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Computable Riesz Representation for Locally Compact Hausdorff Spaces [1]

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Abstract: By the Riesz Representation Theorem for locally compact Hausdorff spaces, for every positive linear functional I on $\mathcal{K}(X)$ there is a measure μ such that $I(f) = \int f\,d\mu$, where $\mathcal{K}(X)$ is the set of continuous real functions with compact support on the locally compact Hausdorff space X. In this article we prove a uniformly computable version of this theorem for computably locally compact computable Hausdorff spaces X. We introduce a representation of the positive linear functionals I on $\mathcal{K}(X)$ and a representation of the Borel measures on X and prove that for every such functional I a measure μ can be computed and vice versa such that $I(f) = \int f\,d\mu$.
Key Words: computable analysis, computable topology, Hausdorff spaces, Riesz representation theorem.
Category: F.0, F.1.0, F.1.1

1 Introduction

Measure and integration can be introduced in two ways: by starting either from a measure and introducing integration as a derived concept or from a "continuous" linear real valued operator, an abstract integral, on a space of functions and considering measure as a derived concept [Coh80, Ped89]. Fundamental theorems relating these two approaches are, for example, the Daniell-Stone theorem [Bau74] or various versions of the Riesz representation theorem [GP65, Con90, Coh80, Rud74]. In this article we study the computational content of one of these theorems, the Riesz representation theorem for locally compact Hausdorff spaces [Coh80]. For this purpose we use the representation approach (TTE), which has turned out to be particulary natural and flexible among the various models for studying computability in Analysis and related fields [Wei00].

There are only few publications on computable measure theory in the framework of TTE [Wei99, Mül99, Wu05, WD05, SS06, WD06, WW06, LW07, Sch07]. In the following four cases the "dual" space is studied:

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1. A computable version of the Daniell-Stone theorem, which characterizes a computable abstract integral on Stone vector lattices of functions $f : X \to \mathbb{R}$ by a computable measure space, has been proved in [WW06].

2. A computable version of the Riesz representation theorem that characterizes the continuous functionals on $C[0; 1]$ by functions of bounded variation has been proved in [LW07].

3. A computable version of the Riesz representation theorem for computable Hilbert spaces that characterizes the dual space of l_2 by itself has been proved in [BY06].

4. In this article we prove a computable correspondence between the positive functionals on the space $\mathcal{K}(X)$ of the continuous functions with compact support on a computable Hausdorff space X and the Borel measures on X.

These four theorems differ by the structure considered on the basic set X: (1) a set X without structure, (2) the real interval $[0; 1]$, (3) the natural numbers, (4) a Hausdorff space. In all the cases the operators on the space of functions are in some sense continuous. Finally, the characterization is by means of (1) "computable measure spaces", (2) functions of bounded variation, (3) the function space itself, (4) "computable Borel measures". In (2) instead of measures functions of bounded variation are considered for (Riemann-Stieltjes) integration. Functions of bounded variation correspond to real-valued measures ($\mu(I)$ can be negative). But neither such measures nor their relation to the functions of bounded variation have been studied in computable analysis. Computable Borel measures have been studied in [Wei99, Mül99, SS06, Sch07]. But their relation to the computable measure spaces considered in [Wu05, WD05, WD06, WW06] is not yet known.

In this article we prove a computable version of the following theorem [Coh80, Rud74].

Theorem 1 (Riesz representation). *Let X be a locally compact σ-compact Hausdorff space. Then for every positive linear functional $I : \mathcal{K}(X) \to \mathbb{R}$ on the space of the continuous real functions with compact support there is a (unique) regular Borel measure μ on X such that*

$$I(f) = \int f \, d\mu \quad \textit{for all} \quad f \in \mathcal{K}(X) \,. \tag{1}$$

We introduce "effective" locally compact Hausdorff spaces and prove that there are computable operators mapping I to μ and vice versa such that (1) holds true. In Section 2 we summarize concepts from Computable Analysis, which we will use in this article. Computability on locally compact Hausdorff spaces has been introduced in [GX07]. The definitions and some results are put together in Section 3. In TTE, computability is defined relative to given representations. In Section 4 we introduce natural representations of the space of positive linear operators $I : \mathcal{K}(X) \to \mathbb{R}$ and of the regular Borel measures on the given topological space X. Finally in Section 5 we prove that with respect to these representations the functions $I \mapsto \mu$ and $\mu \mapsto I$ such that $I(f) = \int f\, d\mu$ are computable.

