for all $n \leq m$. Note that real numbers are sets, i.e., second-order entities. So, the existence of particular real numbers always amounts to set formation (using the available comprehension).

The relations $=$ and $\leq$ in the real numbers can be expressed by formulas of the form $\forall x\varphi$ and the relations $\neq$ and $<$ can be expressed by formulas of the form $\exists x\varphi$ with φ a $\Sigma_0^{1,b}$-formula. We also know that $\alpha(n) - 2^{-n} \leq \alpha \leq \alpha(n) + 2^{-n}$.

Although we can not talk in general about sets of real numbers (the language only allows sets of words), we use expressions of the form $\forall \alpha \in \mathbb{R} \ldots$ or $\alpha \in [\beta, \gamma]$ to stand for $\forall\alpha$*(if* α *is a real number then* $\ldots$*)* or α *is a real number and* $\beta \leq \alpha \leq \gamma$, respectively.

Since the main purpose is to develop analysis, the concept of continuous real function is essential. In [Simpson 1999] and [Yamazaki 2005] we can find different definitions. In this paper we broadly follow [Simpson 1999], adapted to our weak setting (for more information, see [Fernandes and Ferreira 2002]). In the following definition $(x,n)\Phi(y,k)$ can informally be seen as stating that the elements in $]x - 2^{-n}, x + 2^{-n}[$ are applied under Φ in $[y - 2^{-k}, y + 2^{-k}]$.

A *continuous partial function from* $\mathbb{R}$ *to* $\mathbb{R}$ is a set Φ of codes of quintuples (denoted by $\langle w, x, n, y, k\rangle$) satisfying:

- if $\langle w, x, n, y, k\rangle \in \Phi$ then w is a first-order element, $x, y \in \mathbb{D}, n, k \in \mathbb{N}_1$
- if $(x,n)\Phi(y,k)$ and $(x,n)\Phi(y',k')$ then $|y - y'| \leq 2^{-k} + 2^{-k'}$
- if $(x,n)\Phi(y,k)$ and $(x',n') < (x,n)$ then $(x',n')\Phi(y,k)$
- if $(x,n)\Phi(y,k)$ and $(y,k) < (y',k')$ then $(x,n)\Phi(y',k')$,

where $(x,n)\Phi(y,k)$ abbreviates the formula $\exists w \langle w, x, n, y, k\rangle \in \Phi$ and $(x',n') < (x,n)$ abbreviates $|x - x'| + 2^{-n'} < 2^{-n}$.

From [Fernandes and Ferreira 2002], we know that the *identity function Id*, the *constant function* C_γ with $\gamma \in \mathbb{R}$, the *sum* $\Phi_1 + \Phi_2$ and *product* $\Phi_1 \cdot \Phi_2$ of continuous functions can be defined as continuous functions. Next we present another example of a continuous function from $\mathbb{R}$ to $\mathbb{R}$: the modulus function.

We define the *modulus function* $|.|$, by $(x,n)|.|(y,k)$ if $x, y \in \mathbb{D}, n, k \in \mathbb{N}_1$ and $||x| - y| \leq 2^{-k} - 2^{-n}$. Note that set $\{\langle \epsilon, x, n, y, k\rangle : ||x| - y| \leq 2^{-k} - 2^{-n}\}$ exists in TCA^2 and is officially the modulus function. It can be seen that the set is, indeed, a continuous function from $\mathbb{R}$ to $\mathbb{R}$ according to the definition.

Now we present some standard definitions:

- Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$. A real number α *is in the domain of* Φ, denoted by $\alpha \in \mathrm{dom}(\Phi)$, if

  $$\forall k \in \mathbb{N}_1 \exists n \in \mathbb{N}_1 \exists x, y \in \mathbb{D}(|\alpha - x| < 2^{-n} \wedge (x,n)\Phi(y,k)).$$

  Or equivalently: $\forall k \in \mathbb{N}_1 \exists n \in \mathbb{N}_1 \exists y \in \mathbb{D}(\alpha(n+1), n)\Phi(y,k)$.

- Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$, and let α be a real number in the domain of Φ. We say that a real number β *is the value of* α *under the function* Φ, denoted by $\Phi(\alpha) = \beta$, if

$$\forall x, y \in \mathbb{D} \forall n, k \in \mathbb{N}_1 ((x, n)\Phi(y, k) \wedge |\alpha - x| < \tfrac{1}{2^n} \rightarrow |\beta - y| \leq \tfrac{1}{2^k}).$$

From [Fernandes and Ferreira 2002], we know that, if Φ is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and $\alpha \in \mathrm{dom}(\Phi)$, then there is a real number β satisfying $\Phi(\alpha) = \beta$ and it is unique (modulo the equality of reals). The proof in BTFA is not straightforward, but in TCA^2 is much easier since minimization is available (see [Simpson 1999] and [Ferreira 2006]).

Remark 4.2 *Given $\gamma \in \mathbb{R}$ and $\alpha \in \mathrm{dom}(\Phi)$, the formula $\Phi(\alpha) \leq \gamma$ is an abbreviation of $\exists \beta (\Phi(\alpha) = \beta \wedge \beta \leq \gamma)$ $(\star)$. We can prove that such formula is equivalent to $\forall x, y \in \mathbb{D} \forall n, k \in \mathbb{N}_1 ((x, n)\Phi(y, k) \wedge |\alpha - x| < \frac{1}{2^n} \rightarrow y \leq \gamma + \frac{1}{2^k})$ $(\dagger)$. Thus, equivalent to a $\forall \Sigma_0^{1,b}$-formula. We immediately obtain $(\dagger)$ from $(\star)$. The other implication comes from the possibility of fixing β such that $\Phi(\alpha) = \beta$ and from the fact that $\alpha \in \mathrm{dom}(\Phi)$ implies that $\forall k \in \mathbb{N}_1 \exists y \in \mathbb{D}(|\beta - y| \leq \frac{1}{2^k} \wedge y \leq \gamma + \frac{1}{2^k})$, and consequently $\forall k(\beta \leq \gamma + \frac{1}{2^{k-1}})$. So, $\beta \leq \gamma$ because k can be chosen as large as we want. Obviously also $\Phi(\alpha) < \gamma$ is equivalent to a $\exists \Sigma_0^{1,b}$-formula and, by definition, $\Phi(\alpha) = \gamma$ is equivalent to a $\forall \Sigma_0^{1,b}$-formula.*

Of course, we have $\forall \alpha \in \mathbb{R}$, $Id(\alpha) = \alpha$, $C_\gamma(\alpha) = \gamma$, and if $\alpha \in \mathrm{dom}(\Phi_1)$, $\alpha \in \mathrm{dom}(\Phi_2)$, $\Phi_1(\alpha) = \beta_1$ and $\Phi_2(\alpha) = \beta_2$ then $\alpha \in \mathrm{dom}(\Phi_1 + \Phi_2)$, $\alpha \in \mathrm{dom}(\Phi_1 \cdot \Phi_2)$, $(\Phi_1 + \Phi_2)(\alpha) = \beta_1 + \beta_2$ and $(\Phi_1 \cdot \Phi_2)(\alpha) = \beta_1 \cdot \beta_2$.

Proposition 4.1 *The modulus function $|.|$ (introduced before) is a continuous total function and $\forall \alpha \in \mathbb{R}\ \ |.|(\alpha) = |\alpha|$.*

Proof. Take $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_1$. We have $|\alpha - \alpha(k+1)| \leq 2^{-(k+1)} < 2^{-k}$ and $||\alpha(k+1)| - |\alpha(k+1)|| = 0 = 2^{-k} - 2^{-k}$. So, there are $n := k \in \mathbb{N}_1$, $x := \alpha(k+1) \in \mathbb{D}$, $y := |\alpha(k+1)| \in \mathbb{D}$ such that $(|\alpha - x| < 2^{-n} \wedge (x, n)|.|(y, k))$. Thus $\alpha \in \mathrm{dom}(|.|)$. Let us prove now that $|.|(\alpha) = |\alpha|$, i.e., $\forall x, y \in \mathbb{D} \forall n, k \in \mathbb{N}_1((x, n)|.|(y, k) \wedge |\alpha - x| < \frac{1}{2^n} \rightarrow ||\alpha| - y| \leq \frac{1}{2^k})$. Take $x, y \in \mathbb{D}$ and $n, k \in \mathbb{N}_1$ and suppose that $(x, n)|.|(y, k)$ and $|\alpha - x| < \frac{1}{2^n}$. Then $||\alpha| - y| = ||\alpha| - |x| + |x| - y| \leq ||\alpha| - |x|| + ||x| - y| \leq |\alpha - x| + 2^{-k} - 2^{-n} < \frac{1}{2^k}$. Therefore, $|.|(\alpha) = |\alpha|$. $\square$

Definition 4.1 *A continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ is* total *if all the real numbers are in the domain of the function. It is* total in the interval $[\alpha, \beta]$ *if every real number in $[\alpha, \beta]$ is in the domain of the function.*

We finish the study of the continuous functions showing that the usual notion of *composition* of real continuous function can be formalized in TCA^2.

Let Φ_1 and Φ_2 be continuous partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Define $(x, n)(\Phi_1 \circ \Phi_2)(y, k)$ iff $x, y \in \mathbb{D} \wedge n, k \in \mathbb{N}_1 \wedge$

$$\exists x' \in \mathbb{D} \exists n' \in \mathbb{N}_1 \setminus \{0\} (x, n)\Phi_2(x', n') \wedge (x', n' - 1)\Phi_1(y, k).$$

Let $\theta(x,n,y,k)$ be the previous formula. Note that θ is (logically) equivalent to a $\exists\Sigma_0^{1,b}$-formula, i.e., it is of the form $\exists w\theta'(w,x,n,y,k)$, with θ' a $\Sigma_0^{1,b}$-formula. Then, the set $\{\langle w,x,n,y,k\rangle : \theta'(w,x,n,y,k)\}$ exists in TCA^2 and is officially the *composition* $\Phi_1 \circ \Phi_2$.

