Journal of Universal Computer Science, vol. 14, no. 6 (2008), 996-1015
submitted: 31/10/07, accepted: 16/2/08, appeared: 28/3/08 © J.UCS

On the Relationship between Filter Spaces and Weak Limit Spaces

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Abstract: Countably based filter spaces have been suggested in the 1970's as a model for recursion theory on higher types. Weak limit spaces with a countable base are known to be the class of spaces which can be handled by the Type-2 Model of Effectivity (TTE). We prove that the category of countably based proper filter spaces is equivalent to the category of countably based weak limit spaces. This result implies that filter spaces form yet another category from which the category of qcb-spaces inherits its cartesian closed structure. Moreover, we compare the aforementioned categories to other categories of spaces relevant to computability theory.
Key Words: Convenient Categories, Higher Type Computation, QCB-Spaces, Topological Spaces, Filter Spaces, Weak Limit Spaces, Equilogical Spaces
Category: F.1.1

1 Introduction

A category of spaces designed for modelling higher type computation should have the property of being cartesian closed. Cartesian closedness means that a category allows the construction of finite products and function spaces. Computation on non-discrete spaces requires to deal with approximations. One important mathematical tool for modelling approximations are topological spaces. Unfortunately, the category of topological spaces, Top, lacks the property of being cartesian closed. However, there exist several relevant cartesian closed subcategories and supercategories of Top which can be used as an alternative. Examples of cartesian closed supercategories of Top are D. Scott's category of equilogical spaces [Scott et al. 2004] and the category of filter spaces [Hyland 1979], whereas qcb-spaces [Simpson 2003] form a cartesian closed subcategory of Top.

In the 1970's, M. Hyland established the relevance of filter spaces to computation on the continuous functionals over $\mathbb{N}$, see [Hyland 1979]. Several notions of filter spaces (also known as *convergence spaces*) exist in the literature. All notions endow a set X with a convergence relation between filters on X and points of X subject to certain axioms [see Section 2.1]. The most general notion is the one considered in [Hyland 1979]. In this paper we introduce the slightly less general notion of *proper* filter spaces: They enjoy the property that all sets in a converging filter contain its limit(s). This matches with the intuition that a filter should converge to x if it is viewed to contain enough properties of x providing sufficient information about x. The category of proper filter spaces turns out to be equivalent to one of the filter spaces categories which R. Heckmann has investigated in [Heckmann 1998] in terms of their relationship to equilogical spaces.

From the perspective of computability theory, we are particularly interested in filter spaces with a countable basis. In these spaces, a countable set of properties suffices to describe the elements of the space. The presence of a countable basis provides a handle to define computable functions between filter spaces by means of enumeration operators or by means of multirepresentations.

Weak limit spaces [Schröder 2001] are a generalisation of limit spaces [Kuratowski 1966] and thus of sequential topological spaces [Engelking 1989]. The constitutive structure of a weak limit space is a convergence relation between sequences and points [see Section 2.2]. Weak limit spaces play an important role in Weihrauch's representation-based approach to Computable Analysis, the *Type-2 Theory of Effectivity* (TTE) [Weihrauch 2000]: The class of spaces admitting an *admissible* (i.e. continuously well-behaved) multirepresentation is exactly the class of countably based weak limit spaces [Schröder 2001, Schröder 2002].

In [Section 3] we construct an embedding of the weak limit spaces into the category of proper filter spaces and show that it preserves countable products. In [Section 4] we present and prove our main result stating that the category of countably based proper filter spaces is equivalent to the category of countably based weak limit spaces. Thus countably based proper filter spaces are already characterised by the apparently simpler concept of sequence convergence: they are simply weak limit spaces in a different guise. The existence of countable basis is crucial to this result.

An important full cartesian closed subcategory of Top is the category QCB of qcb-spaces [see Section 5.6]. It has the property of inheriting its cartesian closed structure from many interesting cartesian closed supercategories relevant to higher type computability and forms their common core. Examples are the categories of: equilogical spaces, compactly generated spaces, weak limit spaces, Baire space representations, see [Bauer 2002, Escardó et al. 2004, Menni and Simpson 2002, Schröder 2001] and [Section 5]. Countably based equilogical spaces exemplify the domain-theoretic approach to Computable Analysis, whereas Baire space representations describe the TTE approach. Hence QCB qualifies as a convenient category for modelling higher type computation [Simpson et al. 2007]. Our main theorem implies that the aforementioned categories of filter spaces belong to this list [see Section 5]. This answers positively a question in [Simpson 2003, Simpson et al. 2007]. Moreover, at least in topological terms, the approach to higher type computation via countably based filter spaces agrees with the TTE approach. The relationship between the aforementioned categories is discussed in [Section 5].

2 Filter Spaces and Weak Limit Spaces

In this section we recall the definitions of filter spaces and weak limit spaces together with some known facts about these concepts.

2.1 Filter Spaces and Related Notions

2.1.1 Filters

Let X and Y be non-empty sets. A *filter* $\mathcal{F}$ on X is a non-empty family of non-empty subsets of X which is closed under finite intersection and extension to supersets. For $x \in X$, we write $[x]$ for the *principal* ultrafilter $\{A \subseteq X \mid x \in A\}$. A *filter base* $\varPhi$ on X is a non-empty family of non-empty subsets of X such that for all $A, B \in \varPhi$ there is some $C \in \varPhi$ with $\emptyset \neq C \subseteq A \cap B$. For a filter base $\varPhi$ we denote by $[\varPhi]$ the smallest filter containing $\varPhi$, i.e. $\{A \subseteq X \mid \exists B \in \varPhi. B \subseteq A\}$. Given a function $f \colon X \to Y$ and a filter $\mathcal{F}$ on X, $f^*\mathcal{F}$ denotes the filter $[\{f(A) \mid A \in \mathcal{F}\}]$.

2.1.2 Filter spaces

There are several notions of filter spaces (sometimes also called *convergence spaces*) in the literature. Generally, a *filter space* is a pair $(X, \downarrow)$, where X is a set and $\downarrow$ is a relation (called the *convergence relation*) between the filters on X and the points of X satisfying certain axioms. If $\mathcal{F} \downarrow x$ holds, then one says that the filter $\mathcal{F}$ *converges to* x and that x is a *limit of* $\mathcal{F}$. If additionally we have $\mathcal{F} \subseteq [x]$, then we say that $\mathcal{F}$ converges *properly* to x and that x is a *proper* limit of $\mathcal{F}$. For a filter base $\varPhi$, it is convenient to say that $\varPhi$ converges to x iff the filter $[\varPhi]$ generated by $\varPhi$ does. Since any filter $\mathcal{F}$ is equal to $[\mathcal{F}]$, this convention does not cause ambiguity. In the following, we shall use capital Gothic letters like $\mathfrak{X}, \mathfrak{Y}$ for filter spaces of any kind. The carrier set of a filter space $\mathfrak{X}$ will often be denoted by $\mathfrak{X}$ as well. We shall use the symbol '$\downarrow_{\mathfrak{X}}$' or, if no confusion can occur, simply '$\downarrow$' to denote its convergence relation. A function $f \colon \mathfrak{X} \to \mathfrak{Y}$ between filter spaces $\mathfrak{X}$ and $\mathfrak{Y}$ is called *filter-continuous*, if $\mathcal{F} \downarrow_{\mathfrak{X}} x$ implies $f^*\mathcal{F} \downarrow_{\mathfrak{Y}} f(x)$ for every filter $\mathcal{F}$ on $\mathfrak{X}$ and every element x of $\mathfrak{X}$. Obviously, composition of functions preserves filter continuity.

The original notion of filter spaces considered by M. Hyland in [Hyland 1979] requires a filter space $(X, \downarrow)$ to fulfil Axioms (F1) and (F2):

(F1) $[x] \downarrow x$;

(F2) if $\mathcal{F} \downarrow x$ and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G} \downarrow x$,

where $\mathcal{F}, \mathcal{G}$ are filters on X and x is an element of X. We denote the category of filter spaces satisfying Axioms (F1), (F2) and of filter-continuous functions as morphisms by Fil.

2.1.3 Proper and canonical filter spaces

In this paper, we are concerned with *proper* filter spaces. They slightly differ from Hyland's filter spaces in that only proper limits are allowed, i.e., convergence of a filter to a point implies that all sets in the filter contain that point. We define $(X, \downarrow)$ to be a *proper* filter space, if $\downarrow$ is a relation between the filter on X and the points of X that satisfies the Axioms (F1), (F5), (F6):

(F5) if $\mathcal{F} \downarrow x$ and $\mathcal{F} \subseteq \mathcal{G} \subseteq [x]$ then $\mathcal{G} \downarrow x$;

(F6) if $\mathcal{F} \downarrow x$ then $\mathcal{F} \subseteq [x]$.

We denote the category of proper filter spaces as objects and of filter-continuous functions as morphisms by PFil.

The idea behind filter spaces is the following: A filter $\mathcal{F}$ is defined to converge to a point x, if $\mathcal{F}$ is deemed to contain enough properties of x providing sufficient information about x. Here we adopt the usual interpretation of a *property* of x as a set containing x. The original notion of filter slightly mismatches with this intuition, because it allows converging filters to contain sets which are not properties of the limit. It is more general than the notion of proper filter spaces. In fact, PFil is equivalent to the full subcategory CFil (in [Heckmann 1998] denoted by Fil^b) of those filter spaces in Fil that satisfy Axiom (F3):

(F3) if $\mathcal{F} \downarrow x$ then $\mathcal{F} \cap [x] \downarrow x$.

