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# On the Subrecursive Computability of Several Famous Constants

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**Abstract:** For any class $\mathcal{F}$ of total functions in the set $\mathbb{N}$ of the natural numbers, we define the notion of $\mathcal{F}$-computable real number. A real number $\alpha$ is called $\mathcal{F}$-computable if there exist one-argument functions $f$, $g$ and $h$ in $\mathcal{F}$ such that for any $n$ in $\mathbb{N}$ the distance between the rational number $f(n) - g(n)$ over $h(n) + 1$ and the number $\alpha$ is not greater than the reciprocal of $n + 1$. Most concrete real numbers playing a role in analysis can be easily shown to be $\mathcal{E}^3$-computable (as usually, $\mathcal{E}^m$ denotes the $m$-th Grzegorczyk class). Although (as it is proved in Section 5 of this paper) there exist $\mathcal{E}^3$-computable real numbers that are not $\mathcal{E}^2$-computable, we prove that $\pi$, $e$ and other remarkable real numbers are $\mathcal{E}^2$-computable (the number $\pi$ proves to be even $\mathcal{L}$-computable, where $\mathcal{L}$ is the class of Skolem's lower elementary functions). However, only the rational numbers would turn out to be $\mathcal{E}^2$-computable according to a definition of $\mathcal{F}$-computability using $2^n$ instead of $n + 1$.
**Key Words:** computable real number, Grzegorczyk classes, second Grzegorczyk class, lower elementary functions, $\pi$, $e$, Liouville's number, Euler's constant.
**Category:** F.1.3, F.2.1, G.0, G.1.0

## 1 Introduction

Let $\mathcal{F}$ be a class of total functions in the set $\mathbb{N}$ of the natural numbers. We shall call an *$\mathcal{F}$-sequence* any infinite sequence $r_0, r_1, r_2, \ldots$ of rational numbers that has a representation in the form

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, 3, \ldots,$$

with one-argument functions $f$, $g$ and $h$ belonging to $\mathcal{F}$, and a real number $\alpha$ will be called *$\mathcal{F}$-computable* if there exists an $\mathcal{F}$-sequence $r_0, r_1, r_2, \ldots$ such that $|r_n - \alpha| \leq (n + 1)^{-1}$ for all $n$ in $\mathbb{N}$.[1]

In the case when $\mathcal{F}$ is the class of the recursive functions, the $\mathcal{F}$-computable real numbers are exactly the computable ones, although $2^{-n}$ is usually used instead of $(n + 1)^{-1}$ in the definition of computability of a real number (cf.

[1] Under some assumptions about the class $\mathcal{F}$, the $\mathcal{F}$-sequences were called $\mathcal{F}$-expressible in [Skordev 2002], and the $\mathcal{E}^2$-sequences were called $\mathcal{E}^2$-computable in [Skordev 2008]. When $\mathcal{F}$ satisfies the assumptions made in [Skordev 2002], the present definition of $\mathcal{F}$-computability of a real number coincides with the one accepted there.

for instance [Ko 1991, Weihrauch 2000]). Namely the definition obtained from the present one by replacement of $(n+1)^{-1}$ with $2^{-n}$ will be equivalent to it, whenever the class $\mathcal{F}$ is closed under composition and contains some one-argument function that dominates $2^n - 1$. That is the case not only when $\mathcal{F}$ is the class of all recursive functions, but also when it is some Grzegorczyk class $\mathcal{E}^m$ with $m \geq 3$. However, the equivalence is lost, for example, in the case of $\mathcal{F} = \mathcal{E}^2$. Indeed, as seen from the results proved in [Skordev 2002], all real algebraic numbers are $\mathcal{E}^2$-computable in the sense of the present definition[2], whereas only the rational numbers would be $\mathcal{E}^2$-computable in the sense of the definition with $2^{-n}$, as the third statement of the following proposition shows.

**Proposition 1.** *Let $h$ be a one-argument function belonging to the class $\mathcal{E}^2$, and let $r_0, r_1, r_2, \ldots$ be rational numbers such that $(h(n)+1)r_n$ is an integer for any $n \in \mathbb{N}$. Then:*

1. *There exists a polynomial $p(n)$ such that $p(n)|r_n| \geq 1$ holds, whenever $r_n \neq 0$.*
2. *There exists a polynomial $q(n)$ such that $q(n)|r_{n+1} - r_n| \geq 1$ holds, whenever $r_{n+1} \neq r_n$.*
3. *If $\alpha$ is a real number such that $|r_n - \alpha| \leq 2^{-n}$ for all $n$ in $\mathbb{N}$, then $\alpha$ is a rational number.*

*Proof.* The statement 1 follows from the fact that $(h(n)+1)|r_n| \geq 1$, whenever $r_n \neq 0$, and the function $h$ is dominated by some polynomial. The statement 2 can be derived from the statement 1 by taking $r_{n+1} - r_n$ in the role of $r_n$ and using the fact that $(h(n)+1)(h(n+1)+1)(r_{n+1} - r_n)$ is also an integer for any $n \in \mathbb{N}$. To prove the statement 3, suppose $\alpha$ is a real number such that $|r_n - \alpha| \leq 2^{-n}$ for all $n$ in $\mathbb{N}$. Since

$$|r_{n+1} - r_n| \leq |r_{n+1} - \alpha| + |r_n - \alpha| \leq 3 \cdot 2^{-n-1},$$

the polynomial $q(n)$ from the statement 2 will satisfy the inequality

$$3q(n) \geq 2^{n+1}$$

for all $n$ such that $r_{n+1} \neq r_n$, and therefore only finitely many such $n$ can exist.