Let us prove that $\Phi_1 \circ \Phi_2$ is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$.

If $(x,n)(\Phi_1 \circ \Phi_2)(y,k)$ and $(x,n)(\Phi_1 \circ \Phi_2)(y',k')$ then there are $x' \in \mathbb{D}$ and $n' \in \mathbb{N}_1 \setminus \{0\}$ such that $(x,n)\Phi_2(x',n') \wedge (x',n'-1)\Phi_1(y,k)$ and there are $x'' \in \mathbb{D}$ and $n'' \in \mathbb{N}_1 \setminus \{0\}$ such that $(x,n)\Phi_2(x'',n'') \wedge (x'',n''-1)\Phi_1(y',k')$.

Since Φ_2 is a continuous function, $|x'-x''| \leq 2^{-n'} + 2^{-n''}$. If $|x'-x''| < 2^{-n'}+2^{-n''}$ then there are $z \in \mathbb{D}$ and $m \in \mathbb{N}_1$ such that $(z,m) < (x',n') \wedge (z,m) < (x'',n'')$. To see that, suppose that $x' \leq x''$. It is enough to consider the interval $[x''-2^{-n''}, x'+2^{-n'}]$ and to take z as $\frac{x''-2^{-n''}+x'+2^{-n'}}{2}$ and m as an element of $\mathbb{N}_1$ such that $2^{-m} < \frac{x'+2^{-n'}-x''+2^{-n''}}{2}$. Since we have $(x',n') < (x',n'-1) \wedge (x'',n'') < (x'',n''-1)$. We conclude that $(z,m)\Phi_1(y,k) \wedge (z,m)\Phi_1(y',k')$, and so $|y-y'| \leq 2^{-k}+2^{-k'}$. If $|x'-x''| = 2^{-n'}+2^{-n''}$, since $(x',n') < (x',n'-1)$ and $(x'',n'') < (x'',n''-1)$, we have $(x,n)\Phi_2(x',n'-1)$ and $(x,n)\Phi_2(x'',n''-1)$, with $|x'-x''| < 2^{-(n'-1)} + 2^{-(n''-1)}$. Just apply the previous case. The other two conditions can be verified in a straightforward way.

Proposition 4.2 *Take Φ_1 and Φ_2 continuous partial functions from $\mathbb{R}$ to $\mathbb{R}$. If $\alpha \in \mathrm{dom}(\Phi_2)$ and $\Phi_2(\alpha) \in \mathrm{dom}(\Phi_1)$, then $\alpha \in \mathrm{dom}(\Phi_1 \circ \Phi_2)$ and $(\Phi_1 \circ \Phi_2)(\alpha) = \Phi_1(\Phi_2(\alpha))$.*

Proof. Fix $k \in \mathbb{N}_1$ and take $\beta \in \mathbb{R}$ such that $\Phi_2(\alpha) = \beta$. Since $\beta \in \mathrm{dom}(\Phi_1)$, there are $n' \in \mathbb{N}_1$ and $y \in \mathbb{D}$ such that $(\beta(n'+1),n')\Phi_1(y,k)$. Also there are $n \in \mathbb{N}_1$ and $y' \in \mathbb{D}$ such that $(\alpha(n+1),n)\Phi_2(y',n'+3)$ because $\alpha \in \mathrm{dom}(\Phi_2)$. Let us prove that $(\alpha(n+1),n)(\Phi_1 \circ \Phi_2)(y,k)$. By definition of $\Phi_1 \circ \Phi_2$ we have to prove that there are $w \in \mathbb{D}$ and $m \in \mathbb{N}_1 \setminus \{0\}$ such that $(\alpha(n+1),n)\Phi_2(w,m) \wedge (w,m-1)\Phi_1(y,k)$. Take $w := y'$ and $m := n'+3$. Obviously, we have $(\alpha(n+1),n)\Phi_2(w,m)$. It remains to prove that $(y',n'+2)\Phi_1(y,k)$. We know $|\beta(n'+1)-y'| + \frac{1}{2^{n'+2}} \leq |\beta(n'+1)-\beta| + |\beta-y'| + \frac{1}{2^{n'+2}} \leq \frac{1}{2^{n'+1}} + \frac{1}{2^{n'+3}} + \frac{1}{2^{n'+2}} < \frac{1}{2^{n'}}$. So $(y',n'+2) < (\beta(n'+1),n')$. Since $(\beta(n'+1),n')\Phi_1(y,k)$, we have that $(y',n'+2)\Phi_1(y,k)$.

Let us prove that $(\Phi_1 \circ \Phi_2)(\alpha) = \Phi_1(\Phi_2(\alpha))$, i.e., giving λ, β, γ real numbers such that $\lambda = (\Phi_1 \circ \Phi_2)(\alpha)$, $\beta = \Phi_2(\alpha)$ and $\gamma = \Phi_1(\beta)$ we have $\lambda = \gamma$.

From the above, given $k \in \mathbb{N}_1$, there are $n, n' \in \mathbb{N}_1$ and $y \in \mathbb{D}$ such that $(\alpha(n+1),n)(\Phi_1 \circ \Phi_2)(y,k) \wedge (\beta(n'+1),n')\Phi_1(y,k)$. Because $\lambda = (\Phi_1 \circ \Phi_2)(\alpha)$, $(\alpha(n+1),n)(\Phi_1 \circ \Phi_2)(y,k)$ and $|\alpha - \alpha(n+1)| \leq \frac{1}{2^{n+1}} < \frac{1}{2^n}$ we have $|\lambda - y| \leq \frac{1}{2^k}$. Since $\Phi_1(\beta) = \gamma$ and $(\beta(n'+1),n')\Phi_1(y,k)$ and $|\beta - \beta(n'+1)| \leq \frac{1}{2^{n'+1}} < \frac{1}{2^{n'}}$, we have $|\gamma - y| \leq \frac{1}{2^k}$. But $|\lambda-\gamma| = |\lambda-y+y-\gamma| \leq |\lambda-y|+|y-\gamma| \leq \frac{1}{2^k}+\frac{1}{2^k} = \frac{1}{2^{k-1}}$. So $|\lambda-\gamma| \leq 0$. Note that k can be chosen arbitrarily large. Therefore, $\lambda = \gamma$. $\square$

Corollary 4.1 *Take Φ a continuous total function in $[\alpha_1, \alpha_2]$. The function $|.| \circ \Phi$, abbreviated by $|\Phi|$, is a continuous total function in $[\alpha_1, \alpha_2]$ and $|\Phi|(\alpha) = |\Phi(\alpha)|$, $\forall \alpha \in [\alpha_1, \alpha_2]$.*

We need to strengthen the notion of *continuity*. We work with the concept of *modulus of uniform continuity.*

Definition 4.2 *Let Φ be a continuous total function (respectively continuous total in $[\alpha_1, \alpha_2]$). A* modulus of uniform continuity (m.u.c.) for *Φ is a strictly increasing function h from $\mathbb{N}_1$ to $\mathbb{N}_1$ such that for all $n \in \mathbb{N}_1$ and for all $\alpha, \beta \in \mathbb{R}$ (respectively $\alpha, \beta \in [\alpha_1, \alpha_2]$), if $|\alpha - \beta| < 2^{-h(n)}$ then $|\Phi(\alpha) - \Phi(\beta)| < 2^{-n}$.*

The following is easy:

Proposition 4.3 *a) Id, $|.|$, C_γ and $C_\gamma \cdot Id$, with $\gamma \in \mathbb{R}$, are functions with a modulus of uniform continuity.*

b) If Φ_1 and Φ_2 are continuous total functions or continuous total in $[\alpha_1, \alpha_2]$ with a m.u.c then $\Phi_1 + \Phi_2$ has a m.u.c.

c) If Φ_2 is a continuous total function (respectively continuous total in $[\alpha_1, \alpha_2]$) with a m.u.c. and Φ_1 is a continuous total function (respectively continuous total in $[\beta_1, \beta_2]$) with a m.u.c. (and satisfying $\forall \alpha \in [\alpha_1, \alpha_2]\ \Phi_2(\alpha) \in [\beta_1, \beta_2]$), then $\Phi_1 \circ \Phi_2$ has a m.u.c. In particular, if Φ has a m.u.c. then $|\Phi|$ has a m.u.c.

5 On the way to integration: sums

Let us make two preliminary observations that will be used often in the sequel. Take f a function from X to Y (e.g. $f : \mathbb{N}_1 \to \mathbb{D}$, $f : \mathbb{N}_1 \times \mathbb{N}_2 \to \mathbb{N}_2, \ldots$). The formula $\theta(x) := x \in X \wedge \theta'(x, f(x))$, with θ' a $\Sigma^{1,b}_0$-formula is equivalent to a $\Sigma^{1,b}_0$-formula (we allow a set parameter). In fact, $\theta(x)$ can be expressed in the following equivalent forms: $x \in X \wedge \exists y(f(x) = y \wedge \theta'(x, y))$ or $x \in X \wedge \forall y(f(x) = y \to \theta'(x, y))$. So, by recursive comprehension (in TCA^2) we can form the set $Z := \{x \in X : \theta(x)\}$ and $\theta(x)$ is equivalent to $x \in Z$. The second observation is the following. Let $\theta(x, y)$ be a $\exists\Sigma^{1,b}_1$-formula. If it is possible to prove, in TCA^2, that $\forall x \exists y \theta(x, y)$ and that y is unique, then $\theta(x, y)$ is equivalent to a $\Sigma^{1,b}_0$-formula. Indeed, we can form the set $\{\langle x, y\rangle : \theta(x, y)\}$ because we have recursive comprehension and $\underbrace{\theta(x, y)}_{\exists\Sigma^{1,b}_1 - formula} \leftrightarrow \underbrace{\forall y'(\theta(x, y') \to y = y')}_{\forall\Pi^{1,b}_1 - formula}$. Obviously the result is still valid with the restrictions $x \in \mathbb{N}_1$ or $x \in \mathbb{N}_2$.