These filter spaces are sometimes called *canonical* filter spaces. Axiom (F3) ensures that convergence of a filter to a point depends solely on the sets in the filter which contain that point.

Proposition 1. *The category of proper filter spaces is equivalent to the category of canonical filter spaces.*

Proof. One defines functors $C\colon \mathsf{PFil} \to \mathsf{CFil}$ and $P\colon \mathsf{CFil} \to \mathsf{PFil}$ as follows: both functors preserve the underlying sets, C sends a proper filter space $\mathfrak{X}$ to the canonical filter space whose convergence relation is given by $\mathcal{F} \downarrow_{C(\mathfrak{X})} x \;:\Longleftrightarrow\; \mathcal{F} \cap [x] \downarrow_{\mathfrak{X}} x$, and P maps $\mathfrak{Y} \in \mathsf{CFil}$ to the proper filter space endowed with the convergence relation defined by $\mathcal{G} \downarrow_{P(\mathfrak{Y})} y \;:\Longleftrightarrow\; (\mathcal{G} \subseteq [y] \wedge \mathcal{G} \downarrow_{\mathfrak{Y}} y)$. One easily verifies that the object parts of C and P are inverses of each other. $\square$

From the point of view of computability theory, proper filter spaces turn out to be more handy and more natural than canonical filter spaces.

2.1.4 T_0-property for filter spaces

We say that a filter space $\mathfrak{X}$ has the T_0-property, if $[\{\{x, y\}\}] \downarrow_{\mathfrak{X}} x$ and $[\{\{x, y\}\}] \downarrow_{\mathfrak{X}} y$ imply $x = y$ [Heckmann 1998]. This is equivalent to requiring that each filter has at most one *proper* limit.

2.1.5 Products and exponentials

The categories Fil, CFil and PFil have countable products and are cartesian closed, see [Hyland 1979, Heckmann 1998] for Fil and [Heckmann 1998] for CFil. Unfortunately, function spaces in Fil and in its subcategory CFil are constructed differently. However, for canonical filter spaces satisfying the Merging Axiom (F4):

(F4) if $\mathcal{F} \downarrow x$ and $\mathcal{G} \downarrow x$ then $\mathcal{F} \cap \mathcal{G} \downarrow x$

the function space constructions in Fil and CFil agree, see [Heckmann 1998]. We denote the full subcategory of *proper* filter spaces satisfying Axiom (F4) by MFil.

The cartesian closedness of PFil follows from its equivalence to CFil (see Proposition 1). We give explicit constructions for products and exponentials in PFil. Given a sequence of proper filter spaces $(\mathfrak{X}_i)_i$, the carrier set of the product $\prod_{i\in\mathbb{N}} \mathfrak{X}_i$ is the cartesian product of the carrier sets and its convergence relation $\downarrow$ is given by

$$\mathcal{F} \downarrow x \iff \big(\mathcal{F} \subseteq [x] \;\wedge\; \forall i \in \mathbb{N}.\, \mathrm{pr}_i^* \mathcal{F} \downarrow_{\mathfrak{X}_i} \mathrm{pr}_i(x)\big)\,,$$

where pr_i denotes the respective set-theoretic projection function. The exponential $\mathfrak{Y}^{\mathfrak{X}}$ in PFil for proper filter spaces $\mathfrak{X}$ and $\mathfrak{Y}$ has the set $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ of filter-continuous functions from $\mathfrak{X}$ to $\mathfrak{Y}$ as its carrier set and its convergence relation is given as follows: a filter $\mathcal{F}$ on $\mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ converges to a function $f \in \mathcal{C}(\mathfrak{X}, \mathfrak{Y})$ iff

$$\mathcal{F} \subseteq [f] \text{ and } \mathcal{A} \downarrow_{\mathfrak{X}} x \text{ implies } \Big[\big\{\{g(a) \,|\, g \in F,\, a \in A\} \,\big|\, F \in \mathcal{F},\, A \in \mathcal{A}\big\}\Big] \downarrow_{\mathfrak{Y}} f(x)$$

for every filter $\mathcal{A}$ on $\mathfrak{X}$ and every $x \in \mathfrak{X}$.

The importance of filter spaces lies in the fact that each of the aforementionned categories of filter spaces forms a cartesian closed supercategory of the non cartesian closed category Top of topological spaces, see [Heckmann 1998, Hyland 1979]. The embedding functor $\mathcal{I}_{\mathsf{Top}}^{\mathsf{CFil}} \colon \mathsf{Top} \hookrightarrow \mathsf{CFil}$ maps a topological space Z to the canonical filter space which has the same carrier set and whose convergence relation is defined by: $\mathcal{F} \downarrow z$ iff $\mathcal{F}$ contains the neighbourhood filter $[\{U \text{ open} \,|\, z \in U\}]$ of z. This matches with the usual definition of filter convergence in a topological space. The functor $\mathcal{I}_{\mathsf{Top}}^{\mathsf{CFil}}$ is known to preserve products and existing exponentials. By composing $\mathcal{I}_{\mathsf{Top}}^{\mathsf{CFil}}$ and the equivalence functor $P : \mathsf{CFil} \to \mathsf{PFil}$ from the proof of Proposition 1, we obtain an embedding functor $\mathcal{I}_{\mathsf{Top}}$ of Top into MFil which preserves products and existing exponentials as well.

2.1.6 Bases for filter spaces

A family $\mathcal{B}$ of subsets of $\mathfrak{X}$ is called a *basis* for a filter space $\mathfrak{X}$, if for every filter $\mathcal{F}$ converging to some $x \in \mathfrak{X}$ the family $\mathcal{F} \cap \mathcal{B}$ is a filter base such that $[\mathcal{F} \cap \mathcal{B}]$

converges to x. A *subbasis* for $\mathfrak{X}$ is a family of subsets whose closure under finite intersection is a basis for $\mathfrak{X}$. In contrast to topological bases, filter space bases do not characterise filter spaces; indeed, the powerset of Y is a basis for every filter space with carrier set Y. Nevertheless, filter space bases become interesting, when they are countable. By $\omega\mathsf{PFil}$, $\omega\mathsf{CFil}$, $\omega\mathsf{MFil}$ we denote the respective full subcategories of PFil, CFil, MFil consisting of filter spaces with a countable basis.

The categories $\omega\mathsf{PFil}$, $\omega\mathsf{CFil}$, and $\omega\mathsf{MFil}$ are cartesian closed as well, because forming countable products and forming exponentials preserve the existence of a countable basis. Given countable bases $\mathcal{A}$ and $\mathcal{B}$ for proper filter spaces $\mathfrak{X}$ and $\mathfrak{Y}$, one can show similar to [Hyland 1979] that $\big\{A \times B \,\big|\, A \in \mathcal{A},\ B \in \mathcal{B}\big\}$ is a countable basis for the product $\mathfrak{X} \times \mathfrak{Y}$ in PFil and that the family $\big\{\{f \in \mathcal{C}(\mathfrak{X},\mathfrak{Y}) \,|\, f(A) \subseteq B\} \,\big|\, A \in \mathcal{A},\, B \in \mathcal{B}\big\}$ is a countable subbasis for the exponential $\mathfrak{Y}^{\mathfrak{X}}$ in PFil.

Filter space bases relate to topological bases as follows:

Lemma 2. *Any topological base for a topological space Z is a filter space basis for $\mathcal{I}_{\mathsf{Top}}(Z)$. If $\mathcal{B}$ is a filter space basis for $\mathcal{I}_{\mathsf{Top}}(Z)$, then $\{\mathrm{Int}(B) \,|\, B \in \mathcal{B}\}$ is a topological base for Z, where $\mathrm{Int}(B)$ denotes the interior of B.*

Proof. Let $z \in Z$. Let $\mathcal{N} := [\{U \text{ open} \,|\, z \in U\}]$ be the neighbourhood filter of z. Any topological base $\mathcal{A}$ for Z satisfies $[\mathcal{N} \cap \mathcal{A}] = \mathcal{N}$. Since in $\mathcal{I}_{\mathsf{Top}}(Z)$ a filter $\mathcal{F}$ converges to z if, and only if, $\mathcal{N} \subseteq \mathcal{F} \subseteq [z]$ holds, $\mathcal{A}$ is a filter base for $\mathcal{I}_{\mathsf{Top}}(Z)$. Let $\mathcal{B}$ be a filter base for $\mathcal{I}_{\mathsf{Top}}(Z)$. Since the neighbourhood filter $\mathcal{N}$ and thus $[\mathcal{N} \cap \mathcal{B}]$ converge to z in $\mathcal{I}_{\mathsf{Top}}(Z)$, we have $\mathcal{N} \subseteq [\mathcal{N} \cap \mathcal{B}]$. Hence for every open neighbourhood U of z there is some $B \in \mathcal{B}$ and some open set V with $z \in V \subseteq B \subseteq U$, implying $z \in \mathrm{Int}(B) \subseteq U$. Therefore $\{\mathrm{Int}(B) \,|\, B \in \mathcal{B}\}$ is a topological base for Z. $\square$