*Remark.* A weaker result in this direction can be obtained by using Liouville's approximation theorem. Its application proves the statement 3 of the above

[2] Under the assumption that $\mathcal{F}$ contains the successor, projection and product functions, as well as the function $\lambda mn.|m-n|$, and $\mathcal{F}$ is closed under composition and bounded $\mu$-operation, it was proved in [Skordev 2002] that the $\mathcal{F}$-computable real numbers form a field containing the real roots of any non-constant polynomial with coefficients from this field.

proposition under the additional assumption that $\alpha$ is an algebraic number (the possibility of such an application of Liouville's theorem is implicitly indicated in footnote 2 of [Peshev and Skordev 2006]).

Since, as we already mentioned, all real algebraic numbers are $\mathcal{E}^2$-computable, it is natural to ask whether there exist $\mathcal{E}^2$-computable transcendental numbers.[3] A positive answer to this question was given in the paper [Skordev 2008], where, in particular, the numbers $\pi$ and $e$ were shown to be $\mathcal{E}^2$-computable. The present paper is a wholly revised and extended version of the most essential parts of [Skordev 2008]. A radical change is done in the proofs that the considered concrete real numbers are $\mathcal{E}^2$-computable. Namely some general statements about $\mathcal{F}$-computability of sums of series are proved now, and applications of these statements are done instead of the lengthy direct proofs given in [Skordev 2008]. The number $\pi$ is shown to be even $\mathcal{L}$-computable, where $\mathcal{L}$ is the class of Skolem's lower elementary functions studied in [Skolem 1962].

## 2 $\mathcal{F}$-computable real-valued functions with natural arguments

We shall prepare now some tools for facilitating the proofs of $\mathcal{E}^2$-computability of certain real numbers. Throughout this section, a class $\mathcal{F}$ of total functions in $\mathbb{N}$ will be supposed to be given such that $\mathcal{F}$ contains the zero, successor, projection, addition and Kronecker delta functions, and it is closed under composition and bounded summation (any class $\mathcal{E}^m$ with $m \geq 2$ satisfies these assumptions, and the class $\mathcal{L}$ of the lower elementary functions is the smallest class satisfying them).

Let $l$ be a natural number, and $\theta$ be a function from $\mathbb{N}^l$ into the set $\mathbb{R}$ of the real numbers. The function $\theta$ will be called *$\mathcal{F}$-computable* if there exist $l+1$-argument functions $f$, $g$ and $h$ belonging to $\mathcal{F}$ such that

$$\left| \frac{f(i_1, \ldots, i_l, n) - g(i_1, \ldots, i_l, n)}{h(i_1, \ldots, i_l, n) + 1} - \theta(i_1, \ldots, i_l) \right| \leq \frac{1}{n+1}$$

for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$.

Obviously all values of an $\mathcal{F}$-computable real-valued function with natural arguments are $\mathcal{F}$-computable real numbers, and a real-valued function without arguments is $\mathcal{F}$-computable iff its value at the empty tuple is $\mathcal{F}$-computable (thus the 0-argument $\mathcal{F}$-computable real-valued functions can be identified with the $\mathcal{F}$-computable real numbers). Clearly any substitution of functions from

[3] It is quite easy to see that $\pi$, $e$ and many other concrete real numbers playing a part in analysis are $\mathcal{E}^3$-computable. However, there exist $\mathcal{E}^3$-computable real numbers which are not $\mathcal{E}^2$-computable (cf. Section 5).

the class $\mathcal{F}$ into an $\mathcal{F}$-computable real-valued function with natural arguments produces again an $\mathcal{F}$-computable real-valued function with natural arguments.

Since every infinite sequence of real numbers is actually a function from $\mathbb{N}$ into $\mathbb{R}$, the above definition introduces, in particular, the notion of $\mathcal{F}$-computability for such sequences. Obviously, each $\mathcal{F}$-sequence of rational numbers is $\mathcal{F}$-computable as a sequence of real numbers. In general, however, an infinite sequence of rational numbers can be $\mathcal{F}$-computable as a sequence of real numbers without being an $\mathcal{F}$-sequence. For instance, let $\mathcal{F}$ be a subclass of the class of the recursive functions. Then, by a result proved in [Skolem 1962], there exists a two-argument lower elementary function $\varphi$ such that the set $\{n \in \mathbb{N} \mid \exists t(\varphi(n,t) = 0)\}$ is non-recursive. If we set $r_n$ to be $(s+1)^{-1}$ with $s = \mu t(\varphi(n,t) = 0)$ for any $n$ in the set in question, and to be 0 for all other $n$ in $\mathbb{N}$, then the sequence of the rational numbers $r_0, r_1, r_2, \ldots$ will be $\mathcal{F}$-computable as a sequence of real numbers, but without being an $\mathcal{F}$-sequence.