Lemma 5.1 *Let f be a function from $X \times \mathbb{N}_2$ to $\mathbb{N}_2$. Then there is a function g from $X \times \mathbb{N}_2$ to $\mathbb{N}_2$ such that $\forall x \in X \forall n \in \mathbb{N}_2 \forall i \leq_{\mathbb{N}_2} n (f(x, i) \leq g(x, n))$.*

Proof. Take $x \in X$ and $n \in \mathbb{N}_2$. Obviously we have $\forall i \leq_{\mathbb{N}_2} n \exists s f(x,i) \leq s$, just take s as being $f(x,i)$. By bounded collection $\exists r' \forall i \leq_{\mathbb{N}_2} n \exists s \preceq r' f(x,i) \leq s$. So $\exists r \forall i \leq_{\mathbb{N}_2} n f(x,i) \leq r$, for instance $r = 1 \times r'$. Thus, $\forall x \in X \forall n \in \mathbb{N}_2 \exists r \in \mathbb{N}_2 \forall i \leq_{\mathbb{N}_2} n f(x,i) \leq r$. Consider the $\Sigma_0^{1,b}$-formula ϕ defined by $\phi(x,n,r) := \forall i \leq_{\mathbb{N}_2} n f(x,i) \leq_{\mathbb{N}_2} r$ and the set $g := \{\langle\langle x,n\rangle,r\rangle : x \in X \wedge n \in \mathbb{N}_2 \wedge r \in \mathbb{N}_2 \wedge \phi(x,n,r) \wedge \forall r' < r \neg\phi(x,n,r')\}$. Since $\forall x \in X \forall n \in \mathbb{N}_2 \exists r \in \mathbb{N}_2 \phi(x,n,r)$, applying minimization we have $\forall x \in X \forall n \in \mathbb{N}_2 \exists r(\phi(x,n,r) \wedge \forall r' < r \neg\phi(x,n,r'))$. Thus g is a function from $X \times \mathbb{N}_2$ to $\mathbb{N}_2$ satisfying the desired condition. □

Theorem 5.1 *Given $f : X \times \mathbb{N}_2 \to \mathbb{N}_2$, there is Σ_f a function from $X \times \mathbb{N}_2$ to $\mathbb{N}_2$ s.t. $\forall x \in X \forall n \in \mathbb{N}_2[\Sigma_f(x,0) = f(x,0) \wedge \Sigma_f(x,n+1) = \Sigma_f(x,n) + f(x,n+1)]$.*

Proof. Informally notice that, for all x in X, we have $f(x,0) + f(x,1) + \cdots + f(x,n) = \#\{r : r <_{\mathbb{N}_2} f(x,0)\} + \cdots + \#\{r : r <_{\mathbb{N}_2} f(x,n)\} = \#\{\langle r,i\rangle : i \leq_{\mathbb{N}_2} n \wedge r <_{\mathbb{N}_2} f(x,i)\} = \#\{u : \exists i, r \preceq u(u = \langle r,i\rangle \wedge i \leq_{\mathbb{N}_2} n \wedge r <_{\mathbb{N}_2} f(x,i))\}$, where $\#$ is the number of elements in the set that belong to $\mathbb{N}_2$.

Given $x \in X$ and $n \in \mathbb{N}_2$ let Z be the set $\{u : \exists i, r \preceq u(u = \langle r,i\rangle \wedge i \leq_{\mathbb{N}_2} n \wedge r <_{\mathbb{N}_2} f(x,i))\}$ where we suppose that the pairing function has some nice properties, viz. that it is monotone (in the sense of $\preceq$) in both arguments and such that $r, i \preceq \langle r,i\rangle$. Note that from Lemma 5.1, there is a function g such that $u \in Z \to u \preceq \langle g(x,n),n\rangle$. Fix $t := \langle (x,n),n\rangle$. Applying the counting axiom in $\mathbb{N}_2$ (see Remark 4.1), there is an explicit term v_t constructed from t and $C \preceq v_t$ such that $\langle u,j\rangle \in C$ iff there are j elements of $\mathbb{N}_2$ less than or equal to u in Z. Take $\Sigma_f = \{\langle\langle x,n\rangle,s\rangle : x \in X \wedge n \in \mathbb{N}_2 \wedge \exists Z \preceq \langle g(x,n),n\rangle \exists C \preceq v_t(\forall u \preceq \langle g(x,n),n\rangle(u \in Z \leftrightarrow \exists i, r \preceq u(u = \langle r,i\rangle \wedge i \leq n \wedge r < f(x,i))) \wedge Count(C,Z) \wedge \langle\langle g(x,n),n\rangle,s\rangle \in C)\}$. Since for all $x \in X, n \in \mathbb{N}_2$ there is one and only one s in the previous conditions, the set Σ_f exists in TCA^2, and we can easily prove that it does define a function that satisfies $\Sigma_f(x,0) = f(x,0)$ and $\Sigma_f(x,n+1) = \Sigma_f(x,n) + f(x,n+1), \forall x \in X, \forall n \in \mathbb{N}_2$. □

Next, we extend the notion of *sum along* $\mathbb{N}_2$ to elements of $\mathbb{D}_0^+$ (non negative dyadic rational numbers) and then to elements of $\mathbb{D}$.

Proposition 5.1 *Given $f : X \times \mathbb{N}_2 \to \mathbb{D}_0^+$, there is a function $\Sigma_f : X \times \mathbb{N}_2 \to \mathbb{D}_0^+$ s.t. $\Sigma_f(x,0) = f(x,0)$ and $\Sigma_f(x,n+1) = \Sigma_f(x,n) + f(x,n+1), \forall x \in X, \forall n \in \mathbb{N}_2$.*

Proof. We reduce this summation to a summation of the kind discussed in the previous theorem. By the argument of Lemma 5.1, we may take a function $g : X \times \mathbb{N}_2 \to \mathbb{N}_1$ such that $\forall x \in X \forall l \in \mathbb{N}_2 \forall n \leq_{\mathbb{N}_2} l\, (f(x,n) \cdot_{\mathbb{D}} 2^{g(x,l)} \in \mathbb{N}_2)$. Now, define $f' : X \times \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ as follows:

$$f'(x,l,n) = \begin{cases} f(x,n) \cdot_{\mathbb{D}} 2^{g(x,l)} & \text{if } n \leq_{\mathbb{N}_2} l \\ 0_{\mathbb{N}_2} & \text{otherwise} \end{cases}$$

By the previous theorem, there is $\Sigma_{f'} : X \times \mathbb{N}_2 \times \mathbb{N}_2 \to \mathbb{N}_2$ such that:

$$\forall x \in X \forall l \in \mathbb{N}_2 \forall n \in \mathbb{N}_2 \big(\Sigma_{f'}(x,l,0) = f'(x,l,0) \wedge \\ \Sigma_{f'}(x,l,n+1) = \Sigma_{f'}(x,l,n) + f'(x,l,n+1)\big).$$

We now define $\Sigma_f : X \times \mathbb{N}_2 \to \mathbb{D}_0^+$ by $\Sigma_f(x,n) = 2^{-g(x,n)}\Sigma_{f'}(x,n,n)$. It is not difficult to show, using the available forms of induction, that this function has the desired properties. □

Proposition 5.2 *Given $f : X \times \mathbb{N}_2 \to \mathbb{D}$, there is a function $\Sigma_f : X \times \mathbb{N}_2 \to \mathbb{D}$ s.t. $\Sigma_f(x,0) = f(x,0)$ and $\Sigma_f(x,n+1) = \Sigma_f(x,n)+f(x,n+1)$, $\forall x \in X, \forall n \in \mathbb{N}_2$.*

Proof. This can be reduced to the previous case by separating the positive and negative summands. □

Definition 5.1 *Given f a function from $X \times \mathbb{N}_2$ to $\mathbb{D}$, $x \in X$ and $n \in \mathbb{N}_2$, we denote by $\sum_{i=0}^n f(x,i)$ the dyadic rational number $\Sigma_f(x,n)$.*

For simplicity, the properties are presented in functions of only one variable ($f : \mathbb{N}_2 \to \mathbb{D}$). The extension to domains with more variables is straightforward.

Definition 5.2 *Take $f : \mathbb{N}_2 \to \mathbb{D}$, $g : \mathbb{N}_2 \to \mathbb{D}$ and $\lambda \in \mathbb{D}$. We can consider the functions $f+g$, λf and $|f|$ defined respectively by $(f+g)(n) = f(n)+g(n)$, $(\lambda f)(n) = \lambda f(n)$ and $|f|(n) = |f(n)|$, $\forall n \in \mathbb{N}_2$.*

Proposition 5.3 *Take $f : \mathbb{N}_2 \to \mathbb{D}$, $g : \mathbb{N}_2 \to \mathbb{D}$, $n \in \mathbb{N}_2$ and $\lambda \in \mathbb{D}$*

a) $\sum_{i=0}^n (f+g)(i) = \sum_{i=0}^n f(i) + \sum_{i=0}^n g(i)$

b) $\sum_{i=0}^n (\lambda f)(i) = \lambda \sum_{i=0}^n f(i)$

c) $\sum_{i=0}^n \lambda = \lambda \cdot (n+1)$

d) $|\sum_{i=0}^n f(i)| \leq \sum_{i=0}^n |f|(i)$

e) $\sum_{i=0}^n i = \frac{(n+1)n}{2}$

f) If $f(i) \leq g(i)$ for all $i \leq n$, then $\sum_{i=0}^n f(i) \leq \sum_{i=0}^n g(i)$.

Proof. All the clauses can be easily proved by induction on $n \in \mathbb{N}_2$. □

Definition 5.3 *Take $f : \mathbb{N}_2 \to \mathbb{D}$ and $n, m \in \mathbb{N}_2$ such that $n \leq m$. Then*

$\sum_{i=n}^m f(i) := \sum_{i=0}^m f(i) - \sum_{i=0}^{n-1} f(i)$.