2 Computable Analysis

In this article we use the framework of TTE (Type-2 theory of effectivity), see [Wei00] for more details. A partial function from X to Y is denoted by $f :\subseteq X \to Y$. We assume that Σ is a fixed finite alphabet containing the symbols 0 and 1 and consider computable functions on finite and infinite sequences of symbols Σ^* and Σ^ω, respectively, which can be defined, for example, by Type-2 machines, i.e., Turing machines reading from and writing on finite or infinite tapes. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1 a_2 \ldots a_k) := 110a_1 0a_2 0 \ldots a_k 011$ for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly unless $u = v$. We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle \cdot \rangle$. By "$\lhd$" we denote the subword (infix) relation.

We use the concept of multi-functions. A *multi-valued partial function*, or *multi-function* for short, from A to B is a triple $f = (A, B, \mathrm{R}_f)$ such that $\mathrm{R}_f \subseteq A \times B$ (the *graph* of f). Usually we will denote a multi-function f from A to B by $f :\subseteq A \rightrightarrows B$. For $X \subseteq A$ let $f[X] := \{b \in B \mid (\exists a \in X)(a, b) \in \mathrm{R}_f\}$ and for $a \in A$ define $f(a) := f[\{a\}]$. Notice that f is well-defined by the values $f(a) \subseteq B$ for all $a \in A$. We define $\mathrm{dom}(f) := \{a \in A \mid f(a) \neq \emptyset\}$. In the applications we have in mind, in every computation the multi-function $f :\subseteq A \rightrightarrows B$ maps every $a \in \mathrm{dom}(f)$ to some b such that $(a, b) \in R_f$. But generally there is no method to select a specific one. $f(a)$ can be interpreted as the set of all results which are "acceptable" on input $a \in A$. In accordance with this interpretation the "functional" composition $g \circ f :\subseteq A \rightrightarrows D$ of $f :\subseteq A \rightrightarrows B$ and $g :\subseteq C \rightrightarrows D$ is defined by $\mathrm{dom}(g \circ f) := \{a \in A \mid a \in \mathrm{dom}(f) \text{ and } f(a) \subseteq \mathrm{dom}(g)\}$ and $g \circ f(a) := g[f(a)]$ (in contrast to "non-deterministic" or "relational" composition gf defined by $g\,f(a) := g[f(a)]$ for *all* $a \in A$).

Notations $\nu :\subseteq \Sigma^* \to M$ and representations $\delta :\subseteq \Sigma^\omega \to M$ are used for introducing relative continuity and computability on "abstract" sets M. For a representation $\delta :\subseteq \Sigma^\omega \to M$, if $\delta(p) = x$ then the point $x \in M$ can be identified by the "name" $p \in \Sigma^\omega$.

For naming systems $\gamma_i :\subseteq Y_i \to M_i$ $(i = 0, \ldots, k)$, a function $h :\subseteq Y_1 \times \ldots \times Y_k \to Y_0$ is a $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-realization of the multi-function $f :\subseteq M_1 \times \ldots \times M_k \rightrightarrows M_0$, if $\gamma_0 \circ h(p_1, \ldots, p_k) \in f(\gamma_1(p_1), \ldots, \gamma_k(p_k))$ whenever $f(\gamma_1(p_1), \ldots, \gamma_k(p_k))$ exists. The multi-function f is $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-continuous (–computable), if it has a continuous (computable) $(\gamma_1, \ldots, \gamma_k, \gamma_0)$-realization.

For naming systems $\gamma :\subseteq Y \to M$ and $\gamma' :\subseteq Y' \to M'$ ($Y, Y' \in \{\Sigma^*, \Sigma^\omega\}$), let $\gamma \leq_t \gamma'$ (t-reducible) and $\gamma \leq \gamma'$ (reducible) iff the identity $\mathrm{id} : a \mapsto a$ ($a \in M$) is (γ, γ')-continuous and (γ, γ')-computable, respectively. Define t-equivalence and equivalence as follows: $\gamma \equiv_t \gamma' \iff (\gamma \leq_t \gamma'$ and $\gamma' \leq_t \gamma)$ and $\gamma \equiv \gamma' \iff (\gamma \leq \gamma'$ and $\gamma' \leq \gamma)$, respectively. A set $X \subseteq M$ is γ-r.e. iff there is a Type-2 machine such that for all $p \in \mathrm{dom}(\gamma)$: the machine halts on input p iff $\gamma(p) \in M$.