The first statement in the next proposition shows that one can take $h(i_1, \ldots, i_l, n) = n$ in the definition of $\mathcal{F}$-computability of real-valued functions with natural arguments.[4]

**Proposition 2.** *Let $l$ be a natural number, and $\theta$ be a function from $\mathbb{N}^l$ into $\mathbb{R}$. Then:*

1. *If $\theta$ is $\mathcal{F}$-computable, and $c$ is a real number greater than $1/2$, then there exist $l+1$-argument functions $f$ and $g$ belonging to $\mathcal{F}$ such that*

$$\left| \frac{f(i_1, \ldots, i_l, n) - g(i_1, \ldots, i_l, n)}{n+1} - \theta(i_1, \ldots, i_l) \right| \le \frac{c}{n+1} \tag{1}$$

   *for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$.*

2. *If for some $l+1$-argument functions $f$ and $g$ belonging to $\mathcal{F}$ and some real number $c$ the inequality (1) holds for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$, then $\theta$ is $\mathcal{F}$-computable.*

*Proof.* For the proof of the statement 1, suppose $\theta$ is $\mathcal{F}$-computable, and $c$ is a real number greater than $1/2$. One easily shows the existence of $l+1$-argument functions $f_0$, $g_0$ and $h_0$ belonging to $\mathcal{F}$ such that

$$\left| \frac{f_0(i_1, \ldots, i_l, n) - g_0(i_1, \ldots, i_l, n)}{h_0(i_1, \ldots, i_l, n) + 1} - \theta(i_1, \ldots, i_l) \right| \le \frac{c - 1/2}{n+1}$$

[4] This holds, in particular, for 0-argument functions, thus giving a characterization of the $\mathcal{F}$-computable real numbers which is in the spirit of the definition in [Grzegorczyk 1955] for computability of real numbers (that definition, taken literally, defines computability only of non-negative real numbers, but its extension to arbitrary ones is easy).

for all $i_1, \dots, i_l, n$ in $\mathbb{N}$. There exists also a two-argument function $A$ in $\mathcal{F}$ such that

$$\left| A(i,j) - \frac{i}{j+1} \right| \le \frac{1}{2}$$

for all natural numbers $i$ and $j$; for instance, we may set

$$A(i,j) = \left[ \frac{i}{j+1} + \frac{1}{2} \right].$$

Now set

$$f(i_1, \dots, i_l, n) = A((n+1)(f_0(i_1, \dots, i_l, n) \dot{-} g_0(i_1, \dots, i_l, n)), h_0(i_1, \dots, i_l, n)),$$
$$g(i_1, \dots, i_l, n) = A((n+1)(g_0(i_1, \dots, i_l, n) \dot{-} f_0(i_1, \dots, i_l, n)), h_0(i_1, \dots, i_l, n)).$$

Clearly the functions $f$ and $g$ belong to $\mathcal{F}$. It is easy to see that

$$\left| f(i_1, \dots, i_l, n) - g(i_1, \dots, i_l, n) - (n+1)\frac{f_0(i_1, \dots, i_l, n) - g_0(i_1, \dots, i_l, n)}{h_0(i_1, \dots, i_l, n) + 1} \right| \le \frac{1}{2}$$

both in the case of $f_0(i_1, \dots, i_l, n) \ge g_0(i_1, \dots, i_l, n))$ and in the case of $f_0(i_1, \dots, i_l, n) < g_0(i_1, \dots, i_l, n))$. Therefore

$$\left| \frac{f(i_1, \dots, i_l, n) - g(i_1, \dots, i_l, n)}{n+1} - \frac{f_0(i_1, \dots, i_l, n) - g_0(i_1, \dots, i_l, n)}{h_0(i_1, \dots, i_l, n) + 1} \right| \le \frac{1/2}{n+1},$$

hence the inequality (1) holds. To prove the statement 2, suppose $f, g \in \mathcal{F}$, $c \in \mathbb{R}$, and the inequality (1) holds for all $i_1, \dots, i_l, n$ in $\mathbb{N}$. Then, taking a positive integer $k$ such that $k \ge c$, we shall have

$$\left| \frac{f(i_1, \dots, i_l, kn+k-1) - g(i_1, \dots, i_l, kn+k-1)}{(kn+k-1)+1} - \theta(i_1, \dots, i_l) \right| \le \frac{1}{n+1}$$

for all $i_1, \dots, i_l, n$ in $\mathbb{N}$.