Using the notation above and the properties of the sum along $\mathbb{N}_2$, we immediately deduce the following equalities:

– $\sum_{i=n}^n f(i) = f(n)$

- $\sum_{i=0}^{n+m} f(i) = \sum_{i=0}^{n-1} f(i) + \sum_{i=n}^{n+m} f(i)$
- $\sum_{i=n}^{m+1} f(i) = \sum_{i=n}^{m} f(i) + f(m+1)$, with $n \leq m$
- $\sum_{i=n+1}^{m} f(i) = \sum_{i=n}^{m} f(i) - f(n)$, with $n < m$.

The following two properties can also be easily proven with the available induction:

Proposition 5.4 *Let f be a function from $\mathbb{N}_2$ to $\mathbb{D}$ and $n, k \in \mathbb{N}_2$. Then*

$\sum_{i=0}^{n} f(k+i) = \sum_{i=k}^{n+k} f(i)$.

Proposition 5.5 *Given n, k, $m \in \mathbb{N}_2 \backslash \{0\}$, such that $n = k \cdot m$, we have that*

$\sum_{i=0}^{n-1} f(i) = \sum_{j=0}^{k-1} \sum_{i=0}^{m-1} f(jm+i)$.

6 On the way to integration: miscellanea

Definition 6.1 *A* sequence of real numbers *is a function $f : \mathbb{N}_1 \times \mathbb{N}_1 \rightarrow \mathbb{D}$ such that for all $n \in \mathbb{N}_1$ the function $f_n : \mathbb{N}_1 \rightarrow \mathbb{D}$ defined by $f_n(k) = f(k, n)$ is a real number. We denote by $(\alpha_n)_{n \in \mathbb{N}_1}$ the sequence f, with $f_n = \alpha_n$.*

Remark 6.1 *- If $(\alpha_n)_{n \in \mathbb{N}_1}$ and $(\beta_n)_{n \in \mathbb{N}_1}$ are sequences of real numbers and $\lambda \in \mathbb{R}$, then we can form the sequences $(\alpha_n + \beta_n)_{n \in \mathbb{N}_1}$, $(\alpha_n \cdot \beta_n)_{n \in \mathbb{N}_1}$ and $(\lambda \alpha_n)_{n \in \mathbb{N}_1}$. For instance, the* sum *sequence is coded by the set*

$$\{\langle\langle m, n\rangle, d\rangle : m, n \in \mathbb{N}_1 \wedge d \in \mathbb{D} \wedge d = (\alpha_n)_{n \in \mathbb{N}_1}(m+1, n) + (\beta_n)_{n \in \mathbb{N}_1}(m+1, n)\}.$$

- Given $\alpha \in \mathbb{R}$, $X = \{\langle\langle m, n\rangle, d\rangle : m \in \mathbb{N}_1 \wedge n \in \mathbb{N}_1 \wedge d \in \mathbb{D} \wedge \langle m, d\rangle \in \alpha\}$ is the constant sequence (equal to α).

Definition 6.2 *A sequence of real numbers $(\alpha_n)_{n \in \mathbb{N}_1}$ is* bounded *if there is $\alpha \in \mathbb{R}$ such that $\forall n \in \mathbb{N}_1 |\alpha_n| \leq \alpha$; it is* increasing (*respectively* strictly increasing) *if $\forall n \in \mathbb{N}_1(\alpha_n \leq \alpha_{n+1})$ (respectively $\alpha_n < \alpha_{n+1}$) and* decreasing (*respectively* strictly decreasing) *if $\forall n \in \mathbb{N}_1(\alpha_{n+1} \leq \alpha_n)$ (respectively $\alpha_{n+1} < \alpha_n$).*

Definition 6.3 *A sequence of real numbers $(\alpha_n)_{n \in \mathbb{N}_1}$* converges *to the real number α, denoted $\alpha = \lim_n \alpha_n$, if $\forall k \in \mathbb{N}_1 \exists n \in \mathbb{N}_1 \forall i \in \mathbb{N}_1 |\alpha - \alpha_{n+i}| < 2^{-k}$.*

A sequence $(\alpha_n)_{n \in \mathbb{N}_1}$ is convergent *if there is $\alpha \in \mathbb{R}$ such that $\lim_n \alpha_n = \alpha$.*

Let us continue, listing some basic properties of limits with proofs that can be easily formalizable in TCA^2.

Proposition 6.1 *Take $(\alpha_n)_{n \in \mathbb{N}_1}$, $(\beta_n)_{n \in \mathbb{N}_1}$, $(\gamma_n)_{n \in \mathbb{N}_1}$ sequences of real numbers and $\alpha, \beta, \gamma \in \mathbb{R}$. In TCA^2 we can prove the following*

1. *If there is the limit of a sequence of real numbers, then that limit is unique.*
2. *If* $(\alpha_n)_{n\in\mathbb{N}_1}$ *is a constant sequence equal to* α. *Then* $\lim_n \alpha_n = \alpha$.
3. *If* $\alpha_n = \beta_n$ *for all* n *after a certain given value, then if* $\lim_n \alpha_n = \alpha$ *we have* $\lim_n \beta_n = \alpha$.
4. *If* $\lim_n \alpha_n = \alpha$ *and* $\lim_n \beta_n = \beta$, *then* $(\alpha_n+\beta_n)_{n\in\mathbb{N}_1}$ *is a convergent sequence and* $\lim_n(\alpha_n + \beta_n) = \alpha + \beta$.
5. *Every convergent sequence is bounded.*
6. *If* $\lim_n \alpha_n = 0$ *and* $(\beta_n)_{n\in\mathbb{N}_1}$ *is a bounded sequence, then* $(\alpha_n \cdot \beta_n)_{n\in\mathbb{N}_1}$ *is a convergent sequence and* $\lim_n(\alpha_n \cdot \beta_n) = 0$.
7. *If* $\lim_n \alpha_n = \alpha$ *and* $\lim_n \beta_n = \beta$, *then* $(\alpha_n \cdot \beta_n)_{n\in\mathbb{N}_1}$ *is a convergent sequence and* $\lim_n(\alpha_n \cdot \beta_n) = \alpha \cdot \beta$.
8. *If* $\lim_n \alpha_n = \alpha$, *then* $(\lambda\alpha_n)_{n\in\mathbb{N}_1}$ *is a convergent sequence and* $\lim_n(\lambda\alpha_n) = \lambda\alpha$.
9. *If* $\forall n \in \mathbb{N}_1(\alpha_n \leq \gamma_n \leq \beta_n)$ *and* $\lim_n \alpha_n = \lim_n \beta_n = \alpha$, *then* $(\gamma_n)_{n\in\mathbb{N}_1}$ *is a convergent sequence and* $\lim_n \gamma_n = \alpha$.

Remark 6.2 *Given* $\alpha \in \mathbb{R}$ *and* $n \in \mathbb{N}_1$ *the dyadic rational number* $\alpha(n)$ *is well determined and* $\alpha(n) =_{\mathbb{D}} d$ *is a* $\Sigma_0^{1,b}$*-formula, since it abbreviates* $\langle n, d\rangle \in \alpha$.

However, if Φ *is a continuous partial function from* $\mathbb{R}$ *to* $\mathbb{R}$ *and* $\alpha \in \mathrm{dom}(\Phi)$, *although the expression* $\Phi(\alpha)$ *is well defined (modulo the equality of reals) the expression* $\Phi(\alpha)(n)$ *is not. Note that we can have* $\Phi(\alpha) = \beta$ *and* $\Phi(\alpha) = \gamma$, *obviously with* $\beta =_{\mathbb{R}} \gamma$, *but with* $\beta(n) \neq \gamma(n)$.

In order to control the complexity of the formula that defines the integral (see Section 7), it is necessary to introduce the expression $\Phi(\alpha, n)$, which intuitively can be seen as $\lambda(n)$ for a certain $\lambda = \Phi(\alpha)$, avoiding the complexity of examining $\Phi(\alpha)$. Next, we will define $\Phi(\alpha, n)$ in detail.

Let Φ be a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and α a real number in the domain of Φ. Consider the formula

$$\begin{aligned}\varphi(n, r) :\leftrightarrow \exists w \exists k[&\langle w, \alpha(k + 1), k, r, n + 1\rangle \in \Phi \wedge \forall\langle r', w', k'\rangle < \langle r, w, k\rangle \\ &\langle w', \alpha(k' + 1), k', r', n + 1\rangle \notin \Phi].\end{aligned}$$

Given $n \in \mathbb{N}_1$, let us prove that there is a unique $r \in \mathbb{D}$ s.t. $\varphi(n, r)$. Since $\alpha \in \mathrm{dom}(\Phi)$, we have that there are $k \in \mathbb{N}_1$ and $r \in \mathbb{D}$ such that $(\alpha(k+1), k)\Phi(r, n+1)$ i.e., there is w such that $\langle w, \alpha(k + 1), k, r, n + 1\rangle \in \Phi$. If (r, w, k) is not (in code) the least triple such that $\langle w, \alpha(k + 1), k, r, n + 1\rangle \in \Phi$, choose the least in these conditions. Notice that the minimization scheme is available in TCA^2 for $\Delta_1^{1,b}$-extended formulas.

The coding of triples ensures that if $r \neq r'$ then $\langle r', w', k' \rangle \neq \langle r, w, k \rangle$, so the uniqueness follows immediately.