2.1.7 Coded filter spaces and computable functions

For functions between countably based filter spaces one can introduce a reasonable notion of computability by considering numberings of the respective bases. We modify slightly the tentative definition in [Hyland 1979] and call a triple $(X, \downarrow, \alpha)$ a *coded filter space*, if $(X, \downarrow)$ is a proper filter space and α is a numbering of a basis for $(X, \downarrow)$. A total function f between two coded filter spaces $\mathfrak{X} = (X, \downarrow_{\mathfrak{X}}, \alpha)$ and $\mathfrak{Y} = (Y, \downarrow_{\mathfrak{Y}}, \beta)$ is defined to be *computable* iff there is a computable function $g\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

$$[\{\alpha_i \,|\, i \in I\}] \downarrow_{\mathfrak{X}} x \;\text{ implies }\; [\{\beta_j \,|\, j \in g(I)\}] \downarrow_{\mathfrak{Y}} f(x)$$

for all $x \in X$ and all $I \subseteq \mathbb{N}$. This notion of computability is equivalent to $(\delta_{\mathfrak{X}}, \delta_{\mathfrak{Y}})$-computability in the sense of TTE, where $\delta_{\mathfrak{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ is a standard multirepresentation for $\mathfrak{X}$ defined by $\delta_{\mathfrak{X}}(p) \ni x :\Longleftrightarrow [\{\alpha_{p(l)} \,|\, l \in \mathbb{N}\}] \downarrow_{\mathfrak{X}} x$. It comes as no surprise that computable functions between coded filter spaces are filter-continuous.

2.2 Weak Limit Spaces and Related Notions

Let X and Y be sets. We write sequences $x\colon \mathbb{N} \to X$ over X as $(x_n)_n$ and *generalised* sequences $x : \mathbb{N}^+ \to X$ as $(x_n)_{n\leq\infty}$, where $\mathbb{N}^+ := \mathbb{N} \cup \{\infty\}$. By $[(x_n)_n]$ we denote the *Fréchet filter* $\big\{M \subseteq X \,\big|\, \exists i \in \mathbb{N}.\forall j \geq i.\, x_j \in M\big\}$.

A *weak limit space* ([Schröder 2001, Schröder 2002]) is a set X equipped with a *convergence relation* $\to$ between sequences $(x_n)_n$ and points x of X. If $(x_n)_n \to x_\infty$, then we say that $(x_n)_n$ converges to x_∞ in the space $(X, \to)$ and that x_∞ is a limit of the sequence $(x_n)_n$. The convergence relation $\to_{\mathsf{X}}$ of a weak limit space $\mathsf{X} = (X, \to_{\mathsf{X}})$ is required to satisfy the following axioms:

(L1) $(x)_n \to_{\mathsf{X}} x$;

(L4) if $(x_n)_n \to_{\mathsf{X}} x_\infty$ and $(\xi_n)_n \to \infty$ in $(\mathbb{N}^+, \tau_{\mathbb{N}^+})$ then $(x_{\xi_n})_n \to_{\mathsf{X}} x_\infty$;

(L5) if $(x_{n+1})_n \to_{\mathsf{X}} x_\infty$ and $x_0 \in X$ then $(x_n)_n \to_{\mathsf{X}} x_\infty$.

Here $\tau_{\mathbb{N}^+}$ denotes the standard topology $\big\{U \subseteq \mathbb{N}^+ \,\big|\, \infty \in U \Longrightarrow U \in [(n)_n]\big\}$ on $\mathbb{N}^+$. Note that the members of the sequence $(\xi_n)_n$ in (L4) may be equal to ∞. By means of Axiom (L4'):

(L4') if $(y_n)_n \to_{\mathsf{X}} x$ and $[(y_n)_n] \cap [x] \subseteq [(z_n)_n]$ then $(z_n)_n \to_{\mathsf{X}} x$

we can characterise weak limit spaces.

Lemma 3. *A pair* $\mathsf{X} = (X, \to_{\mathsf{X}})$*, where is* $\to_{\mathsf{X}}$ *is a relation between sequences and points of* X*, is a weak limit spaces if, and only if,* X *satisfies Axioms* (L1) *and* (L4').

Proof. The *if-part* follows from the fact that $[(x_{n+1})_n] = [(x_n)_n]$ and $[(x_n)_n] \cap [x_\infty] \subseteq [(x_{\xi_n})_n]$ hold, if $(\xi_n)_n$ converges to ∞. For the *only-if-part*, let $(x_n)_n$ converge to x_∞ and let $(z_n)_n$ be a sequence with $[(x_n)_n] \cap [x_\infty] \subseteq [(z_n)_n]$. Then there is a sequence $l_0 < l_1 < l_2 < \ldots \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$, $\{z_n \,|\, n \geq l_i\} \subseteq \{x_\infty, x_n \,|\, n \geq i\}$. For every $m \in \{l_i, \ldots, l_{i+1} - 1\}$ let ξ_{m-l_0} be the least number in $\{\infty\} \cup \{i, i+1, \ldots\}$ with $x_{\xi_{m-l_0}} = z_m$. By Axiom (L4), the sequence $(z_{n+l_0})_n = (x_{\xi_n})_n$ converges to x_∞. Axiom (L5) implies $(z_n)_n \to_{\mathsf{X}} x_\infty$. □

Weak limit spaces are a generalisation of Kuratowski's *limit spaces* [Kuratowski 1966] (called *L-spaces* in [Hyland 1979]). The convergence relation $\to_{\mathsf{X}}$ of a limit space X is subject to the axioms (L1), (L2), (L3):

(L2) if $(x_n)_n \to_{\mathsf{X}} x_\infty$ then $(y_n)_n \to_{\mathsf{X}} x_\infty$ for every subsequence $(y_n)_n$ of $(x_n)_n$;

(L3) if $(x_n)_n \not\to_{\mathsf{X}} x_\infty$ then $(x_n)_n$ has a subsequence $(y_n)_n$ such that $(z_n)_n \not\to_{\mathsf{X}} x_\infty$ for every subsequence $(z_n)_n$ of $(y_n)_n$.

Every weak limit space fulfils Axiom (L2), but not necessarily Axiom (L3), whereas every limit space is a weak limit space. The convergence relation of a topological space Z satisfies the axioms of a limit space. We denote the corresponding (weak) limit space by $\mathcal{L}_{\mathsf{Top}}(Z)$. A weak limit space is called *topological*, if it lies in the image of $\mathcal{L}_{\mathsf{Top}}$. The class of weak limit spaces in which every converging sequence has only one limit turns out to be exactly the class of $\mathcal{L}^+$–spaces defined in [Dudley 1964].

A function f between two weak limit spaces X and Y is called *sequentially continuous*, if it preserves convergence of sequences, i.e., $(x_n)_n \to_{\mathsf{X}} x_\infty$ implies $(f(x_n))_n \to_{\mathsf{Y}} f(x_\infty)$. By WLim we denote the category whose objects are the weak limit spaces and whose morphisms are the (total) sequentially continuous functions.

Given a weak limit space X, we call a family Φ of subsets of X a *witness of convergence* for an element x in X, if $\Phi \subseteq [x]$ and, for every sequence $(y_n)_n$, $\Phi \subseteq [(y_n)_n]$ implies $(y_n)_n \to x$ in X. For example, any neighbourhood base of a point z in a topological space is a witness of convergence for z. Lemma 3 implies that $(y_n)_n$ converges in X to x if, and only if, the Fréchet filter $[(y_n)_n] \cap [x]$ is a witness of convergence for x in X.

A *limit base* for a weak limit space X is a family $\mathcal{B}$ of subsets of X such that for every element $x \in \mathsf{X}$ and every sequence $(y_n)_n$ converging to x, $\mathcal{B}$ contains a witness of convergence Φ for x in X such that $\Phi \subseteq [(y_n)_n]$. This implies that for every element x_∞, for every sequence $(x_n)_n$ converging to x_∞ and and for every sequence $(z_n)_n$ that does not converge to x_∞ there is some $B \in \mathcal{B}$ such that $x_\infty \in B$, $x_n \in B$ for almost all n and $z_m \notin B$ for infinitely many m. Similar to bases of filter spaces, the powerset of a set Y is a limit base for every weak limit space with carrier set Y. By $\omega\mathsf{WLim}$ we denote the full subcategory of WLim consisting of all weak limit spaces admitting a countable limit base.