**Proposition 3.** *Let $k$ be a natural number, $\theta$ be a $k+1$-argument real-valued function with natural arguments, and $\theta^{\Sigma}$ be the mapping of $\mathbb{N}^{k+1}$ into $\mathbb{R}$ defined by*

$$\theta^{\Sigma}(i_1, \dots, i_k, t) = \sum_{s=0}^{t} \theta(i_1, \dots, i_k, s).$$

*Then $\theta^{\Sigma}$ is $\mathcal{F}$-computable iff $\theta$ is $\mathcal{F}$-computable.*

*Proof.* Suppose $\theta$ is $\mathcal{F}$-computable. By Proposition 2, there exist $k+2$-argument functions $f$ and $g$ belonging to $\mathcal{F}$ such that

$$\left| \frac{f(i_1, \dots, i_k, s, n) - g(i_1, \dots, i_k, s, n)}{n+1} - \theta(i_1, \dots, i_k, s) \right| \le \frac{1}{n+1}$$

for all $i_1, \ldots, i_k, s, n$ in $\mathbb{N}$. We consider the functions

$$f^{\Sigma}(i_1, \ldots, i_k, t, n) = \sum_{s=0}^{t} f(i_1, \ldots, i_k, s, nt + n + t),$$
$$g^{\Sigma}(i_1, \ldots, i_k, t, n) = \sum_{s=0}^{t} g(i_1, \ldots, i_k, s, nt + n + t).$$

They also belong to the class $\mathcal{F}$, since this class is closed under bounded summation. In addition, for any $i_1, \ldots, i_k, s, n, t$ in $\mathbb{N}$ the number

$$\left| \frac{f(i_1, \ldots, i_k, s, tn + t + n) - g(i_1, \ldots, i_k, s, tn + t + n)}{tn + t + n + 1} - \theta(i_1, \ldots, i_k, s) \right|$$

does not exceed the reciprocal of $(t + 1)(n + 1)$, hence

$$\left| \frac{f^{\Sigma}(i_1, \ldots, i_k, t, n) - g^{\Sigma}(i_1, \ldots, i_k, t, n)}{nt + n + t + 1} - \theta^{\Sigma}(i_1, \ldots, i_k, t) \right| \leq \frac{1}{n + 1}$$

for all $i_1, \ldots, i_l, t, n$ in $\mathbb{N}$. Thus the $\mathcal{F}$-computability of $\theta$ implies the $\mathcal{F}$-computability of $\theta^{\Sigma}$. The converse implication follows from the equality

$$\theta(i_1, \ldots, i_k, t) = \begin{cases} \theta^{\Sigma}(i_1, \ldots, i_k, t) & \text{if } t = 0 \\ \theta^{\Sigma}(i_1, \ldots, i_k, t) - \theta^{\Sigma}(i_1, \ldots, i_k, t - 1) & \text{otherwise.} \end{cases}$$

**Theorem.** *Let $k$ be a natural number, $\theta$ be such an $\mathcal{F}$-computable $k+1$-argument real-valued function with natural arguments that the series*

$$\sum_{s=0}^{\infty} \theta(i_1, \ldots, i_k, s)$$

*converges for all $i_1, \ldots, i_k$ in $\mathbb{N}$, and $\sigma(i_1, \ldots, i_k)$ be the sum of this series. Let there exist a $k + 1$-argument function $p$ belonging to $\mathcal{F}$ and such that*

$$\left| \sum_{s=t+1}^{\infty} \theta(i_1, \ldots, i_k, s) \right| \leq \frac{1}{n + 1}$$

*for any natural numbers $i_1, \ldots, i_k, n$ and $t = p(i_1, \ldots, i_k, n)$. Then the function $\sigma$ is also $\mathcal{F}$-computable.*

*Proof.* Let $\theta^{\Sigma}$ be as in Proposition 3. Since $\theta^{\Sigma}$ is $\mathcal{F}$-computable, there exist $k + 2$-argument functions $f_1$, $g_1$ and $h_1$ belonging to $\mathcal{F}$ such that

$$\left| \frac{f_1(i_1, \ldots, i_k, t, n) - g_1(i_1, \ldots, i_k, t, n)}{h_1(i_1, \ldots, i_k, t, n) + 1} - \theta^{\Sigma}(i_1, \ldots, i_k, t) \right| \leq \frac{1}{n + 1}$$

for all $i_1, \ldots, i_k, t, n$ in $\mathbb{N}$. If we set

$$f(i_1, \ldots, i_k, n) = f_1(i_1, \ldots, i_k, p(i_1, \ldots, i_k, 2n + 1), 2n + 1),$$
$$g(i_1, \ldots, i_k, n) = g_1(i_1, \ldots, i_k, p(i_1, \ldots, i_k, 2n + 1), 2n + 1),$$
$$h(i_1, \ldots, i_k, n) = h_1(i_1, \ldots, i_k, p(i_1, \ldots, i_k, 2n + 1), 2n + 1),$$