If the representations of the sets under consideration are fixed, we will say simply "computable" instead of "(γ, δ)-computable" etc. Two representations induce the same continuity or computability iff they are t-equivalent or equivalent, respectively. If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. In particular, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on represented sets are closed under flowchart programming with indirect addressing [Wei08]. This result allows convenient informal construction of new computable functions on multi-represented sets from given ones.

Let $\nu_{\mathbb{N}}$ and $\nu_{\mathbb{Q}}$ be standard notations of the natural numbers and the rational numbers, respectively and let ρ be the Cauchy representation of the real numbers.

For any two representations $\gamma :\subseteq \Sigma^\omega \to M$ and $\delta :\subseteq \Sigma^\omega \to N$ there is a canonical representation $[\gamma \to \delta]$ of the set of (γ, δ)-continuous (total) functions $f : M \to N$ [Wei00, Definition 3.3.13] which can be characrerized up to equivalence as follows [Wei00, Theorem 3.3.14]: For every representation $\tilde{\delta}$ of the of (γ, δ)-continuous (total) functions $f : M \to N$, the function

$$\mathrm{eval} : (F, x) \mapsto F(x) \text{ is } (\tilde{\delta}, \gamma, \delta)\text{-computable} \iff \tilde{\delta} \leq [\gamma \to \delta]\,. \tag{2}$$

3 Computable Topology

For the basic concepts of topology the reader is referred, for example, to [Eng89] or the corresonding sections in [Coh80] and [Rud74]. The definitions and results on computability in this section are from [GX07, GSW07].

On a *second countable T_0-space*, that is, a topological space with countable base such that every point $x \in X$ can be identified by its (open) neighbourhoods, we introduce computability by means of a notation of a base.

Definition 2 (computable T_0-space [GSW07]). A computable T_0-space is a tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that (X, τ) is a topological T_0-space, β is a base of τ ($U \neq \emptyset$ for all $U \in \beta$) and $\nu :\subseteq \Sigma^* \to \beta$ is a notation of the base with recursive domain and computable intersection, i.e., there is an r.e. set $B \subseteq (\mathrm{dom}(\nu))^3$ with

$$\nu(u) \cap \nu(v) = \bigcup_{(u,v,w) \in B} \nu(w)\,.$$

In [GW05] computable T_0-spaces are called "computable T_0-spaces with computable intersection" and the relation to the similar "computable topological spaces" from [Wei00] is discussed.

In the following we assume that (X,τ) is a Hausdorff space (that is, for any $x \neq y$ there are disjoint open sets $U, V \in \tau$ such that $x \in U$ and $y \in V$) and that the topology is locally compact (that is, for every point x there is some $U \in \tau$ such that $x \in U$ and the closure $\overline{U}$ of U is compact). On the set X, the set β^* of the finite unions of base elements, the topology τ, the set $\tilde{\tau}$ of the closed subsets of X and the set $\mathrm{Cp}(X)$ of compact subsets of X we introduce computability via the following naming systems.

Definition 3 (some standard representations [GSW07, GX07]). Define

1. the representation $\delta :\subseteq \Sigma^\omega \to X$ by
$$\delta(p) = x, \quad \text{iff} \quad \{u \mid x \in \nu(u)\} = \{u \mid \iota(u) \lhd p\}\,.$$
2. the notation $\nu^* :\subseteq \Sigma^* \to \beta^*$ by $\nu^*(w) := \bigcup\{\nu(u) \mid \iota(u) \lhd w\}$,
3. the representation $\theta :\subseteq \Sigma^\omega \to \tau$ by $\theta(p) := \bigcup\{\nu(u) \mid \iota(u) \lhd p\}$,
4. the representation $\psi :\subseteq \Sigma^\omega \to \tilde{\tau}$ by $\psi(p) := X \setminus \theta(p)$,
5. the representation $\kappa :\subseteq \Sigma^\omega \to \mathrm{Cp}(X)$ by
$$\kappa(p) = K, \quad \text{iff} \quad \{u \mid K \subseteq \nu^*(u)\} = \{u \mid \iota(u) \lhd p\}\,.$$

Notice that in 1. and 5., a name $p \in \Sigma^\omega$ is a list of *all* u such that $x \in \nu(u)$ and $K \subseteq \nu^*(u)$, respectively, while in 3. and 4. a name p must list only sufficiently many base elements. The representations δ and κ are topologically *admissible* [Wei00], θ and ψ are admissible representations of natural limit spaces [Sch02, Sch03].