Definition 6.4 *Given Φ a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and α a real number in the domain of Φ, we let $\Phi(\alpha, n) = r :\leftrightarrow \varphi(n, r)$.*

Remark 6.3 *Accordingly, $\forall n \in \mathbb{N}_1 \exists^1 r \in \mathbb{D}\, \Phi(\alpha, n) = r$. $\Phi(\alpha, n)$ denotes the unique dyadic rational number satisfying $\varphi(n, \Phi(\alpha, n))$. Of course, $\Phi(\alpha, n) = r$ is a $\Delta_1^{1,b}$-extended formula (with parameter α).*

Proposition 6.2 *If Φ is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$, $\alpha \in \mathrm{dom}(\Phi)$ and $n \in \mathbb{N}_1$, then $|\Phi(\alpha, n) - \Phi(\alpha)| \leq \frac{1}{2^{n+1}}$, independently of the representative chosen for $\Phi(\alpha)$, i.e., $\forall \beta \in \mathbb{R}(\Phi(\alpha) = \beta \rightarrow |\Phi(\alpha, n) - \beta| \leq \frac{1}{2^{n+1}})$.*

Proof. Since $\varphi(n, \Phi(\alpha, n))$, there is k such that $(\alpha(k+1), k)\Phi(\Phi(\alpha, n), n+1)$. But $|\alpha - \alpha(k+1)| \leq \frac{1}{2^{k+1}} < \frac{1}{2^k}$, so, by definition of $\Phi(\alpha) = \beta$, we have $|\beta - \Phi(\alpha, n)| \leq \frac{1}{2^{n+1}}$. Thus, $|\Phi(\alpha, n) - \beta| \leq \frac{1}{2^{n+1}}$.

□

The next proposition emphasizes the possibility of considering a canonical representative for the value of a real under a continuous function.

Proposition 6.3 *Take Φ a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$ and $\alpha \in \mathrm{dom}(\Phi)$. The function $\lambda : \mathbb{N}_1 \rightarrow \mathbb{D}$ defined by $\lambda(n) = \Phi(\alpha, n)$ is a real number and $\Phi(\alpha) = \lambda$.*

Proof. The existence of a set, in TCA^2, that codes the function λ, was ensured in the previous observations. Let us prove that λ is a real number. Take $n \leq m$. $|\lambda(n) - \lambda(m)| = |\Phi(\alpha, n) - \Phi(\alpha, m)| = |\Phi(\alpha, n) - \Phi(\alpha) + \Phi(\alpha) - \Phi(\alpha, m)| \leq |\Phi(\alpha, n) - \Phi(\alpha)| + |\Phi(\alpha) - \Phi(\alpha, m)|$, independently of the representative chosen for $\Phi(\alpha)$. By Proposition 6.2, $|\Phi(\alpha, k) - \Phi(\alpha)| \leq \frac{1}{2^{k+1}}$, $\forall k \in \mathbb{N}_1$. So, $|\lambda(n) - \lambda(m)| \leq \frac{1}{2^{n+1}} + \frac{1}{2^{m+1}} \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$. Therefore, λ is a real number. Also note that for all n, $|\Phi(\alpha) - \lambda| \leq |\Phi(\alpha) - \Phi(\alpha, n)| + |\Phi(\alpha, n) - \lambda| \leq \frac{1}{2^{n+1}} + |\lambda(n) - \lambda| \leq \frac{1}{2^{n-1}}$. Since n can be chosen arbitrarily large, $|\Phi(\alpha) - \lambda| = 0$, i.e. $\Phi(\alpha) = \lambda$.

□

7 The Riemann integral

We now introduce the notion of the Riemann integral. In order to simplify notation, we restrict the definition to the interval $[0, 1]$.

Definition 7.1 *Take Φ a continuous total function in $[0,1]$, with a modulus of uniform continuity, h. We define* the integral between 0 and 1 of Φ*, denoted by $\int_0^1 \Phi(t)\ dt$, in the following way:*

$$\int_0^1 \Phi(t)\ dt :=_{\mathbb{R}} \lim_n S_n$$

where, for all $n \in \mathbb{N}_1$, $S_n = \sum_{i=0}^{2^{h(n)}-1} \frac{1}{2^{h(n)}} \Phi(\frac{i}{2^{h(n)}}, n)$.

Remark 7.1 *By the definition of the expressions of the form $\Phi(\alpha, n) = r$ (and subsequent discussions), it is easy to see that $f : \mathbb{N}_1 \times \mathbb{N}_2 \to \mathbb{D}$, defined by $f(n,i) = \frac{1}{2^{h(n)}} \Phi(\frac{i}{2^{h(n)}}, n)$ is indeed a function in TCA^2. Observe also that it is possible to consider (in TCA^2) sums of the form $\sum_{i=0}^{2^{h(n)}-1} f(n,i)$, with f a function from $\mathbb{N}_1 \times \mathbb{N}_2$ to $\mathbb{D}$. In fact, $\sum_{i=0}^{2^{h(n)}-1} f(n,i) = \Sigma_f(n, 2^{h(n)} - 1)$, with $h(n) \in \mathbb{N}_1$, so, $2^{h(n)} - 1$ is a dyadic rational number which can be seen as an element of $\mathbb{N}_2$ (see Notation 4.1).*

From the previous remark, the equality $d =_{\mathbb{D}} S_n$ is given by a $\Sigma_0^{1,b}$-formula. So the set $X = \{\langle n, d\rangle : n \in \mathbb{N}_1 \wedge d = S_n\}$ exists in TCA^2 and it makes sense to consider the sequence $(S_n)_{n\in\mathbb{N}_1} = \{\langle\langle m, n\rangle, d\rangle : m, n \in \mathbb{N}_1 \wedge d \in \mathbb{D} \wedge \langle n, d\rangle \in X\}$.

In order to ensure that the integral is well defined we have to prove that the sequence $(S_n)_{n\in\mathbb{N}_1}$ is convergent.

Proposition 7.1 *The sequence $(S_n)_{n\in\mathbb{N}_1}$ is a Cauchy sequence, i.e.,*

$$\forall n \in \mathbb{N}_1 \exists p \in \mathbb{N}_1 \forall k \in \mathbb{N}_1 (p < k \to |S_p - S_k| < 2^{-n}).$$

Proof. Take $p < k$. We have $h(p) < h(k)$, so, by the sum properties,

$$\sum_{i=0}^{2^{h(p)}-1} \frac{1}{2^{h(p)}} \Phi(\frac{i}{2^{h(p)}}, p) = \sum_{i=0}^{2^{h(p)}-1} \frac{1}{2^{h(k)}} \Phi(\frac{i}{2^{h(p)}}, p) 2^{h(k)-h(p)} =$$

$$\sum_{i=0}^{2^{h(p)}-1} \frac{1}{2^{h(k)}} \Phi(\frac{i}{2^{h(p)}}, p) \sum_{j=0}^{2^{h(k)-h(p)}-1} 1 =$$

$$\sum_{i=0}^{2^{h(p)}-1} \sum_{j=0}^{2^{h(k)-h(p)-1}} \frac{1}{2^{h(k)}} \Phi(\frac{i}{2^{h(p)}}, p).$$

Since $2^{h(k)} = 2^{h(p)} \cdot 2^{h(k)-h(p)}$, by Proposition 5.5, $\sum_{i=0}^{2^{h(k)}-1} \frac{1}{2^{h(k)}} \Phi(\frac{i}{2^{h(k)}}, k) = \sum_{i=0}^{2^{h(p)}-1} \sum_{j=0}^{2^{h(k)-h(p)}-1} \frac{1}{2^{h(k)}} \Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}}, k)$. So,

$$|S_p - S_k| = |\sum_{i=0}^{2^{h(p)}-1} \frac{1}{2^{h(p)}} \Phi(\frac{i}{2^{h(p)}}, p) - \sum_{i=0}^{2^{h(k)}-1} \frac{1}{2^{h(k)}} \Phi(\frac{i}{2^{h(k)}}, k)| =$$

$$|\sum_{i=0}^{2^{h(p)}-1} \sum_{j=0}^{2^{h(k)-h(p)}-1} \frac{1}{2^{h(k)}} [\Phi(\frac{i}{2^{h(p)}}, p) - \Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}}, k)]| \leq$$

$$\sum_{i=0}^{2^{h(p)}-1} \sum_{j=0}^{2^{h(k)-h(p)}-1} \frac{1}{2^{h(k)}} |\Phi(\frac{i}{2^{h(p)}}, p) - \Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}}, k)|.$$

But $\frac{1}{2^{h(k)}}|\Phi(\frac{i}{2^{h(p)}},p)-\Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}},k)| \leq \frac{1}{2^{h(k)}}(|\Phi(\frac{i}{2^{h(p)}},p)-\Phi(\frac{i}{2^{h(p)}})| + |\Phi(\frac{i}{2^{h(p)}})-\Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}})| + |\Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}})-\Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}},k)|)$ $(\star)$.

Given $\alpha \in \mathbb{R}$, we know that $|\Phi(\alpha)-\Phi(\alpha,n)| \leq 2^{-n}$. Since, by hypothesis, h is a m.u.c for Φ, and $\forall i \in \{0,\ldots,2^{h(p)}-1\}\forall j \in \{0,\ldots,2^{h(k)-h(p)}-1\}$ $\frac{i}{2^{h(p)}} \in [0,1]$, $\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}} \in [0,1]$ and $|\frac{i}{2^{h(p)}}-\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}}| = |\frac{j}{2^{h(k)}}| \leq \frac{2^{h(k)-h(p)}-1}{2^{h(k)}} = \frac{1}{2^{h(p)}}-\frac{1}{2^{h(k)}} < \frac{1}{2^{h(p)}}$, we know that $|\Phi(\frac{i}{2^{h(p)}})-\Phi(\frac{i2^{h(k)-h(p)}+j}{2^{h(k)}})| < \frac{1}{2^p}$. So $(\star) < \frac{1}{2^{h(k)}}(\frac{1}{2^p}+\frac{1}{2^p}+\frac{1}{2^k})$.