then $f, g, h \in \mathcal{F}$ and

$$\left|\frac{f(i_1,\ldots,i_k,n)-g(i_1,\ldots,i_k,n)}{h(i_1,\ldots,i_k,n)+1}-\sigma(i_1,\ldots,i_k)\right| \leq$$

$$\left|\frac{f(i_1,\ldots,i_k,n)-g(i_1,\ldots,i_k,n)}{h(i_1,\ldots,i_k,n)+1}-\theta^{\Sigma}(i_1,\ldots,i_k,p(i_1,\ldots,i_k,2n+1))\right|+$$

$$\left|\theta^{\Sigma}(i_1,\ldots,i_k,p(i_1,\ldots,i_k,2n+1))-\sigma(i_1,\ldots,i_k)\right| \leq \frac{1}{2n+2}+\frac{1}{2n+2}=\frac{1}{n+1}$$

for all $i_1, \ldots, i_k, n$ in $\mathbb{N}$.

**Corollary.** *Let $\theta$ be such an $\mathcal{F}$-computable real-valued function of one natural argument that the series*

$$\sum_{s=0}^{\infty}\theta(s)$$

*converges, and $\alpha$ be the sum of this series. Let there exist a one-argument function $p$ belonging to $\mathcal{F}$ and such that*

$$\left|\sum_{s=t+1}^{\infty}\theta(s)\right| \leq \frac{1}{n+1}$$

*for any natural number $n$ and $t = p(n)$. Then the number $\alpha$ is also $\mathcal{F}$-computable.*

## 3 $\mathcal{E}^2$-computability of the numbers $\pi$ and $e$, of Liouville's number and of Euler's constant

### 3.1 $\mathcal{E}^2$-computability of the number $\pi$

The terms of the series in the well-known formula

$$\frac{\pi}{4}=\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1} \tag{2}$$

form an $\mathcal{L}$-sequence, since

$$(-1)^s = (s+1) \bmod 2 - s \bmod 2$$

for any $s \in \mathbb{N}$. In addition,

$$\left|\sum_{s=t+1}^{\infty}\frac{(-1)^s}{2s+1}\right| < \frac{1}{2t+3}$$

for any $t \in \mathbb{N}$. An application of the corollary from Section 2 immediately shows that $\pi/4$ is $\mathcal{L}$-computable, hence $\pi$ is also $\mathcal{L}$-computable (thus it is $\mathcal{E}^2$-computable).

Due to the slow convergence of the series in the formula (2), this formula is not convenient for the numerical computation of $\pi$. There are formulas that are much more appropriate for this, e.g. Machin's formula

$$\frac{\pi}{4} = 4\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)5^{2s+1}} - \sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)239^{2s+1}}. \tag{3}$$

The sums of the two series in (3) also turn out to be $\mathcal{E}^2$-computable. Of course any of the two series has a modulus of convergence of the sort required by the corollary in Section 2. Unfortunately, the sequences of their terms are not $\mathcal{E}^2$-recursive, as it is seen from Proposition 1. Nevertheless, the corollary is applicable to these series, since the sequences in question are still $\mathcal{E}^2$-computable (as sequences of real numbers). To prove their $\mathcal{E}^2$-computability, we may, for example, consider the three-argument function $f_0$ in $\mathbb{N}$ defined by

$$f_0(i,s,t) = \left[\frac{t}{(s+1)^i}\right].$$

This function belongs to $\mathcal{E}^2$, since

$$f_0(0,s,t) = t, \quad f_0(i+1,s,t) = \left[\frac{f_0(i,s,t)}{s+1}\right], \quad f_0(i,s,t) \le t$$

for all $i, s, t$ in $\mathbb{N}$. In addition,

$$\left|\frac{f_0(i,s,n+1)}{n+1} - \frac{1}{(s+1)^i}\right| = \frac{1}{n+1}\left|f_0(i,s,n+1) - \frac{n+1}{(s+1)^i}\right| < \frac{1}{n+1},$$

for any $i, s, n$ in $\mathbb{N}$, hence also

$$\left|\frac{(-1)^i f_0(2i+1,s,n+1)}{(2i+1)(n+1)} - \frac{(-1)^i}{(2i+1)(s+1)^{2i+1}}\right| < \frac{1}{n+1}.$$

Thus we may complete the proof by using the instances for values 4 and 238 of $s$ of the above inequality and by representing $(-1)^i f_0(2i+1,s,n+1)$ as the difference $((i+1) \bmod 2) f_0(2i+1,s,n+1) - (i \bmod 2) f_0(2i+1,s,n+1)$.