Both categories WLim and $\omega\mathsf{WLim}$ have countable products and are cartesian closed [Schröder 2001, Schröder 2002]. The product $\prod_{i\in\mathbb{N}} \mathsf{X}_i$ of a sequence $(\mathsf{X}_i)_i$ of weak limit spaces is constructed as one expects. The exponential Y^{X} is obtained by equipping the set $\mathcal{C}(\mathsf{X},\mathsf{Y})$ of sequentially continuous functions from X to Y with the convergence relation $\Rrightarrow$ of *continuous convergence* defined by: $(f_n)_n \Rrightarrow f_\infty$ iff $(f_{\xi_n}(x_n))_n \to_{\mathsf{Y}} f(x_\infty)$ holds for all $(x_n)_n \to_{\mathsf{X}} x_\infty$ and all $(\xi_n)_n \to \infty$ in $(\mathbb{N}^+, \tau_{\mathbb{N}^+})$. For limit spaces this definition is equivalent to the usual definition of continuous convergence. Given countable limit bases $\mathcal{B}_i$ for weak limit spaces X_i, countable limit bases for the product $\prod_{i\in\mathbb{N}} \mathsf{X}_i$ and the exponential $\mathsf{X}_2^{\mathsf{X}_1}$ are constructed by $\bigl\{\prod_{i\in\mathbb{N}} B_i \bigm| k \in \mathbb{N},\, B_k \in \mathcal{B}_k,\, \forall i \neq k.\, B_i = \mathsf{X}_i\bigr\}$ and $\bigl\{\{f \in \mathcal{C}(\mathsf{X}_1,\mathsf{X}_2) \,|\, f(\bigcap_{i=1}^k A_i) \subseteq B\} \bigm| \{A_1,\dots,A_k\} \subseteq \mathcal{B}_1,\, B \in \mathcal{B}_2\bigr\}$, respectively. Proofs can be found in [Schröder 2001, Schröder 2002].

3 Embedding Weak Limit Spaces into Filter Spaces

In this section we construct an embedding $\mathcal{J}$ of the category of weak limit spaces WLim into the category PFil of proper filter spaces. It preserves countable products and maps countably based weak limit spaces to countably based filter spaces. However, it fails to preserve exponentials.

Let $\mathsf{X} = (X, \to)$ be a weak limit space. We define a filter convergence relation $\downarrow_{\mathsf{X}}$ on X by

$$\mathcal{F} \downarrow_{\mathsf{X}} x :\Longleftrightarrow \begin{cases} \mathcal{F} \subseteq [x] \text{ and} \\ \mathcal{F} \text{ contains a countable witness of convergence for } x \text{ in } \mathsf{X}. \end{cases}$$

Clearly, $\mathcal{J}(\mathsf{X}) := (X, \downarrow_{\mathsf{X}})$ is a proper filter space. We call $\mathcal{J}(\mathsf{X})$ the filter space *associated to* X. From Lemma 3 we can deduce the following characterisation of sequence convergence in a weak limit space in terms of filter convergence in its associated filter space.

Lemma 4. *Let X be a weak limit space. Then a sequence $(y_n)_n$ converges to a point x in X if, and only if, the Fréchet filter $[(y_n)_n] \cap [x]$ converges to x in $\mathcal{J}(\mathsf{X})$.*

By setting $\mathcal{J}(f) := f$ for every morphism f in WLim, we obtain an embedding functor from WLim to PFil.

Proposition 5. *Let X and Y be weak limit spaces. Then a function $f \colon \mathsf{X} \to \mathsf{Y}$ is sequentially continuous if, and only if, f is a filter-continuous function from $\mathcal{J}(\mathsf{X})$ to $\mathcal{J}(\mathsf{Y})$.*

Proof. Only-if-part: Let f be sequentially continuous. Let $\mathcal{F}$ be a filter that converges to some x in $\mathcal{J}(\mathsf{X})$. There is a sequence $(F_i)_i$ of sets in $\mathcal{F}$ such that $\{F_i \mid i \in \mathbb{N}\}$ is a witness of convergence for x. We define $G_j := f(\bigcap_{i=0}^{j} F_i) \in f^*\mathcal{F}$. Let $(y_n)_n$ be a sequence with $\{G_j \mid j \in \mathbb{N}\} \subseteq [(y_n)_n]$. Then there are natural numbers $m_0 < m_1 < m_2 \ldots$ such that $y_n \in G_j$ for all $j \in \mathbb{N}$ and $n \geq m_j$. For $j \in \mathbb{N}$ and $n \in \{m_j, \ldots, m_{j+1}-1\}$ we choose some $x_n \in \bigcap_{i=0}^{j} F_i$ with $f(x_n) = y_n$. Then $(x_{n+m_0})_n$ converges x, because we have $\{F_i \mid i \in \mathbb{N}\} \subseteq [(x_{n+m_0})_n] \cap [x]$. By sequential continuity of f, $(y_n)_n$ converges to $f(x)$. Hence the family $\{G_j \mid j \in \mathbb{N}\}$ constitutes a countable witness of convergence for $f(x)$ contained in $f^*\mathcal{F} \subseteq [f(x)]$, implying $f^*\mathcal{F} \downarrow_{\mathsf{Y}} f(x)$. Therefore f is filter-continuous.

If-part: Let f be filter-continuous. Let $(x_n)_n$ be a sequence converging in X to some x_∞. By Lemma 4 the filter $\mathcal{F} := [(x_n)_n] \cap [x_\infty]$ converges to x_∞ in $\mathcal{J}(\mathsf{X})$. Hence $f^*\mathcal{F}$ converges to $f(x_\infty)$. Since the Fréchet filter $[(f(x_n))_n] \cap [f(x_\infty)]$ contains $f^*\mathcal{F}$ as a subset, it converges to $f(x_\infty)$ by Axiom (F5). Hence $(f(x_n))_n$ converges to $f(x_\infty)$ by Lemma 4. □

The functor $\mathcal{J}$ gives rise to an alternative embedding of the sequential topological spaces into PFil, using the functor $\mathcal{L}_{\mathsf{Top}}\colon \mathsf{Top} \to \mathsf{Lim}$ mentioned in [Section 2.2]. We characterise the class of topological spaces on which $\mathcal{J} \circ \mathcal{L}_{\mathsf{Top}}$ agrees with the standard embedding $\mathcal{I}_{\mathsf{Top}}$ from [Section 2.1].

Proposition 6. *A topological space Z satisfies $\mathcal{J}\mathcal{L}_{\mathsf{Top}}(Z) = \mathcal{I}_{\mathsf{Top}}(Z)$ if, and only if, Z is first-countable (i.e. every element has a countable neighbourhood base).*

Proof. If-part: Let $z \in Z$, and let Φ be a countable neighbourhood base of z. Let $\mathcal{F}$ be a filter converging to z in $\mathcal{I}_{\mathsf{Top}}(Z)$. By definition of $\mathcal{I}_{\mathsf{Top}}$, $\mathcal{F}$ contains Φ. Since Φ is a countable witness of convergence for z in $\mathcal{L}_{\mathsf{Top}}(Z)$, $\mathcal{F}$ converges to z in $\mathcal{J}\mathcal{L}_{\mathsf{Top}}(Z)$.
Conversely, let $\mathcal{F}$ be a filter converging to z in $\mathcal{J}\mathcal{L}_{\mathsf{Top}}(Z)$. Then $\mathcal{F}$ contains a witness of convergence $\{F_i \,|\, i \in \mathbb{N}\}$ for z. Assume for contradiction that there is an open neighbourhood U of z that is not contained in $\mathcal{F}$. Then for every n there is some $z_n \in \bigcap_{i=0}^{n} F_i \setminus U$. Since $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(z_n)_n]$, $(z_n)_n$ converges to z in Z. This contradicts $\{z_n \,|\, n \in \mathbb{N}\} \cap U = \emptyset$. We conclude $\{U \text{ open} \,|\, z \in U\} \subseteq \mathcal{F}$. Hence $\mathcal{F}$, being a subset of $[z]$, converges to z in $\mathcal{I}_{\mathsf{Top}}(Z)$.
Only-if-part: Let $z \in Z$. Since $\mathcal{F} := [\{U \text{ open} \,|\, z \in U\}]$ converges to z, $\mathcal{F}$ contains a countable witness of convergence $\{F_i \,|\, i \in \mathbb{N}\}$ for z. For every i there is an open set U_i with $z \in U_i \subseteq F_i$. Let V be an open set containing z. Assume for contradiction that for every $n \in \mathbb{N}$ there is some $z_n \in \bigcap_{i=0}^{n} U_i \setminus V$. Since $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(z_n)_n]$, $(z_n)_n$ converges to z in Z. This contradicts $\{z_n \,|\, n \in \mathbb{N}\} \cap V = \emptyset$. Hence $\{U_i \,|\, i \in \mathbb{N}\}$ is a neighbourhood base for z. Therefore Z is first-countable. $\square$

The functor $\mathcal{J}$ preserves the existence of a countable base. We denote the arising functor from $\omega\mathsf{WLim}$ to $\omega\mathsf{PFil}$ by $\mathcal{J}_\omega$.