Therefore, $|S_p - S_k| < \sum_{i=0}^{2^{h(p)}-1}\sum_{j=0}^{2^{h(k)-h(p)}-1}\frac{1}{2^{h(k)}}(\frac{1}{2^p}+\frac{1}{2^p}+\frac{1}{2^k}) = \frac{1}{2^{h(k)}}(\frac{1}{2^{p-1}}+\frac{1}{2^k})2^{h(p)}2^{h(k)-h(p)} = \frac{1}{2^{p-1}}+\frac{1}{2^k} < \frac{1}{2^{p-1}}+\frac{1}{2^p} < \frac{1}{2^{p-2}}$.

We proved that, given $n \in \mathbb{N}_1$, there is $p = n+2 \in \mathbb{N}_1$ such that $\forall k \in \mathbb{N}_1(p < k \rightarrow |S_p - S_k| < \frac{1}{2^n})$. In conclusion, $(S_n)_{n\in\mathbb{N}_1}$ is a Cauchy sequence. □

Remark 7.2 *In the previous argument, we actually prove that* $(S_n)_{n\in\mathbb{N}_1}$ *has a* modulus of Cauchy convergence*, i.e., there is a function* $p : \mathbb{N}_1 \rightarrow \mathbb{N}_1$ *strictly increasing (*$p(n) = n+2$*), such that*

$$\forall n \in \mathbb{N}_1 \forall k \in \mathbb{N}_1 (p(n) < k \rightarrow |S_{p(n)} - S_k| < 2^{-n}).$$

It is not true that every Cauchy sequence is convergent. Such an assertion is equivalent to ACA_0 over RCA_0 (see [Simpson 1999]). However:

Lemma 7.1 *Let* $(\beta_n)_{n\in\mathbb{N}_1}$ *be a sequence of real numbers and* $p : \mathbb{N}_1 \rightarrow \mathbb{N}_1$ *a modulus of Cauchy convergence for* $(\beta_n)_{n\in\mathbb{N}_1}$*. Then the function* $\alpha : \mathbb{N}_1 \rightarrow \mathbb{D}$ *defined by* $\alpha(n) = \beta_{p(n+3)}(n+3)$ *is a real number and* $\lim_n \beta_n = \alpha$*.*

Proof. Let us prove that α is a real number.

Take $n, m \in \mathbb{N}_1$ such that $n \leq m$. We have that $|\alpha(n)-\alpha(m)| = |\beta_{p(n+3)}(n+3)-\beta_{p(m+3)}(m+3)| \leq |\beta_{p(n+3)}(n+3)-\beta_{p(n+3)}| + |\beta_{p(n+3)}-\beta_{p(m+3)}| + |\beta_{p(m+3)}-\beta_{p(m+3)}(m+3)| \leq \frac{1}{2^{n+3}}+\frac{1}{2^{n+3}}+\frac{1}{2^{m+3}} \leq \frac{1}{2^n}$. Note that, in order to ensure that $|\beta_{p(n+3)}-\beta_{p(m+3)}| < 2^{-(n+3)}$, we use the fact that p is a modulus of Cauchy convergence for $(\beta_n)_{n\in\mathbb{N}_1}$ and satisfies $p(n+3) \leq p(m+3)$.

Let us prove that $(\beta_n)_{n\in\mathbb{N}_1}$ converges for α, i.e., given $k \in \mathbb{N}_1$ we want to prove that $\exists n \in \mathbb{N}_1 \forall i \in \mathbb{N}_1 |\alpha - \beta_{n+i}| < 2^{-k}$.

For all $m > k$ and $n, i \in \mathbb{N}_1$, $|\alpha - \beta_{n+i}| \leq |\alpha - \alpha(m)| + |\alpha(m) - \beta_{n+i}(m)| + |\beta_{n+i}(m) - \beta_{n+i}| \leq \frac{1}{2^m} + |\beta_{p(m+3)}(m+3) - \beta_{n+i}(m)| + \frac{1}{2^m} \leq \frac{1}{2^m} + |\beta_{p(m+3)}(m+3) - \beta_{p(m+3)}| + |\beta_{p(m+3)} - \beta_{n+i}| + |\beta_{n+i} - \beta_{n+i}(m)| + \frac{1}{2^m}$. For $n = p(k+3)$, $|\alpha - \beta_{n+i}| \leq \frac{1}{2^m} + \frac{1}{2^{m+3}} + |\beta_{p(m+3)} - \beta_{p(k+3)+i}| + \frac{1}{2^m} + \frac{1}{2^m} \leq \frac{1}{2^{m-2}} + |\beta_{p(m+3)} - \beta_{p(k+3)}| + |\beta_{p(k+3)} - \beta_{p(k+3)+i}|$. Since $(\beta_n)_{n\in\mathbb{N}_1}$ is a sequence with a modulus of Cauchy convergence, $|\alpha - \beta_{n+i}| < \frac{1}{2^{m-2}} + 2^{-(k+3)} + 2^{-(k+3)} = \frac{1}{2^{m-2}} + \frac{1}{2^{k+2}} \leq \frac{1}{2^{m-2}} + \frac{1}{2^k}$. Since m can be chosen arbitrarily large, $|\alpha - \beta_{n+i}| < \frac{1}{2^k}$. □

By Remark 7.2 and Lemma 7.1, we know that $(S_n)_{n\in\mathbb{N}_1}$ is a convergent sequence. So, we can consider the real number $\lim_n S_n$.

It remains to prove that the Riemann integral does not depend on the function chosen as a modulus of uniform continuity.

Proposition 7.2 *Let Φ be a continuous total function in $[0,1]$ and h_1, h_2 m.u.c for Φ. Then*

$$\lim_n \sum_{i=0}^{2^{h_1(n)}-1} \frac{1}{2^{h_1(n)}}\Phi(\frac{i}{2^{h_1(n)}}, n) = \lim_n \sum_{i=0}^{2^{h_2(n)}-1} \frac{1}{2^{h_2(n)}}\Phi(\frac{i}{2^{h_2(n)}}, n).$$

Proof. It is easy to check that, for all $n \in \mathbb{N}_1$,

$$\left|\sum_{i=0}^{2^{h_1(n)}-1} \frac{1}{2^{h_1(n)}}\Phi(\frac{i}{2^{h_1(n)}}, n) - \sum_{i=0}^{2^{h_2(n)}-1} \frac{1}{2^{h_2(n)}}\Phi(\frac{i}{2^{h_2(n)}}, n)\right| \leq \frac{1}{2^{n-2}}.$$

$\square$

Therefore, the Riemann integral is well defined.

Remark 7.3 - *Take Φ and $(S_n)_{n\in\mathbb{N}_1}$ as in Definition 7.1. By Lemma 7.1 and Remark 7.2, we know that $(S_n)_{n\in\mathbb{N}_1}$ is convergent. Furthermore, we also know that it converges for α, the real number defined by $\alpha(n) = S_{p(n+3)}(n+3)$, i.e., $\alpha(n) = S_{n+5}(n+3)$. Since S_{n+5} is a dyadic rational number, we have $\alpha(n) = S_{n+5}$, $\forall n \in \mathbb{N}_1$. So, $\int_0^1 \Phi(t)\,dt = \alpha$. Note that the real number α, defined by $\alpha(n) = S_{n+5}$, exists in TCA^2, since $d = \sum_{i=0}^{2^{h(n+5)}-1} \frac{1}{2^{h(n+5)}}\Phi(\frac{i}{2^{h(n+5)}}, n+5)$ can be expressed — see Remark 7.1 — by means of a $\Sigma_0^{1,b}$-formula.*

- $\int_0^1 \Phi(t)\,dt = \beta$, with $\beta \in \mathbb{R}$, is equivalent to a $\forall\Sigma_0^{1,b}$-formula. Note that it is equivalent to the formula $\alpha = \beta$, with α the real number defined above.

- In a similar way, $\int_0^1 \Phi(t)\,dt < \beta$ and $\int_0^1 \Phi(t)\,dt \leq \beta$ are equivalent to $\exists\Sigma_0^{1,b}$ and $\forall\Sigma_0^{1,b}$-formulas respectively.

Next, we establish some of the usual integral properties.

Proposition 7.3 *Let Φ and Ψ be continuous total functions in the interval $[0,1]$ with a modulus of uniform continuity and let γ be a real number:*

a) $\int_0^1 \gamma\,dt = \gamma$

b) $\int_0^1 t\,dt = \frac{1}{2}$

c) $\int_0^1 (\Phi+\Psi)(t)\,dt = \int_0^1 \Phi(t)\,dt + \int_0^1 \Psi(t)\,dt$

d) $|\int_0^1 \Phi(t)dt| \leq \int_0^1 |\Phi|(t)dt$

e) *If $\Phi(t) = \Psi(t)$ for all $t \in [0,1]$, then* $\int_0^1 \Phi(t)\,dt = \int_0^1 \Psi(t)\,dt$

f) If $\Phi(t) \leq \Psi(t)$ for all $t \in [0,1]$, then $\int_0^1 \Phi(t)\,dt \leq \int_0^1 \Psi(t)\,dt$

g) $\int_0^1 \gamma\Phi(t)\,dt = \gamma \int_0^1 \Phi(t)\,dt$.

Proof. We present the proof of assertion a). Similar demonstrations work for the other assertions (see [Ferreira 2006]).

a) By Proposition 4.3, fix h a m.u.c. for C_γ. By Remark 7.3, we know that $\int_0^1 \gamma\,dt = \int_0^1 C_\gamma(t)\,dt$ is equal to α, with α the real number defined by $\alpha(n) = S_{n+5} = \sum_{i=0}^{2^{h(n+5)}-1} \frac{1}{2^{h(n+5)}} C_\gamma(\frac{i}{2^{h(n+5)}}, n+5)$. We prove that $\alpha = \gamma$.