The $\mathcal{E}^2$-computability of the sequences of the terms of the two series in (3) can be proved also as follows. One considers the three-argument function $g_0$ in $\mathbb{N}$ defined by

$$g_0(i,s,t) = \min((s+1)^i, t+1). \tag{4}$$

This function belongs to $\mathcal{E}^2$ since

$$g_0(0,s,t) = t+1, \quad g_0(i+1,s,t) = \min(g_0(i,s,t)(s+1), t+1), \quad g_0(i,s,t) \le t+1$$

for all $i, s, t$ in $\mathbb{N}$. On the other hand,

$$\left|\frac{1}{g_0(i,s,n)} - \frac{1}{(s+1)^i}\right| < \frac{1}{n+1}$$

for any $i, s, n$ in $\mathbb{N}$, hence also

$$\left|\frac{(-1)^i}{(2i+1)g_0(2i+1,s,n)} - \frac{(-1)^i}{(2i+1)(s+1)^{2i+1}}\right| < \frac{1}{n+1}.$$

*Remark.* The $\mathcal{E}^2$-computability of $f_0$ can be derived also from the $\mathcal{E}^2$-computability of $g_0$, since

$$f_0(i,s,t) = \left[\frac{t}{g_0(i,s,t)}\right]$$

for all $i, s, t$ in $\mathbb{N}$.

### 3.2 $\mathcal{E}^2$-computability of the number $e$

To prove the $\mathcal{E}^2$-computability of the number $e$, we shall use the equality

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} \tag{5}$$

by showing that the series in it is $\mathcal{E}^2$-convergent, and the sequence of its terms is $\mathcal{E}^2$-computable (although, as seen from Proposition 1, this sequence is not $\mathcal{E}^2$-recursive). The $\mathcal{E}^2$-convergence of this series follows from the fact that

$$\sum_{i=n+1}^{\infty} \frac{1}{i!} < \frac{1}{n!n} \le \frac{1}{n}$$

for any positive integer $n$. To prove the $\mathcal{E}^2$-computability of the sequence of the terms of the series, we shall use the following two-argument function in $\mathbb{N}$:

$$f_1(i,t) = \left[\frac{t}{i!}\right].$$

This function belongs to $\mathcal{E}^2$ since

$$f_1(0,t) = t, \quad f_1(i+1,t) = \left[\frac{f_1(i,t)}{i+1}\right], \quad f_1(i,t) \le t.$$

for all $i, t$ in $\mathbb{N}$. In addition,

$$\left|\frac{f_1(i,n+1)}{n+1} - \frac{1}{i!}\right| = \frac{1}{n+1}\left|f_1(i,n+1) - \frac{n+1}{i!}\right| < \frac{1}{n+1}$$

for any $i, n$ in $\mathbb{N}$.

The $\mathcal{E}^2$-computability of the sequences of the terms of the series in (5) can be proved also as follows. One considers the two-argument function $g_1$ in $\mathbb{N}$ defined by

$$g_1(i,t) = \min(i!, t+1). \tag{6}$$

This function belongs to $\mathcal{E}^2$ since

$$g_1(0,t) = t+1, \quad g_1(i+1,t) = \min(g_1(i,t)(i+1), t+1), \quad g_1(i,t) \le t+1$$

for all $i, t$ in $\mathbb{N}$. On the other hand,

$$\left|\frac{1}{g_1(i,n)} - \frac{1}{i!}\right| < \frac{1}{n+1}$$

for any $i, n$ in $\mathbb{N}$.

*Remark.* The $\mathcal{E}^2$-computability of $f_1$ can be derived also from the $\mathcal{E}^2$-computability of $g_1$, since

$$f_1(i,t) = \left[\frac{t}{g_1(i,t)}\right]$$

for all $i, t$ in $\mathbb{N}$.

The proof in [Skordev 2008] of the $\mathcal{E}^2$-computability of the number $e$ can be briefly described as follows. Let $r_0, r_1, r_2, r_3, \ldots$ be the sequence of the partial sums of the series in (5), i.e.

$$r_n = \sum_{i=0}^{n} \frac{1}{i!}$$

for any $n \in \mathbb{N}$. Although this sequence of rational numbers is not $\mathcal{E}^2$-recursive, there exists a monotonically increasing sequence $k_0, k_1, k_2, \ldots$ of natural numbers such that

$$|r_{k_n} - e| < \frac{1}{n+1}$$

for any $n \in \mathbb{N}$, and the sequence $r_{k_0}, r_{k_1}, r_{k_3}, \ldots$ is $\mathcal{E}^2$-recursive. Clearly the idea of the present proof is rather different from the so described one.