Proposition 7. *Any countable limit base $\mathcal{B}$ for a weak limit space X is a subbasis for the associated filter space $\mathcal{J}(\mathsf{X})$.*

Proof. Let $\mathcal{F}$ be a filter that converges to x in $\mathcal{J}(\mathsf{X})$. In order to show that $\mathcal{F} \cap \mathcal{B}$ is a witness of convergence for x, let $(z_n)_n$ be a sequence that does not converge to x in X. We choose a numbering $i \mapsto \alpha_i$ of the non-empty family $\mathcal{A} := \mathcal{B} \cap [x] \setminus [(z_n)_n]$ such that $\forall i. \exists j > i. \alpha_j = \alpha_i$. Assume for contradiction $\mathcal{F} \cap \mathcal{A} = \emptyset$. Let $\{F_i \,|\, i \in \mathbb{N}\}$ be the countable witness for x contained in $\mathcal{F}$. Then for every $n \in \mathbb{N}$ there exists some $y_n \in \bigcap_{i=0}^{n} F_i \setminus \alpha_n$, as $\alpha_n \notin \mathcal{F}$. Since $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(y_n)_n]$, $(y_n)_n$ converges to x. As $\mathcal{B}$ is a limit base, there exist some $A \in \mathcal{A}$ and $n_1 \in \mathbb{N}$ with $y_n \in A$ for all $n \geq n_1$. Moreover, there is some $n_2 \geq n_1$ with $\alpha_{n_2} = A$. This contradicts $y_{n_2} \notin \alpha_{n_2}$. Thus we have $\mathcal{F} \cap \mathcal{B} \not\subseteq [(z_n)_n]$ showing that $\mathcal{F} \cap \mathcal{B}$ is a witness of convergence for x. Hence the filter base $\mathcal{F} \cap \mathcal{B}^\cap$, where $\mathcal{B}^\cap$ denotes the closure of $\mathcal{B}$ under finite intersection, converges to x in $\mathcal{J}(\mathsf{X})$. Therefore $\mathcal{B}$ is a subbasis for $\mathcal{J}(\mathsf{X})$. $\square$

We obtain from Propositions 5 and 7:

Theorem 8. *The functor $\mathcal{J}$ embeds* WLim *into* PFil*. Its restriction $\mathcal{J}_\omega$ embeds* $\omega\mathsf{WLim}$ *into* $\omega\mathsf{PFil}$*.*

We prove that $\mathcal{J}$ preserves countable products, but not exponentials. By contrast, the restriction $\mathcal{J}_\omega$ preserves exponentials by being an equivalence functor, as we shall see in [Section 4].

Proposition 9. *For a sequence $(\mathsf{X}_i)_i$ of weak limit spaces, we have $\prod_{i\in\mathbb{N}} \mathcal{J}(\mathsf{X}_i) = \mathcal{J}(\prod_{i\in\mathbb{N}} \mathsf{X}_i)$.*

Proof. Let $\mathcal{F}$ be a filter converging to x in $\mathcal{J}(\prod_{i\in\mathbb{N}} \mathsf{X}_i)$. For every i, the projection pr_i is a filter-continuous function from $\mathcal{J}(\prod_{i\in\mathbb{N}} \mathsf{X}_i)$ to $\mathcal{J}(\mathsf{X}_i)$ by being sequentially continuous, hence $\mathrm{pr}_i^*\mathcal{F} \downarrow \mathrm{pr}_i(x)$ in X_i. By definition this means that $\mathcal{F}$ converges to x in the product filter space $\prod_{i\in\mathbb{N}} \mathcal{J}(\mathsf{X}_i)$.
Conversely, let $\mathcal{F}$ be a filter converging to x in $\prod_{i\in\mathbb{N}} \mathcal{J}(\mathsf{X}_i)$. For every i we have $\mathrm{pr}_i^*\mathcal{F} \downarrow_{\mathsf{X}_i} \mathrm{pr}_i(x)$, hence $\mathrm{pr}_i^*\mathcal{F}$ contains a countable witnesses of convergence Φ_i for $\mathrm{pr}_i(x)$ in X_i. The countable family

$$\Psi := \{\mathsf{X}_0 \times \ldots \times \mathsf{X}_{i-1} \times A \times \mathsf{X}_{i+1} \times \mathsf{X}_{i+2} \ldots \mid i \in \mathbb{N},\, A \in \Phi_i\}$$

is contained in $\mathcal{F}$, as $\mathcal{F}$ is a filter. For every $i \in \mathbb{N}$ and every sequence $(y_n)_n$ in $\prod_{j\in\mathbb{N}} \mathsf{X}_j$, $\Psi \subseteq [(y_n)_n]$ implies $\Phi_i \subseteq [(\mathrm{pr}_i(y_n))_n]$ and thus $(\mathrm{pr}_i(y_n))_n \to \mathrm{pr}_i(x)$ in X_i. Hence $(y_n)_n$ converges to x in the weak limit space $\prod_{i\in\mathbb{N}} \mathsf{X}_i$. Thus Ψ is a countable witness of convergence for x. We conclude that $\mathcal{F}$ converges to x in $\mathcal{J}(\prod_{i\in\mathbb{N}} \mathsf{X}_i)$. □

Example 1. The functor $\mathcal{J}$ does not preserve exponentials. As an example we consider an uncountable discrete[1] limit space $\mathsf{D} \in \mathsf{WLim}$ as domain space and as codomain space the two point discrete limit space $\mathsf{2}$ with carrier set $\{0, 1\}$. The only filter to converge in $\mathcal{J}(\mathsf{D})$ to a point x is the principal filter $[x]$. This implies that the filter

$$\mathcal{F} := \big[\big\{F \subseteq \mathcal{C}(\mathsf{D}, \mathsf{2}) \,\big|\, \exists E \subseteq \mathsf{X} \text{ finite}. \forall f \in F.\, f(E) = \{0\}\big\}\big]$$

converges in $\mathcal{J}(\mathsf{2})^{\mathcal{J}(\mathsf{D})}$ to the constant zero function $\mathbf{0}$. Assume that $\mathcal{F}$ converges to $\mathbf{0}$ in $\mathcal{J}(\mathsf{2}^\mathsf{D})$. Then $\mathcal{F}$ contains a countable witness of convergence, $\{F_i \mid i \in \mathbb{N}\}$, for $\mathbf{0}$ in $\mathsf{2}^\mathsf{D}$. For every $i \in \mathbb{N}$ there is a finite set E_i with $\big\{f \colon \mathsf{D} \to \mathsf{2} \,\big|\, f(E_i) = \{0\}\big\} \subseteq F_i$. The function $f_n \colon \mathsf{D} \to \mathsf{2}$ defined by $f_n(x) = 0 \;:\Longleftrightarrow\; x \in \bigcup_{i=0}^n E_i$ is sequentially continuous and hence filter-continuous by Proposition 5. As D is uncountable, there exists some $x_0 \in \mathsf{D} \setminus \bigcup_{i\in\mathbb{N}} E_i$. Since $(f_n(x_0))_n \not\to \mathbf{0}(x_0)$ in $\mathsf{2}$, $(f_n)_n$ does not converge to $\mathbf{0}$ in $\mathsf{2}^\mathsf{D}$. This contradicts $\{F_i \mid i \in \mathbb{N}\} \subseteq [(f_n)_n]$. Hence $\mathcal{J}(\mathsf{2})^{\mathcal{J}(\mathsf{D})}$ is not isomorphic to $\mathcal{J}(\mathsf{2}^\mathsf{D})$.

[1] i.e., $(x_n)_n \to_\mathsf{D} x_\infty$ if, and only if, $x_n = x_\infty$ for almost all $n \in \mathbb{N}$

4 Equivalence of $\omega\mathsf{WLim}$ and $\omega\mathsf{PFil}$

We prove in this section our main result stating that the categories $\omega\mathsf{WLim}$ and $\omega\mathsf{PFil}$ are equivalent. Actually we show the stronger result that the object part of $\mathcal{J}_\omega$ is a bijection between the weak limit spaces with a countable limit base and the countably based proper filter spaces.

In order to obtain the inverse of $\mathcal{J}_\omega$, we construct at first a retraction $\mathcal{W}$ from the proper filter spaces back to the weak limit spaces. Any filter space $\mathfrak{X}$ induces a natural sequence convergence relation $\to_{\mathfrak{X}}$. It is defined by

$$(x_n)_n \to_{\mathfrak{X}} x_\infty \;:\Longleftrightarrow\; [(x_n)_n] \cap [x_\infty] \text{ converges to } x_\infty \text{ in } \mathfrak{X}\,.$$

We define $\mathcal{W}(\mathfrak{X}) := (X, \to_{\mathfrak{X}})$ and $\mathcal{W}(f) := f$ for any morphism f in PFil and prove that $\mathcal{W}$ constitutes a functor from the category of proper filter spaces to the category of weak limit spaces.

Proposition 10. *For any proper filter space $\mathfrak{X}$, $\mathcal{W}(\mathfrak{X})$ is a weak limit space. Moreover, any filter-continuous function f from $\mathfrak{X}$ to a proper filter space $\mathfrak{Y}$ is a sequentially continuous function from $\mathcal{W}(\mathfrak{X})$ to $\mathcal{W}(\mathfrak{Y})$.*

Proof. We have to show that $\mathcal{W}(\mathfrak{X})$ satisfies Axioms (L1), (L4), (L5). Axiom (L1) follows from the validity of (F1) in $\mathfrak{X}$. Axiom (L5) holds, because $[(x_n)_n]$ is equal to $[(x_{n+1})_n]$. Axiom (L4) follows from Axiom (F5) for $\mathfrak{X}$ along with the fact that $[(x_n)_n] \cap [x_\infty] \subseteq [(x_{\xi_n})_n] \cap [x_\infty]$ holds for all $x\colon \mathbb{N}^+ \to \mathfrak{X}$ and all sequences $(\xi_n)_n$ converging to ∞ in $(\mathbb{N}^+, \tau_{\mathbb{N}^+})$.
Now let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be filter-continuous. Let $(x_n)_n$ be a sequence converging in $\mathcal{W}(\mathfrak{X})$ to some x_∞. Then the Fréchet filter $[(x_n)_n] \cap [x_\infty]$ converges to x_∞. By filter continuity, $f^*\big([(x_n)_n] \cap [x_\infty]\big)$ converges to $f(x_\infty)$. Since the Fréchet filter $[(f(x_n))_n] \cap [f(x_\infty)]$ contains the filter $f^*\big([(x_n)_n] \cap [x_\infty]\big)$, $(f(x_n))_n$ converges to $f(x_\infty)$ in $\mathcal{W}(\mathfrak{Y})$ by Axiom (F5). Hence f is sequentially continuous. $\square$

By the next lemma, $\mathcal{W}$ preserves the existence of a countable basis. We denote the arising functor from $\omega\mathsf{PFil}$ to $\omega\mathsf{WLim}$ by $\mathcal{W}_\omega$.