Take $n \in \mathbb{N}_1$. We have $|\alpha(n) - \gamma(n)| = |\sum_{i=0}^{2^{h(n+5)}-1} \frac{1}{2^{h(n+5)}} C_\gamma(\frac{i}{2^{h(n+5)}}, n+5) - \gamma(n)| = |\sum_{i=0}^{2^{h(n+5)}-1} \frac{1}{2^{h(n+5)}} C_\gamma(\frac{i}{2^{h(n+5)}}, n+5) - \sum_{i=0}^{2^{h(n+5)}-1} \frac{1}{2^{h(n+5)}}(\gamma - \gamma + \gamma(n))| \leq \frac{1}{2^{h(n+5)}} \sum_{i=0}^{2^{h(n+5)}-1} |C_\gamma(\frac{i}{2^{h(n+5)}}, n+5) - \gamma + \gamma - \gamma(n)|$. Since, for all i, $|C_\gamma(\frac{i}{2^{h(n+5)}}, n+5) - \gamma + \gamma - \gamma(n)| \leq |C_\gamma(\frac{i}{2^{h(n+5)}}, n+5) - C_\gamma(\frac{i}{2^{h(n+5)}})| + |\gamma - \gamma(n)| \leq \frac{1}{2^{n+5}} + \frac{1}{2^n}$, we have $|\alpha(n) - \gamma(n)| \leq \frac{1}{2^{h(n+5)}} 2^{h(n+5)}(\frac{1}{2^{n+5}} + \frac{1}{2^n}) = \frac{1}{2^{n+5}} + \frac{1}{2^n} \leq \frac{1}{2^{n-1}}$. So $\alpha = \gamma$. □

In a quite similar way, we can (as we sketch next) introduce the Riemann integral with arbitrary dyadic rational limits.

Definition 7.2 *Take $x, y \in \mathbb{D}$ such that $x < y$ and let Φ be a continuous total function in the interval $[x, y]$, with a modulus of uniform continuity h. We define the* integral between x and y *of Φ, denoted by $\int_x^y \Phi(t)\,dt$, in the following way:*

$$\int_x^y \Phi(t)\,dt :=_{\mathbb{R}} \lim_n S_n$$

where, for all $n \in \mathbb{N}_1$, $S_n = \sum_{i=0}^{2^{h(n)}-1} \frac{y-x}{2^{h(n)}} \Phi(x + \frac{(y-x)i}{2^{h(n)}}, n)$.

With a strategy completely similar to the one used in the context of $[0,1]$, we can see that the previous notion of integral is well defined and has no ambiguities. We just call the attention for some minor adaptations needed in this context:

- Fix $l \in \mathbb{N}_1$ such that $y - x \leq 2^l$. It is possible to prove that $p : \mathbb{N}_1 \to \mathbb{N}_1$ defined by $p(n) = n+2l+2$ is a modulus of Cauchy convergence for $(S_n)_{n\in\mathbb{N}_1}$. The proof is similar to that of Proposition 7.1.
- Apropos the independence of the integral relatively to the modulus of uniform continuity chosen, note that a result similar to Proposition 7.2 is still valid. Fixing $l \in \mathbb{N}_1$ we consider the upper bound $\frac{1}{2^{n-2l-2}}$.
- The extension of the integral has the purpose of studying, in the next section, the fundamental theorem of calculus for functions defined in $[0,1]$. So, we will just work with integrals with limits $x, y \in \mathbb{D}$ such that $0 \leq x < y \leq 1$. Therefore, adapting the study developed in $[0,1]$ is still easier.

- Considerations analogous to the ones stated in Remark 7.3 can still be made within the context $[x, y]$. In particular, we have that $\int_x^y \Phi(t)\, dt = \alpha$, with α the real number defined by $\alpha(n) = \sum_{i=0}^{2^{h(n+5)}-1} \frac{y-x}{2^{h(n+5)}} \Phi(x + \frac{(y-x)i}{2^{h(n+5)}}, n+5)$ and the complexity of the formulas involving integrals is still controlled. The results described below are similar to the case $[0, 1]$.

Proposition 7.4 *$\int_x^y \Phi(t)\, dt = \beta$ and $\int_x^y \Phi(t)\, dt \leq \beta$ are equivalent to $\forall\Sigma_0^{1,b}$-formulas and $\int_x^y \Phi(t)\, dt < \beta$ is equivalent to a $\exists\Sigma_0^{1,b}$-formula.*

Proposition 7.5 *Let Φ and Ψ be continuous total functions in the interval $[x, y]$, with a modulus of uniform continuity and $\gamma \in \mathbb{R}$:*

a) $\int_x^y \gamma\, dt = \gamma \cdot (y - x)$

b) $\int_x^y t\, dt = \frac{y^2-x^2}{2}$

c) $\int_x^y (\Phi + \Psi)(t)\, dt = \int_x^y \Phi(t)\, dt + \int_x^y \Psi(t)\, dt$

d) $|\int_x^y \Phi(t)\, dt| \leq \int_x^y |\Phi|(t)\, dt$

e) If $\forall t \in [x, y](\Phi(t) = \Psi(t))$ *then* $\int_x^y \Phi(t)\, dt = \int_x^y \Psi(t)\, dt$

f) If $\forall t \in [x, y](\Phi(t) \leq \Psi(t))$ *then* $\int_x^y \Phi(t)\, dt \leq \int_x^y \Psi(t)\, dt$

g) $\int_x^y \gamma\Phi(t)\, dt = \gamma \int_x^y \Phi(t)\, dt$.

In the remaining lines of this section we prove the additivity of the Riemann integral. For the sake of simplicity, we consider x and y dyadic rational numbers in the interval $[0, 1]$. We will use the following technical result:

Lemma 7.2 *Let x and y be dyadic rational numbers with $x < y$ and let Φ be a continuous total function on $[x, y]$ with a m.u.c. h. Then, for tally n sufficiently large, $(y - x)2^{h(n+n)}$ is in $\mathbb{N}_2$ and*

$$\int_x^y \Phi(t)dt = \lim_n \sum_{i=0}^{(y-x)2^{h(n+n)}-1} \frac{1}{2^{h(n+n)}} \Phi(x + \frac{i}{2^{h(n+n)}}, n).$$

Proof. Since x and y are dyadic rational numbers, there are functions k^x and k^y from $\mathbb{N}_1$ to $\mathbb{N}_2$ such that, for tally n sufficiently large, $x = \frac{k^x(n)}{2^n}$ and $y = \frac{k^y(n)}{2^n}$. Note that, for such n, $(y - x)2^{h(n+n)} = 2^{h(n)}q(n)$, where

$$q(n) := (k^y(n) - k^x(n))2^{h(n+n)-(h(n)+n)}.$$

By Proposition 5.5 we have that $\sum_{i=0}^{(y-x)2^{h(n+n)}-1} \frac{1}{2^{h(n+n)}}\Phi(x+\frac{i}{2^{h(n+n)}},n)$ is

$$\sum_{r=0}^{2^{h(n)}-1}\sum_{j=0}^{q(n)-1}\frac{1}{2^{h(n+n)}}\,\Phi(x+\frac{rq(n)+j}{2^{h(n+n)}},n).$$

We will see that this value differs form the following by an amount that goes to zero when $n\to\infty$:

$$\sum_{r=0}^{2^{h(n)}-1}\sum_{j=0}^{q(n)-1}\frac{1}{2^{h(n+n)}}\,\Phi(x+\frac{rq(n)}{2^{h(n+n)}},n).$$

The above is, of course, equal to

$$\sum_{r=0}^{2^{h(n)}-1} q(n)\frac{1}{2^{h(n+n)}}\,\Phi(x+\frac{rq(n)}{2^{h(n+n)}},n),$$

and this simplifies to

$$\sum_{r=0}^{2^{h(n)}-1}\frac{y-x}{2^{h(n)}}\,\Phi(x+r\,\frac{y-x}{2^{h(n)}},n).$$

It remains to verify that the amount mentioned above does indeed converge to 0 as $n\to\infty$. Using the majorization of Proposition 6.2 it is clear that

$$|\Phi(x+\tfrac{rq(n)+j}{2^{h(n+n)}},n)-\Phi(x+\tfrac{rq(n)}{2^{h(n+n)}},n)|\le|\Phi(x+\tfrac{rq(n)+j}{2^{h(n+n)}})-\Phi(x+\tfrac{rq(n)}{2^{h(n+n)}})|+\tfrac{1}{2^n}.$$

Notice that $\frac{q(n)}{2^{h(n+n)}}=\frac{y-x}{2^{h(n)}}\le\frac{1}{2^{h(n)}}$. Therefore, for $j<q(n)$ and using the properties of the m.u.c. h, we get that

$$|\Phi(x+\tfrac{rq(n)+j}{2^{h(n+n)}},n)-\Phi(x+\tfrac{rq(n)}{2^{h(n+n)}},n)|\le\tfrac{1}{2^n}+\tfrac{1}{2^n}=\tfrac{1}{2^{n-1}}.$$

Hence, the amount mentioned above does not exceed $2^{h(n)}q(n)\frac{1}{2^{h(n+n)}}\frac{1}{2^{n-1}}$, which is less than or equal to $\frac{1}{2^{n-1}}$. □

With the aid of the above lemma, the additivity of the Riemann integral is now immediate:

Proposition 7.6 *If z is a dyadic rational number such that $x<z<y$ and Φ is a continuous total function in $[x,y]$ with a modulus of uniform continuity, then*

$$\int_x^z\Phi(t)\,dt+\int_z^y\Phi(t)\,dt=\int_x^y\Phi(t)\,dt.$$

8 The fundamental theorem of calculus

Let Φ be a continuous total function in $[0,1]$ with a modulus of uniform continuity. In what follows, we define Ψ, a continuous total function in $[0,1]$, satisfying $\Psi(x) = \int_0^x \Phi(t)\ dt$ for all dyadic rational number $x \in [0,1]$.