In the following let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable locally compact Hausdorff space defined as follows.

Definition 4 [GX07]. A computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ is called

1. computable Hausdorff space, iff there is an r.e. set $H \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ such that $\nu(u) \cap \nu(v) = \emptyset$ for all $(u, v) \in H$, and for all $x \neq y$ there is some $(u, v) \in H$ such that $x \in \nu(u)$ and $y \in \nu(v)$,
2. computable locally compact Hausdorff space, iff it is a computable Hausdorff space and $\overline{U}$ is compact for all $U \in \beta$ such that the function $U \mapsto \overline{U}$ is (ν, κ)-computable.

Notice that $\overline{V}$ is compact for all $V \in \beta^*$. The following results are from [GX07]:

$$- \text{ intersection is } (\theta,\theta,\theta)\text{-computable on } \tau, \quad (3)$$

$$- \nu^* \le \theta\,, \quad (4)$$

$$- V \mapsto \overline{V} \text{ for } V \in \beta^* \text{ is } (\nu^*,\kappa)\text{-computable}, \quad (5)$$

$$- \kappa \le \psi\,, \quad (6)$$

$$- x \in U \text{ is } (\delta,\theta)\text{-r.e.}\,, \quad (7)$$

$$- K \subseteq U \text{ is } (\kappa,\theta)\text{-r.e.}\,. \quad (8)$$

For $A \subseteq X$ let $\chi_A : X \to \mathbb{R}$ be the characteristic function of A ($\chi_A(x) = 1 \iff x \in A$). For $f : X \to \mathbb{R}$ let $\mathrm{supp}(f) := \overline{\{x \mid f(x) \neq 0\}}$ be the the support of f. Let $\mathcal{K}(X)$ be the set of all continuous functions $f : X \to \mathbb{R}$ with compact support. For compact K, open U and $f \in \mathcal{K}(X)$ such that $\mathrm{range}(f) \subseteq [0;1]$ we define

$$K \prec f :\iff \chi_K \le f, \quad \text{and} \quad f \prec U :\iff \mathrm{supp}(f) \subseteq U\,. \quad (9)$$

Obviously, $f \le \chi_U$ if $f \prec U$.

For a locally compact Hausdorff space with countable base, for every compact set K and every open set U such that $K \subseteq U$ there are some $V \in \beta^*$ and continuous $f : X \to \mathbb{R}$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$ and $K \prec f \prec U$ (Urysohn theorem) [Coh80, Eng89]. We need a computable version.

Lemma 5. *1. The multifunction $(K,U) \rightrightarrows V$ mapping each compact K and each open U such that $K \subseteq U$ to some $V \in \beta^*$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$ is (κ,θ,ν^*)-computable.*

2. (computable Urysohn) The multifunction $(K,U) \rightrightarrows f$ mapping each compact K and each open U such that $K \subseteq U$ to some $f \in \mathcal{K}(\mathbb{R})$ such that $K \prec f \prec U$ is $(\kappa,\theta,[\delta \to \rho])$-computable.

Proof: 1. This has been proved in [GX07].

2. In [GX07] it is also shown that every computable locally compact Hausdorff space is computably T_3. The computable Urysohn theorem for such spaces has been proved in [GSW07]. □

4 The Representations of Functions, Functionals and Measures.

Computable Analysis studies, which functions are computable with respect to given representations. Since almost all representations of a set are completely useless the investigations are concentrated on "effective" representations, that is, representations which are related to some given algebraic or topological structure on the set. In many cases TTE can explain why some representations are useful or "natural" (admissible representations [KW85, Her99, Wei00, Sch02]).

In our situation we have a bijection $I \leftrightarrow \mu$ and try to *find* reasonable or "natural" representations such that the function and its inverse become computable. Such a problem may have many solutions. For example, the real function $x \mapsto 3x$ and its inverse are (ρ, ρ)-computable as well as $(\rho_<, \rho_<)$-computable [Wei00].