Lemma 11. *Any basis $\mathcal{B}$ for a proper filter space $\mathfrak{X}$ is a limit base for the weak limit space $\mathcal{W}(\mathfrak{X})$.*

Proof. Let $(x_n)_n$ be a sequence converging to some x_∞ in $\mathcal{W}(\mathfrak{X})$. Then the filter $[(x_n)_n] \cap [x_\infty] \cap \mathcal{B}$ converges to x_∞ in $\mathfrak{X}$. Thus for every sequence $(z_n)_n$ failing to converge to x_∞ in $\mathcal{W}(\mathfrak{X})$, there is some $B \in [(x_n)_n] \cap [x_\infty] \cap \mathcal{B}$ with $B \notin [(z_n)_n]$, because otherwise we would have $[(x_n)_n] \cap [x_\infty] \cap \mathcal{B} \subseteq [(z_n)_n] \cap [x_\infty]$ and hence $[(z_n)_n] \cap [x_\infty] \downarrow_{\mathfrak{X}} x_\infty$ by Axiom (F5), a contradiction. Therefore $[(x_n)_n] \cap [x_\infty] \cap \mathcal{B}$ is a witness of convergence for x. $\square$

Lemma 4 implies:

Proposition 12. *Every weak limit space* X *satisfies* $\mathcal{W}\mathcal{J}(\mathsf{X}) = \mathsf{X}$.

Yet $\mathcal{W}$ is not left adjoint to $\mathcal{J}$: Consider the filter space $\mathcal{J}(2)^{\mathcal{J}(\mathsf{D})}$, where D and 2 are the discrete weak limit spaces from Example 1. The identity function of $\mathcal{C}(\mathsf{D}, 2)$ is a sequentially continuous function from $\mathcal{W}(\mathcal{J}(2)^{\mathcal{J}(\mathsf{D})})$ to 2^{D}, but it is not a filter-continuous function from $\mathcal{J}(2)^{\mathcal{J}(\mathsf{D})}$ to $\mathcal{J}(2^{\mathsf{D}})$, as we have seen in Example 1. This implies that the functors $\mathrm{Hom}_{\mathsf{WLim}}(\mathcal{W}(-), -)$ and $\mathrm{Hom}_{\mathsf{CFil}}(-, \mathcal{J}(-))$ are not naturally isomorphic. Neither is $\mathcal{W}$ a right adjoint to $\mathcal{J}$. To see this, we equip the set of real numbers, $\mathbb{R}$, with the proper filter convergence relation $\Downarrow$ given by

$$\mathcal{F} \Downarrow x \;:\Longleftrightarrow\; \exists (y_n)_n.\big((y_n)_n \to_{\mathbb{R}} x \text{ and } [(y_n)_n] \cap [x] \subseteq \mathcal{F} \subseteq [x]\big)\,.$$

It is easy to verify that $\mathcal{W}(\mathbb{R}, \Downarrow)$ is endowed with the usual Euclidean convergence relation $\to_{\mathbb{R}}$ and that the neighbourhood filter $\mathcal{N}_{\tau_{\mathbb{R}}}(x)$ of any real number x with respect the Euclidean topology $\tau_{\mathbb{R}}$ converges to x in $\mathcal{J}\mathcal{W}(\mathbb{R}, \Downarrow)$. Since $\mathcal{N}_{\tau_{\mathbb{R}}}(x)$ does not converge to x in $(\mathbb{R}, \Downarrow)$, the identity function on $\mathbb{R}$ is not filter-continuous from $\mathcal{J}\mathcal{W}(\mathbb{R}, \Downarrow)$ to $(\mathbb{R}, \Downarrow)$. Hence $\mathcal{W}$ is not right adjoint to $\mathcal{J}$.

Proper filter spaces $\mathfrak{X}$ do not necessarily satisfy $\mathcal{J}\mathcal{W}(\mathfrak{X}) = \mathfrak{X}$. Counterexamples are $\mathcal{J}(2)^{\mathcal{J}(\mathsf{D})}$ and $(\mathbb{R}, \Downarrow)$. Surprisingly, $\mathcal{J}\mathcal{W}(\mathfrak{X}) = \mathfrak{X}$ does hold true, if $\mathfrak{X}$ has countable basis.

Proposition 13. *Any proper filter space* $\mathfrak{X}$ *with a countable basis fulfils* $\mathcal{J}\mathcal{W}(\mathfrak{X}) = \mathfrak{X}$.

Proof. Let $\mathcal{B}$ be a countable basis for $\mathfrak{X}$. We have to show that a filter $\mathcal{F}$ on $\mathfrak{X}$ converges in $\mathfrak{X}$ to an element x if, and only if, it converges in $\mathcal{J}\mathcal{W}(\mathfrak{X})$ to x.
First, let $\mathcal{F}$ converge to x in $\mathfrak{X}$. Then $\mathcal{F} \subseteq [x]$ and the countable filter base $\varPhi := \mathcal{F} \cap \mathcal{B}$ converges to x in $\mathfrak{X}$. By Axiom (F5), every sequence $(y_n)_n$ with $\varPhi \subseteq [(y_n)_n]$ converges to x in $\mathcal{W}(\mathfrak{X})$. Hence $\varPhi$ is a countable witness of convergence for x contained in $\mathcal{F}$. Therefore $\mathcal{F}$ converges to x in $\mathcal{J}\mathcal{W}(\mathfrak{X})$.
Conversely, assume that $\mathcal{F}$ converge to x in $\mathcal{J}\mathcal{W}(\mathfrak{X})$. Then we have $\mathcal{F} \subseteq [x]$ and $\mathcal{F}$ contains a countable witness of convergence $\{F_i \,|\, i \in \mathbb{N}\}$ for x in $\mathcal{W}(\mathfrak{X})$. Let $\mathcal{D} := \{\emptyset\} \cup \{B \in \mathcal{B} \,|\, \forall n \in \mathbb{N}.\ \bigcap_{i=0}^{n} F_i \not\subseteq B\}$, and let $i \mapsto \beta_i$ be a numbering of $\mathcal{D}$ with $\forall i. \exists j > i.\, \beta_j = \beta_i$. We choose for every n some $y_n \in \bigcap_{i=0}^{n} F_i \setminus \beta_n$. Since $\{F_i \,|\, i \in \mathbb{N}\} \subseteq [(y_n)_n]$, $(y_n)_n$ converges to x in $\mathcal{W}(\mathfrak{X})$. Hence the filter $[(y_n)_n] \cap [x]$ and the filter base $\varPhi := [(y_n)_n] \cap [x] \cap \mathcal{B}$ converge to x in $\mathfrak{X}$. Since $\mathcal{D} \cap [(y_n)_n] = \emptyset$, for every $B \in \varPhi$ there is some $n \in \mathbb{N}$ with $\bigcap_{i=0}^{n} F_i \subseteq B$, implying $[\varPhi] \subseteq \mathcal{F}$. Hence $\mathcal{F}$ converges to x in $\mathfrak{X}$ by Axiom (F5). $\square$

By Propositions 12 and 13, the object part of the functor $\mathcal{J}_\omega$ is a bijection between the countably based weak limit spaces and the countably based canonical filter spaces. By Proposition 5, for all $\mathsf{X}, \mathsf{Y} \in \omega\mathsf{WLim}$ the morphism part of $\mathcal{J}_\omega$ constitutes a bijection between the morphisms from X to Y and the morphisms from $\mathcal{J}_\omega(\mathsf{X})$ to $\mathcal{J}_\omega(\mathsf{Y})$. This implies our main result:

Theorem 14. *The category $\omega\mathsf{PFil}$ of countably based proper filter spaces is equivalent to the category $\omega\mathsf{WLim}$ of weak limit spaces with a countable limit base.*

From [Schröder 2002] we know that $\omega\mathsf{WLim}$ is locally cartesian closed and has all countable limits and all countable colimits. Hence:

Corollary 15. *The categories $\omega\mathsf{PFil}$ and $\omega\mathsf{CFil}$ are locally cartesian closed and have all countable limits and all countable colimits.*

It is not difficult to verify that $\mathcal{J}$ maps a weak limit space X to a filter space satisfying the Merging Axiom (F4) if, and only if, X fulfils the Merging Axiom for weak limit spaces:

(L6) if $(y_{2n})_n \to x$ and $(y_{2n+1})_n \to x$ then $(y_n)_n \to x$.

We obtain:

Theorem 16. *The category $\omega\mathsf{MFil}$ is equivalent to the category of countably based weak limit spaces satisfying the Merging Axiom (L6).*

5 Comparison of Categories Relevant to Computability Theory

In this section, we repeat the definition of other categories relevant to computability theory and investigate their relationships to filter spaces and weak limit spaces.