### 3.3 $\mathcal{E}^2$-computability of Liouville's number

As well-known, the first examples of transcendental real numbers were constructed by Liouville. The most famous of them is the sum of the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{10^{i!}}.$$

This number is called now Liouville's number or Liouville's constant. It is sometimes denoted by $L$, and we shall adopt this notation here. We shall prove that $L$ is $\mathcal{E}^2$-computable. For technical convenience, we shall actually prove the equivalent statement that $L + 1/10$ is $\mathcal{E}^2$-computable. Since

$$L + 1/10 = \sum_{i=0}^{\infty} \frac{1}{10^{i!}}, \tag{7}$$

we shall proceed by proving the $\mathcal{E}^2$-convergence of the series in the above equality and the $\mathcal{E}^2$-computability of the sequence of its terms. The $\mathcal{E}^2$-convergence follows from the inequalities

$$\sum_{i=n+1}^{\infty} \frac{1}{10^{i!}} < \frac{1}{10^{n!n}} \leq \frac{1}{n+1}.$$

To prove the $\mathcal{E}^2$-computability of the sequence of the terms, we consider the function

$$g_2(i,t) = \min(10^{i!}, t+1).$$

It is easy to check that

$$g_2(i,t) = g_0(9, g_1(i,t)),$$

where $g_0$ and $g_1$ are the functions defined by (4) and (6), respectively. Therefore $g_2 \in \mathcal{E}^2$. On the other hand,

$$\left| \frac{1}{g_2(i,n)} - \frac{1}{10^{i!}} \right| < \frac{1}{n+1}$$

for any $i, n$ in $\mathbb{N}$.

Another way to prove the $\mathcal{E}^2$-computability of the sequence of the terms of the series in (7) is by considering the function

$$f_2(i,t) = \left[ \frac{t}{10^{i!}} \right].$$

This function also belongs to $\mathcal{E}^2$ since

$$f_2(i,t) = \left[ \frac{t}{g_2(i,t)} \right]$$

for all $i, t$ in $\mathbb{N}$, and

$$\left| \frac{f_2(i,n+1)}{n+1} - \frac{1}{10^{i!}} \right| = \frac{1}{n+1} \left| f_2(i,n+1) - \frac{n+1}{10^{i!}} \right| < \frac{1}{n+1}$$

for any $i, n$ in $\mathbb{N}$.

### 3.4 $\mathcal{E}^2$-computability of Euler's constant

To prove that Euler's constant $\gamma$ is $\mathcal{E}^2$-computable, we shall use its representation

$$\gamma = \sum_{i=0}^{\infty} \left( \frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right)\right), \tag{8}$$

as well as the fact that for any $i \in \mathbb{N}$ we have the equality

$$\frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right) = \sum_{j=0}^{\infty} u(i,j), \tag{9}$$

where

$$u(i,j) = \frac{(-1)^j}{(j+2)(i+1)^{j+2}}.$$

The series in (9) is $\mathcal{E}^2$-convergent thanks to the inequality

$$\left| \sum_{j=n+1}^{\infty} u(i,j) \right| < \frac{1}{n+3}.$$

The function $u$ is $\mathcal{E}^2$-computable, since for all $i, j, k \in \mathbb{N}$ the inequality

$$\left| \frac{(-1)^j}{(j+2)g_0(j+2,i,k)} - u(i,j) \right| < \frac{1}{2(k+1)}$$

holds, where $g_0$ is the function defined by (4). Therefore the sum of the series is an $\mathcal{E}^2$-computable function of $i$. Thus the $\mathcal{E}^2$-computability of Euler's constant will be proved if we prove the $\mathcal{E}^2$-convergence of the series in (8). To do this, we note that, by the equality (9), we have the inequalities

$$0 < \frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right) < \frac{1}{2(i+1)^2}$$

for any $i \in \mathbb{N}$, hence

$$0 < \sum_{i=n+1}^{\infty} \left( \frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right)\right) < \sum_{i=n+1}^{\infty} \frac{1}{2(i+1)^2} < \frac{1}{2(n+1)}$$

for all $n \in \mathbb{N}$.

## 4 Some comments

Although our proofs concern only four concrete real numbers, the methods used in the proofs or similar ones can be applied in many other cases. It seems that

$\mathcal{E}^2$-computability of real numbers is present much more often than one could expect.

One of the four considered numbers turned out to be $\mathcal{L}$-computable. It seems that the other of them are also $\mathcal{L}$-computable, but the proof of this requires a greater amount of technical work.

Several characterizations of the class $\mathcal{E}^2$ are known that are in the terms of computational complexity, for instance the characterization from [Ritchie 1963] according to which a function belongs to $\mathcal{E}^2$ iff it can be computed on a linear tape bounded Turing machine in the case of binary encoding of inputs and outputs. As the referee of the preliminary version [Skordev 2008] of the paper indicated, such characterizations could be useful for comparison with already known results and for further studies, and, in particular, the characterization from [Ritchie 1963] allows relating complexity of real functions as in [Ko 1991, Weihrauch 2000] to $\mathcal{E}^2$-computability.