The result below follows immediately from the m.u.c. definition:

Lemma 8.1 *Every continuous total function (respectively continuous total in* $[\alpha_1, \alpha_2]$*), Φ, with a m.u.c. is* uniformly continuous*, i.e.,* $\forall k \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall \alpha, \beta \in \mathbb{R}(|\alpha - \beta| < \frac{1}{2^m} \rightarrow |\Phi(\alpha) - \Phi(\beta)| < \frac{1}{2^k})$ *(respectively* $\forall k \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall \alpha, \beta \in [\alpha_1, \alpha_2](|\alpha - \beta| < \frac{1}{2^m} \rightarrow |\Phi(\alpha) - \Phi(\beta)| < \frac{1}{2^k})$*.*

From [Fernandes and Ferreira 2005], in BTFA (so also in TCA^2), we have the following:

Proposition 8.1 *If Φ is a continuous total function in* $[0,1]$*, uniformly continuous in that interval, then there is $m \in \mathbb{N}_1$ such that for all $\alpha \in [0,1]$, $\Phi(\alpha) \leq 2^m$.*

Remember, we fixed Φ a continuous total function in $[0,1]$ with a m.u.c. By Corollary 4.1 and Proposition 4.3-*c*), we know that $|\Phi|$ is also a continuous total function in $[0,1]$ with a m.u.c., so applying the previous proposition it is possible to take $m \in \mathbb{N}_1$ such that $\forall \alpha \in [0,1], |\Phi|(\alpha) \leq 2^m$.

Let $d : \mathbb{D} \rightarrow \mathbb{D}$ be the function defined by $d(x) = \begin{cases} x \text{ if } 0 \leq x \leq 1 \\ 0 \text{ if } x < 0 \\ 1 \text{ if } 1 < x. \end{cases}$

We define $(x,n)\Psi(y,k)$ as:

$$x, y \in \mathbb{D} \wedge n, k \in \mathbb{N}_1 \wedge |\int_0^{d(x)} \Phi(t)\ dt - y| < \frac{1}{2^k} - \frac{1}{2^{n-m-1}}.$$

Remark 8.1 *The quaternary relation $(x,n)\Psi(y,k)$ is equivalent to a $\exists\Sigma_0^{1,b}$-formula $\exists w \theta'(w,x,n,y,k)$. So, the set $\{\langle w,x,n,y,k\rangle : \theta'(w,x,n,y,k)\}$ is officially the function Ψ.*

Theorem 8.1 (Indefinite integral) *Given Φ a continuous total function from* $[0,1]$ *to* $\mathbb{R}$ *with a m.u.c., let us take Ψ as above. Then Ψ is a continuous total function from* $[0,1]$ *to* $\mathbb{R}$*, and for all dyadic rational number r in* $[0,1]$*, $\Psi(r) = \int_0^r \Phi(t)\ dt$.*

Proof. We first show that Ψ is a *partial* continuous function from $[0,1]$ to $\mathbb{R}$. The first two conditions of the definition of partial continuous function are clear. Let us now prove that if $(x,n)\Psi(y,k) \wedge (x',n') < (x,n)$ then $(x',n')\Psi(y,k)$. We know that $|\int_0^{d(x')} \Phi(t)\ dt - y| \leq |\int_0^{d(x')} \Phi(t)\ dt - \int_0^{d(x)} \Phi(t)\ dt| + |\int_0^{d(x)} \Phi(t)\ dt - y|$. The first term is less than or equal to $|d(x') - d(x)|2^m$ which, by the way we defined $d(x)$, is less than or equal to $|x' - x|2^m$. Since $(x',n') < (x,n)$, we have $|x' - x|2^m < (\frac{1}{2^n} - \frac{1}{2^{n'}})2^m$. By hypothesis, the second term is less than $\frac{1}{2^k} - \frac{1}{2^{n-m-1}}$. So $|\int_0^{d(x')} \Phi(t)\ dt - y| < \frac{1}{2^k} + 2^m(2^{-n} - 2^{-n'} - 2^{n+1}) = \frac{1}{2^k} + 2^m(-2^{-n} - 2^{-n'}) < \frac{1}{2^k} - 2^{m-n'+1}$.

Let us prove the totality of Ψ. Take $\alpha \in \mathbb{R}$ and fix $k \in \mathbb{N}_1$. We must show that $\exists n \in \mathbb{N}_1 \exists y \in \mathbb{D}(\alpha(n+1), n)\Psi(y, k)$. Take $n := m + k + 2$. We know that $\int_0^{d(\alpha(n+1))} \Phi(t)\ dt = \beta$, with $\beta(n) = \sum_{i=0}^{2^{h(n+5)}-1} \frac{d(\alpha(n+1))}{2^{h(n+5)}} \Phi(\frac{d(\alpha(n+1))i}{2^{h(n+5)}}, n+5)$. Take $y := \beta(k+2)$. $|\int_0^{d(\alpha(n+1))} \Phi(t)\ dt - y| = |\beta - \beta(k+2)| \leq \frac{1}{2^{k+2}} < \frac{1}{2^{k+1}} = \frac{1}{2^k} - \frac{1}{2^{n-m-1}}$. So $(\alpha(n+1), n)\Psi(y, k)$.

Finally, we must show that $\Psi(r) = \int_0^r \Phi(t)\ dt$. Let us take x, y, n, k such that $(x, n)\Psi(y, k) \wedge |r - x| < \frac{1}{2^n}$ in order to prove that $|\int_0^r \Phi(t)\ dt - y| \leq \frac{1}{2^k}$. We have $|\int_0^r \Phi(t)\ dt - y| \leq |\int_0^r \Phi(t)\ dt - \int_0^{d(x)} \Phi(t)\ dt| + |\int_0^{d(x)} \Phi(t)\ dt - y| < |r - d(x)|2^m + \frac{1}{2^k} - \frac{1}{2^{n-m-1}} \leq |r - x|2^m + \frac{1}{2^k} - \frac{1}{2^{n-m-1}} < 2^{m-n} + \frac{1}{2^k} - 2^{m-n+1} < \frac{1}{2^k}$. □

Remark 8.2 *Although the function Ψ, defined as a set, depends on the* tally *number m chosen, the values of the dyadic rational numbers in* $[0,1]$ *under Ψ do not depend (modulo equality of reals) on such a choice.*

The previous theorem permits to give a meaning to $\int_0^\alpha \Phi(t)dt$ also for real numbers $\alpha \in [0,1]$. It is just defined as $\Psi(\alpha)$. It is also easy to define $\int_\alpha^\beta \Phi(t)dt$, e.g., by taking an appropriate difference. By working with approximations, Propositions 7.5 and 7.6 can be easily extended to integrals with real limits.

Definition 8.1 *Let Φ be a continuous total function in* $[0,1]$, $\alpha \in [0,1]$ *and* $\beta \in \mathbb{R}$. β is the derivative of Φ at α, *denoted by $\Phi'(\alpha) = \beta$, if*

$$\forall n \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall h \neq 0\, (0 \leq \alpha + h \leq 1 \wedge |h| < \tfrac{1}{2^m} \rightarrow |\tfrac{\Phi(\alpha+h) - \Phi(\alpha)}{h} - \beta| \leq \tfrac{1}{2^n}).$$

Definition 8.2 *Let Φ and Ψ be continuous total functions in* $[0,1]$. Φ is the derivative of Ψ *if $\Phi(\alpha) = \Psi'(\alpha)$,* $\forall \alpha \in [0,1]$.

Theorem 8.2 (The fundamental theorem of calculus) *If Φ is a continuous total function in* $[0,1]$ *with a m.u.c. and Ψ is such that $\Psi(\alpha) = \int_0^\alpha \Phi(t)\ dt, \forall \alpha \in [0,1]$, then Φ is the derivative of Ψ.*

Proof. The usual proof of the theorem goes through in TCA^2. Take $\alpha \in [0,1]$. Let us prove that, if Φ is a continuous total function in $[0,1]$ with a m.u.c. and Ψ is a continuous total function in $[0,1]$ such that $\Psi(\alpha) = \int_0^\alpha \Phi(t)\ dt, \forall \alpha \in [0,1]$ then $\Phi(\alpha) = \Psi'(\alpha)$, i.e., given $n \in \mathbb{N}_1$ there is $m \in \mathbb{N}_1$ such that

$$\forall h \neq 0 \left(0 \leq \alpha + h \leq 1 \wedge |h| < \tfrac{1}{2^m} \rightarrow |\tfrac{\Psi(\alpha+h) - \Psi(\alpha)}{h} - \Phi(\alpha)| \leq \tfrac{1}{2^n}\right).$$

Consider p a m.u.c. for Φ. Given $n \in \mathbb{N}_1$, take $m := p(n)$. Let $h \neq 0$ be such that $0 \leq \alpha + h \leq 1 \wedge |h| < \frac{1}{2^m} = \frac{1}{2^{p(n)}}$. Since p is a m.u.c. for Φ, we have $|\Phi(\alpha) - \Phi(\alpha + k)| < \frac{1}{2^n}$ for all k such that $|k| \leq |h|$.

If $0 < h$, we have $h(\Phi(\alpha) - \frac{1}{2^n}) \leq \int_0^{\alpha+h} \Phi(t)\, dt - \int_0^{\alpha} \Phi(t)\, dt \leq h(\Phi(\alpha) + \frac{1}{2^n})$ and if $h < 0$ we know that $h(\Phi(\alpha) + \frac{1}{2^n}) \leq \int_0^{\alpha+h} \Phi(t)\, dt - \int_0^{\alpha} \Phi(t)\, dt \leq h(\Phi(\alpha) - \frac{1}{2^n})$. Therefore, in each case, $\Phi(\alpha) - \frac{1}{2^n} \leq \frac{\int_0^{\alpha+h} \Phi(t)\, dt - \int_0^{\alpha} \Phi(t)\, dt}{h} \leq \Phi(\alpha) + \frac{1}{2^n}$. We proved that $|\frac{\int_0^{\alpha+h} \Phi(t)\, dt - \int_0^{\alpha} \Phi(t)\, dt}{h} - \Phi(\alpha)| \leq \frac{1}{2^n}$, i.e., $|\frac{\Psi(\alpha+h) - \Psi(\alpha)}{h} - \Phi(\alpha)| \leq \frac{1}{2^n}$. □

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