5.1 Equilogical spaces

The largest of the categories we shall consider is D. Scott's category Equ of equilogical spaces and of equivariant maps [Scott et al. 2004]. An *equilogical space*[2] is a pair $\mathcal{X} = (S, \equiv_\mathcal{X})$, where S is a topological space and $\equiv_\mathcal{X}$ is an equivalence relation on the carrier set of S. We say that $\mathcal{X}$ has the *T_0-property*, if S is a T_0-space. By $q_\mathcal{X} \colon S \to S/{\equiv_\mathcal{X}}$ we denote the quotient function mapping any $s \in S$ to its equivalence class modulo $\equiv_\mathcal{X}$. An *equivariant* map $f \colon \mathcal{X} \to \mathcal{Y}$ from $\mathcal{X}$ to an equilogical space $\mathcal{Y} = (T, \equiv_\mathcal{Y})$ is a set-theoretical function between the quotient sets $S/{\equiv_\mathcal{X}}$ and $T/{\equiv_\mathcal{Y}}$ which is *tracked* by a continuous function $g \colon S \to T$, meaning that $q_\mathcal{Y}(g(s)) = f(q_\mathcal{X}(s))$ holds for all $s \in S$. We denote by $\omega\mathsf{Equ}$ the

[2] We follow here the more general definition in [Menni and Simpson 2002]. The original definition in [Scott et al. 2004] requires the topological space S to be T_0.

full subcategory of those equilogical spaces whose underlying topological space has a countable topological basis. Note that a non-countably-based equilogical space may be isomorphic to a countably-based one, for example, if its equivalence relation defines all elements to be equivalent.

From Corollary 4.2 in R. Heckmann's paper [Heckmann 1998] we already know the T_0-version of the following theorem.

Theorem 17. *The category* PFil *is equivalent to a full subcategory of* Equ *that is closed under countable limits and exponentiation.*

Proof. (Sketch) One proves that PFil is an exponential ideal of Equ. In order to show this, by [Freyd and Scedrov 1990, Section 1.857] it suffices to construct an inclusion functor $\mathcal{E}\colon \mathsf{PFil} \to \mathsf{Equ}$ which has a left adjoint $\mathcal{P}\colon \mathsf{Equ} \to \mathsf{PFil}$ that preserves finite products. In order to define $\mathcal{E}$ and $\mathcal{P}$, we adapt the construction ideas for the respective functors in [Heckmann 1998].

Given a proper filter space $\mathfrak{X}$, the equilogical space $\mathcal{E}(\mathfrak{X})$ is defined to have as underlying topological space the set $Z := \big\{(\mathcal{F}, x) \,\big|\, \mathcal{F} \downarrow_{\mathfrak{X}} x\big\}$ endowed with the topology which is induced by the base consisting of the sets $\big\{(\mathcal{F}, x) \in Z \,\big|\, \mathcal{F} \ni M\big\}$, where M runs over all nonempty subsets of $\mathfrak{X}$. The equivalence relation of $\mathcal{E}(\mathfrak{X})$ is given by $(\mathcal{F}, x) \equiv_{\mathcal{E}(\mathfrak{X})} (\mathcal{F}', x') :\Longleftrightarrow x = x'$, allowing us to identify the quotient set $Z/{\equiv_{\mathcal{E}(\mathfrak{X})}}$ with the carrier set of $\mathfrak{X}$. It is not difficult to see that a function $f\colon \mathfrak{X} \to \mathfrak{Y}$ is filter-continuous if, and only if, f is an equivariant map from $\mathcal{E}(\mathfrak{X})$ to $\mathcal{E}(\mathfrak{Y})$. So defining the morphism part of $\mathcal{E}$ by $\mathcal{E}(f) := f$ yields an inclusion functor from PFil to Equ.

Conversely, for an equilogical space $\mathcal{X} = (S, \equiv_{\mathcal{X}})$ we define the carrier set of the filter space $\mathcal{P}(\mathcal{X})$ to be the quotient set $S/{\equiv_{\mathcal{X}}}$ and the convergence relation of $\mathcal{P}(\mathcal{X})$ by

$$\mathcal{F} \downarrow_{\mathcal{P}(\mathcal{X})} x :\Longleftrightarrow \exists s \in S.\big(q_{\mathcal{X}}(s) = x \wedge \{q_{\mathcal{X}}(U) \,|\, U \text{ open},\ s \in U\} \subseteq \mathcal{F} \subseteq [x]\big)\,.$$

The morphism part of $\mathcal{P}$ is defined to map a function to itself. We omit the proof of $\mathcal{P}$ being a left adjoint to $\mathcal{E}$ that preserves countable limits. $\square$

The countably based version of Theorem 17 holds as well.

Theorem 18. *The category* $\omega\mathsf{PFil}$ *is equivalent to a full subcategory of* $\omega\mathsf{Equ}$ *that is closed under countable limits and exponentiation.*

Proof. (Sketch) The reflection functor $\mathcal{P}$ in the proof of Theorem 17, which preserves countable limits, maps a countably based equilogical space $\mathcal{X}$ to a countably based filter space: if $\mathcal{B}$ is a countable base for the underlying topological space, then $\big\{q_{\mathcal{X}}(B) \,|\, B \in \mathcal{B}\big\}$ is a subbasis for the filter space $\mathcal{P}(\mathcal{X})$. Conversely, for a countably based filter space $\mathfrak{X}$ the equilogical space $\mathcal{E}(\mathfrak{X})$ is isomorphic to a countably based equilogical space $\mathcal{E}_\omega(\mathfrak{X})$: given a countable basis $\mathcal{B}$ for $\mathfrak{X}$, the underlying topological space of $\mathcal{E}_\omega(\mathfrak{X})$ has $S := \{([\mathcal{F} \cap \mathcal{B}], x) \mid \mathcal{F} \downarrow_{\mathfrak{X}} x\}$ as its carrier set and the family of sets $\{(\mathcal{G}, x) \in S \mid B \in \mathcal{G}\}$, where B runs over $\mathcal{B}$, as a topological base. So $\omega\mathsf{PFil}$ is an exponential ideal of $\omega\mathsf{Equ}$ by [Freyd and Scedrov 1990, Section 1.857], as $\mathcal{P}$ preserves finite products. □

5.2 Assemblies

The category $\omega\mathsf{Equ}$ of countably based equilogical spaces is equivalent to the category $\omega\mathsf{Ass}$ of *assemblies* over countably-based algebraic lattices, see [Menni and Simpson 2002]. It is also equivalent to the category $\mathsf{Asm}(\mathbb{P})$ of assemblies over Scott's graph model $\mathbb{P}$, see [Bauer et al. 2002]. The carrier set of the topological space $\mathbb{P}$ is the powerset of $\mathbb{N}$, topologised by the Scott topology on the dcpo $(\mathbb{P}, \subseteq)$. An object X of $\mathsf{Asm}(\mathbb{P})$ is a set X together with a function ϕ_X from X to the non-empty subsets of $\mathbb{P}$. The elements in $\phi_\mathsf{X}(x)$ are called the *realisers* of x. A morphism f between two assemblies X, Y is a function between their carriers sets which is *tracked* by a continuous function $h \colon \mathbb{P} \to \mathbb{P}$, meaning that $p \in \phi_\mathsf{X}(x)$ implies $h(p) \in \phi_\mathsf{Y}(f(x))$. An equivalence functor $\mathcal{E}_{\mathsf{Asm}(\mathbb{P})}$ from $\mathsf{Asm}(\mathbb{P})$ to $\omega\mathsf{Equ}$ can be constructed by mapping an assembly $\mathsf{X} = (X, \mathbb{P}, \phi_\mathsf{X}) \in \mathsf{Asm}(\mathbb{P})$ to the countably based equilogical space $(S_\mathsf{X}, \equiv_\mathsf{X})$ defined as follows: S_X has $\{(p, x) \in \mathbb{P} \times X \mid p \in \phi_\mathsf{X}(x)\}$ as its carrier set, topologised by the subspace topology inherited from the topological product of $\mathbb{P}$ and the indiscrete space over X; the equivalence relation $\equiv_\mathsf{X}$ is given by $(p, x) \equiv_\mathsf{X} (p', x')$ iff x equals x'. For the converse direction, one defines the object part of a functor $\mathcal{A} \colon \omega\mathsf{Equ} \to \mathsf{Asm}(\mathbb{P})$ as follows: given a countably based equilogical space $\mathcal{X} = (S, \equiv_\mathcal{X})$ and a numbering β of a countable base of S, $\mathcal{A}$ sends $\mathcal{X}$ to the assembly $(S/{\equiv_\mathcal{X}}, \mathbb{P}, \phi_\mathcal{X})$, where $\phi_\mathcal{X}(q_\mathcal{X}(s)) := \{\{i \in \mathbb{N} \mid s' \in \beta(i)\} \mid s' \equiv_\mathcal{X} s\}$. It is not too difficult to see that $\mathcal{A}(\mathcal{E}_{\mathsf{Asm}(\mathbb{P})}(\mathsf{X}))$ is isomorphic to X in $\mathsf{Asm}(\mathbb{P})$ and that $\mathcal{E}_{\mathsf{Asm}(\mathbb{P})}(\mathcal{A}(\mathcal{X}))$ is isomorphic to $\mathcal{X}$ in $\omega\mathsf{Equ}$.