We still assume that $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable locally compact Hausdorff space. For a computable version of the Riesz representation theorem we need representations of the set $\mathcal{K}(X)$ of continuous functions $f : X \to \mathbb{R}$ with compact support, of the set LP of linear positive functionals on $\mathcal{K}(X)$ and of the set RBM of regular Borel measures.

We consider the representations from Definition 3. Since δ and ρ are admissible representations for the topologies τ and $\tau_\mathbb{R}$ (the standard topology on the real numbers), respectively, a function $f : X \to \mathbb{R}$ is continuous, iff it is (δ, ρ)-continuous by Theorem 3.2.11 in [Wei00]. For the (δ, ρ)-continuous functions we have the canonical representation $[\delta \to \rho]$ which by (2) is tailor-made for computing the evaluation $(f, x) \mapsto f(x)$.

Let $\hat{\delta}$ be the restriction of $[\delta \to \rho]$ to $\mathcal{K}(X)$, the continuous functions with compact support. The representation $[\hat{\delta} \to \rho]$ of the set of $(\hat{\delta}, \rho)$-continuous operators is tailor-made for evaluation $(I, h) \mapsto I(h)$. But in general range($[\hat{\delta} \to \rho]$) does not contain all positive linear functionals $I : \mathcal{K}(X) \to \mathbb{R}$.

Example 1. Consider the space $\mathbf{X} := (\mathbb{R}, \tau_\mathbb{R}, \tilde{J}, \nu_J)$ where ν_J is a canonical notation of the set $\tilde{J}$ of open intervals with rational end-points, which is a computable locally compact Hausdorff space. Let $I(h) := \int h \, d\lambda$ (λ the Lebesgue measure) be the usual Riemann integral. Then I is positive and linear on $\mathcal{K}(\mathbb{R})$.

Suppose that Riemann integration I is $(\hat{\delta}, \rho)$-continuous, hence $([\delta \to \rho], \rho)$-continuous on the set $\mathcal{K}(\mathbb{R})$. Since $\delta \equiv \rho$ (δ from Definition 3), $[\delta \to \rho] \equiv [\rho \to \rho]$. By [Wei00, Lemma 6.1.7], $[\rho \to \rho] \equiv \delta_{co}$ where $\delta_{co}(p) = f$ iff p is a list of all pairs $(u, v) \in \Sigma^* \times \Sigma^*$ such that $f[\overline{\nu_J(u)}] \subseteq \nu_J(v)$ (compact-open representation). Therefore, Riemann integration is (δ_{co}, ρ)-continuous on $\mathcal{K}(\mathbb{R})$. Since the representation δ_{co} is admissible with respect to the compact-open topology on $C(\mathbb{R}, \mathbb{R})$ [Wei00], integration must be continuous on the subset $\mathcal{K}(\mathbb{R})$ of $C(\mathbb{R}, \mathbb{R})$, in particular in the "point" f, $f(x) = 0$ for all x. Since $I(f) = 0$, f must have an open neighborhood U in the compact-open topology such that $I[U] \subseteq (-1; 1)$. Since the finite intersections of subbase elements $\{f \in C(\mathbb{R}, \mathbb{R}) \mid f[\overline{\nu_J(u)}] \subseteq \nu_J(v)\}$ form a basis, there are open rational intervals $I_1, J_1, \ldots, I_k, J_k$, such that $0 \in J_1 \cap \ldots \cap J_k$ and $I(g) \in (-1; 1)$ whenever $g \in \mathcal{K}(\mathbb{R})$ and $g[\overline{I}_m] \subseteq J_m$ for $1 \le m \le k$. But there is some $g \in \mathcal{K}(\mathbb{R})$ such that $g[\overline{I}_m] \subseteq J_m$ for $1 \le m \le k$ and $I(g) = \int g \, d\lambda > 1$.

Therefore, Riemann integration I is a linear positive operator on $\mathcal{K}(\mathbb{R})$ which is not $(\hat{\delta}, \rho)$-continuous, hence not in range$([\hat{\delta} \to \rho])$. $\square$

We solve the problem by adding to each $[\delta \to \rho]$-name of $f \in \mathcal{K}(\mathbb{R})$ information about its support.