## 5 Existence of $\mathcal{E}^3$-computable real numbers which are not $\mathcal{E}^2$-computable

It is shown in [Skordev 2002] (cf. footnote 9 there) that for any integer $m$ greater than 2 there exist $\mathcal{E}^{m+1}$-computable real numbers which are not $\mathcal{E}^m$-computable. The proof from [Skordev 2002] cannot be used in the case of $m = 2$. We shall give now another proof that covers also the case of $m = 2$. Namely we shall make use of the fact that for any natural number $m \geq 2$ there exists a two-argument function in $\mathcal{E}^{m+1}$ which is universal for the one-argument functions in $\mathcal{E}^m$, i.e. each one-argument functions belonging to $\mathcal{E}^m$ can be obtained from the two-argument function in question by replacement of the first argument with some natural number.[5]

Let $m \in \mathbb{N}$, $m \geq 2$, and let $h$ be a two-argument function from $\mathcal{E}^{m+1}$ which is universal for the one-argument functions in $\mathcal{E}^m$. We define a one-argument function $g$ in $\mathbb{N}$ as follows: $g(0) = 0$, and, for any $k \in \mathbb{N}$,

$$g(k+1) = \begin{cases} 3g(k) & \text{if } 6g(k) + 3 \leq h(k, 2 \cdot 3^{k+1} - 1) \\ 3g(k) + 2 & \text{otherwise} \end{cases}$$

(thus $g(k+1) - 3g(k) \in \{0, 2\}$ for all $k \in \mathbb{N}$). Making use of the inequality

[5] In [Grzegorczyk 1953] a proof of this is sketched for $m > 2$. As Lars Kristiansen indicated, the truth of the statement for the case of $m = 2$ follows straightforwardly from what is written in section 6 of [Ritchie 1963], and the statement in question can be derived also from the equality $LINSPACE = \mathcal{E}^2_*$, the inclusion $ESPACE \subset \mathcal{E}^3_*$ and the fact that $ESPACE$ contains a universal function for $LINSPACE$.

$g(k) \leq 3^k - 1$, one sees that $g \in \mathcal{E}^{m+1}$. Now let

$$\alpha = \sum_{k=0}^{\infty} \frac{g(k+1) - 3g(k)}{3^{k+1}}.$$

For any natural number $k$, the sum of the first $k$ terms of the above series is equal to $g(k)/3^k$, and, making use of this, we see that

$$0 \leq \alpha - \frac{g(k)}{3^k} \leq \frac{1}{3^k}$$

for all $k \in \mathbb{N}$, hence the real number $\alpha$ is $\mathcal{E}^{m+1}$-computable. We shall show that $\alpha$ is not $\mathcal{E}^m$-computable. Suppose the contrary. Then, by the case $l = 0$ of Proposition 2, one-argument functions $f$ and $g$ belonging to $\mathcal{E}^m$ exist such that

$$\left| \frac{f(n) - g(n)}{n+1} - \alpha \right| < \frac{1}{n+1}$$

for all $n \in \mathbb{N}$. The function $|f(n) - g(n)|$ also belongs to $\mathcal{E}^m$, and

$$\left| \frac{|f(n) - g(n)|}{n+1} - \alpha \right| < \frac{1}{n+1}$$

also holds for all $n \in \mathbb{N}$, since $\alpha \geq 0$. Let $k$ be a natural number such that $|f(n) - g(n)| = h(k, n)$ for all $n \in \mathbb{N}$. Then

$$\left| \frac{h(k,n)}{n+1} - \alpha \right| < \frac{1}{n+1}$$

for all $n \in \mathbb{N}$. In particular, we shall have

$$\left| \frac{h(k, 2 \cdot 3^{k+1} - 1)}{2 \cdot 3^{k+1}} - \alpha \right| < \frac{1}{2 \cdot 3^{k+1}}.$$

We shall now consider separately the case, when $6g(k) + 3 \leq h(k, 2 \cdot 3^{k+1} - 1)$, and the case, when $6g(k) + 3 > h(k, 2 \cdot 3^{k+1} - 1)$. We shall get a contradiction in both of them. In the first of these cases, we have

$$\begin{aligned} \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k} \leq \frac{h(k, 2 \cdot 3^{k+1} - 1)}{2 \cdot 3^{k+1}} < \alpha + \frac{1}{2 \cdot 3^{k+1}} \\ \leq \frac{g(k+1)}{3^{k+1}} + \frac{1}{3^{k+1}} + \frac{1}{2 \cdot 3^{k+1}} = \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k}, \end{aligned}$$

and this is impossible. In the second of the cases, we have

$$\begin{aligned} \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k} > \frac{h(k, 2 \cdot 3^{k+1} - 1)}{2 \cdot 3^{k+1}} > \alpha - \frac{1}{2 \cdot 3^{k+1}} \\ \geq \frac{g(k+1)}{3^{k+1}} - \frac{1}{2 \cdot 3^{k+1}} = \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k}, \end{aligned}$$

and this is again impossible.

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An observation made by Peter Peshev in 2005 stimulated the author to look for extending his knowledge about $\mathcal{E}^2$-computability in analysis. The observation was that almost all constructions described in [Rosenbloom 1945] can actually be accomplished by means of operators which are not only $\mathcal{E}^3$-computable, but even $\mathcal{E}^2$-computable (sufficiency of the $\mathcal{E}^3$-computable operators was the initially expected result of Peshev's study).

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