5.3 Multirepresentations

A *multirepresentation* [Schröder 2002] of a set X is a surjective partial multifunction from the Baire space $\mathbb{N}^\mathbb{N}$ onto X. Multirepresentations generalise representations [Weihrauch 2000], which can be viewed as single-valued multirepresentations. Given two multirepresentations $\delta :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X$ and $\gamma :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows Y$, we say that a function $f \colon X \to Y$ is *relatively continuous* w.r.t. δ and γ, if there is a partial continuous function $g \colon \mathbb{N}^\mathbb{N} \rightharpoonup \mathbb{N}^\mathbb{N}$ satisfying $x \in \delta(p) \Longrightarrow f(x) \in \gamma(g(p))$ for every $p \in \mathbb{N}^\mathbb{N}$ and $x \in X$. The category MRep of multirepresentations as objects and of relatively continuous functions as morphisms is known to be cartesian closed. This can be shown similar to the cartesian closedness of its full subcategory Rep of Baire space representations, see [Bauer 2002]. Moreover, MRep is equivalent to the cartesian closed category $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$ of assemblies over the Baire space, see [Lietz 2004]. This category is defined like $\mathsf{Asm}(\mathbb{P})$, except for using $\mathbb{N}^{\mathbb{N}}$ as the set of realisers and *partial* continuous functions on $\mathbb{N}^{\mathbb{N}}$ as tracking functions.

The category MRep embeds into the category of countably based equilogical spaces. The corresponding functor maps a multirepresentation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows X$ to an equilogical space whose underlying topological space has the graph $\{(p,x) \mid x \in \delta(p)\}$ of δ as its carrier set and as its topology the subspace topology inherited from the product of the Baire space and the indiscrete space over X. This functor preserves finite products, but not exponentials (see [Bauer 2002] for the respective inclusion functor for Rep into $\omega\mathsf{Equ}_0$).

The category ωWLim embeds into MRep via a functor that preserves countable products and exponentials. This functor maps a countably based weak limit space X to an admissible multirepresentation of X, see [Schröder 2002].

5.4 Spaces based on net convergence

The existence of countable bases are crucial to the equivalence results in [Section 4]. A generalisation of weak limit spaces that gives rise to a category equivalent to the category of all proper filter spaces can be obtained by considering nets instead of sequences. A *net* over a set X is function from some directed set D to X written as $(x_d)_{d\in D}$, see [Engelking 1989]. A *net convergence relation* on X is a relation between nets over X and points of X. We define NLim to be the category of pairs $(X,\to)$, where $\to$ is a net convergence relation on a set X satisfying Axioms (L1) and (N2):

(N2) if $(y_d)_{d\in D} \to x$ and $[(y_d)_{d\in D}] \cap [x] \subseteq [(z_e)_{e\in E}]$ then $(z_e)_{e\in E} \to x$.

Here $[(y_d)_{d\in D}]$ denotes the induced filter $\big[\big\{\{y_{d'} \mid d \le_D d'\} \,\big|\, d \in D\big\}\big]$. Morphisms of NLim are the functions which preserve convergence of nets. The category NLim can be proven to be equivalent to PFil. The equivalence functor from NLim to PFil maps an object $(X,\to)$ to the proper filter space $(X,\downarrow)$ defined by

$$\mathcal{F} \downarrow x \;:\Longleftrightarrow\; \exists (y_d)_{d\in D}.\big((y_d)_{d\in D} \to x \text{ and } [(y_d)_{d\in D}] \cap [x] \subseteq \mathcal{F} \subseteq [x]\big)\,.$$

Conversely, a proper filter space $\mathfrak{X} = (X, \downarrow_{\mathfrak{X}})$ is mapped to that space in NLim in which a net $(y_d)_{d\in D}$ converges to a point $x \in X$ if, and only if, the filter $[(y_d)_{d\in D}] \cap [x]$ converges to x in $\mathfrak{X}$. We omit the details.

5.5 Sequential spaces, Kelley spaces, Core compactly generated spaces

We give some examples of full cartesian closed subcategories of the category Top of topological spaces. The first is the category Seq of sequential spaces. A topological space Z is called *sequential*, if any sequentially closed subset of Z is closed, see [Engelking 1989]. The category Seq is known to be equivalent to a full reflective subcategory of the category Lim of limit spaces and inherits its cartesian closed structure from it, see [Hyland 1979, Menni and Simpson 2002]. Another supercategory of Seq is the category kTop of compactly generated spaces. A topological space Z is called *compactly generated*, if any subset V is open in Z, whenever $p^{-1}(V)$ is open in K for every compact Hausdorff space K and every continuous function $p\colon K \to Z$. Compactly generated Hausdorff spaces are referred to as *Kelley spaces*. The inclusion functor from Seq to kTop preserves products, but not exponentials. An even larger cartesian closed subcategory of Top is the category of *core compactly generated spaces*. Core compactly generated spaces are characterised as topological quotients of core compact (i.e. exponentiable) topological spaces. None of the inclusion functors of these categories into Top preserves finite products (nor exponentials). Details can be found in [Escardó et al. 2004].

5.6 QCB-spaces

A *qcb-space* [Simpson 2003] is a topological space Z that is a *q*uotient of a *c*ountably *b*ased topological space[3]. The class of qcb-spaces is exactly the class of sequential topological spaces on which a reasonable computability theory is possible. Moreover, the category QCB of qcb-spaces (with topological continuous functions as morphisms) is cartesian closed, see [Schröder 2002]. It is a full subcategory of several categories in such a way that the respective embeddings preserve finite products and exponentials. Examples are the categories of: equilogical spaces (as an example of a cartesian closed supercategory of Top), sequential spaces, compactly generated spaces, core compactly generated spaces (as examples of cartesian closed subcategories of Top), Baire space multirepresentations and assemblies over $\mathbb{P}$ (as examples of effective categories), limit spaces and weak limit spaces. The corresponding proofs are due to several authors, see [Bauer 2002, Escardó et al. 2004, Menni and Simpson 2002, Schröder 2001]. Theorems 14 and 16 imply that QCB also lives inside the filter space categories PFil and MFil inheriting the respective constructions of products and exponentials. The same holds true for the filter space category Fil in [Hyland 1979], because function spaces in MFil are constructed as in Fil, see [Heckmann 1998]. This gives the expected positive answer to the question in [Simpson 2003, Simpson et al. 2007] whether this is the case.

The following diagram depicts embeddings between the aforementioned categories as described before. Solid arrows denote embeddings that preserve finite products as well as existing exponentials, dashed arrows indicate embeddings that preserve finite products, but not exponentials, and dotted arrows stand for

[3] i.e., there is a surjection $q\colon A \to Z$ from a countably based space A onto Z satisfying: V open in $Z \iff q^{-1}[V]$ open in A.

embeddings that do not preserve finite products. The symbol ωTop denotes the category of topological spaces with a countable base. Note that the diagram would commute, were the two embeddings into Top omitted.

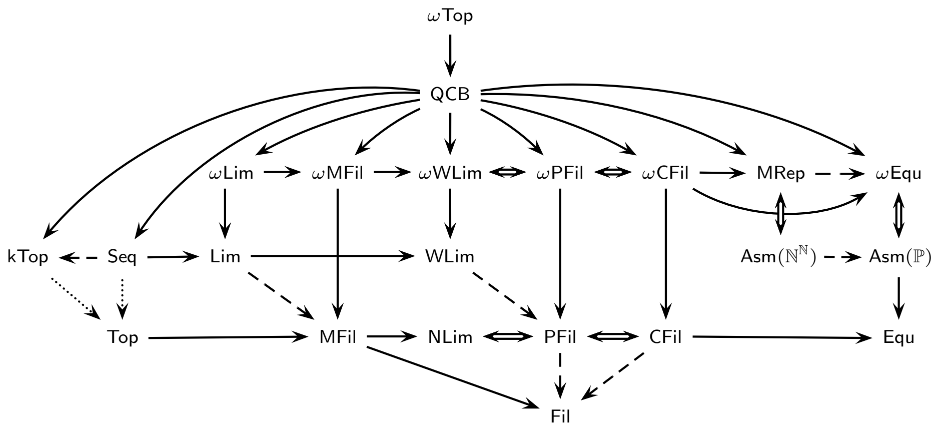

Figure 1: *The relationship between relevant categories*

6 Discussion

We have seen that countably based proper filter spaces are basically the same mathematical objects as weak limit spaces. In [Schröder 2002] computability for functions between countably based weak limits spaces is introduced by endowing the weak limit spaces with multirepresentations obtained from numberings of the respective limit bases. One can show that the induced computability notion is equivalent to the one generated by coded filter spaces, see [Section 2.1]. However, the corresponding category of coded filter spaces as objects and computable functions as morphisms does not have all finite colimits and seems not to be cartesian closed. In [Schröder 2002], an effective cartesian closed category (denoted by EffWeakLim) with finite limits and colimits is defined by imposing an effectivity condition on the used multirepresentations. It would be interesting to investigate effectivity notions on filter space bases which lead to a category of "effectively coded filter spaces" that is equivalent to EffWeakLim or to a cartesian closed subcategory of EffWeakLim.

Acknowledgement

I thank Alex Simpson for fruitful discussions and the anonymous referees for valuable remarks.

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