Definition 6. Define the representation δ_K of $\mathcal{K}(X)$ by

$$\delta_K(p) = f \iff (\exists w, q)\big(p = \langle w, q\rangle,\ \mathrm{supp}(f) \subseteq \nu^*(w) \ \text{ and } \ [\delta \to \rho](q) = f\big). \tag{10}$$

Lemma 7. *Every positive linear operator $I : \mathcal{K}(X) \to \mathbb{R}$ is in the range of $[\delta_K \to \rho]$.*

Proof: We will show this in the proof of Theorem 10 below. $\square$

In Theorem 1 the space must be σ-compact, that is, a countable union of compact sets. In our case X is σ-compact since $X = \bigcup\{\overline{U} \mid U \in \beta\}$. The set of *Borel sets* $\mathcal{B}(X)$ is the smallest σ-algebra containing the set τ of open sets. A measure on $\mathcal{B}(X)$ is called a Borel measure. We consider regular Borel measures as defined in [Coh80].

Definition 8 (regular Borel measure). A Borel measure $\mu : \mathcal{B}(X) \to \mathbb{R}$ on a Hausdorff space is *regular*, iff

1. $\mu(K) < \infty$ for all compact K,
2. $\mu(U) = \sup\{\mu(K) \mid K \text{ compact}, \ K \subseteq U\}$ for all open $U \in \tau$,
3. $\mu(A) = \inf\{\mu(U) \mid A \subseteq U, \ U \text{ open}\}$ for all $A \in \mathcal{B}(X)$.

Let $\mathcal{M}$ be the set of all regular Borel measures on $\mathbf{X}$.

We need an appropriate representation of $\mathcal{M}$ such that a name p of a measure μ supplies sufficient information for computing the positive linear operator $f \mapsto \int f\,d\mu$ for $f \in \mathcal{K}(\mathbb{R})$ represented by δ_K. By Definition 8.3 a regular Borel measure is uniquely defined by its values $\mu(U)$ for open sets U.

Since $U = \bigcup\{V \in \beta^* \mid V \subseteq U\}$, $\mu(U) = \sup\{\mu(V) \mid V \in \beta^*, V \subseteq U\}$, hence the measure is defined already by its values on the countable set β^*. In [Wei99] a representation of the probability measures on the unit interval is defined by names, which for every rational open interval approximate its measure from below. This information and the fact that the probability measure of the whole space, the compact unit interval, is 1 allows to show that $(\mu, f) \mapsto \int f\,d\mu$ for continuous f becomes computable. Since our space is only locally compact we need a list of arbitrarily big open sets with *known* measure. In the following definition a name of a measure contains information to compute the measure of every open set from below and instead of the single compact space $[0;1]$ with measure 1 in [Wei99] it contains information about an exhausting sequence $\theta(r_i)$ of open sets with their measures.

Definition 9 (representation of measures). Define a representation $\delta_\mathcal{M}$ of the regular Borel measures on $\mathbf{X}$ as follows: $\delta_\mathcal{M}(p) = \mu$, iff there are $q \in \Sigma^\omega$ and $r_i, s_i \in \Sigma^\omega$ for $i \in \mathbb{N}$ such that

1. $p = \langle q, r_0, s_0, r_1, s_1, \ldots \rangle$,
2. q is a list of all $\langle u, v \rangle$ such that $\nu_\mathbb{Q}(u) < \mu(\nu^*(v))$,
3. $(\forall w)(\exists i)\, \overline{\nu^*(w)} \subseteq \theta(r_i)$ and
4. $\mu \circ \theta(r_i) = \rho(s_i)$.

For every compact set K there is some w such that $K \subseteq \nu^*(w)$. Therefore, if $p \in \mathrm{dom}(\delta_\mathcal{M})$, then for every compact K there is some i such that $K \subseteq \theta(r_i)$. In general the sets $\nu^*(w)$ as well as their closures $\overline{\nu^*(w)}$ may have non-computable measures even if μ corresponds to a computable operator I (Theorem 1).

Example 2. Consider the space $\mathbf{X} := (\mathbb{R}, \tau_\mathbb{R}, \tilde{J}, \nu_J)$ from Example 1. We define a measure μ on the Borel subsets. Let $a_1, a_2, \ldots$ be a computable one-one enumeration of an r.e. set $A \subseteq \mathbb{N}$, which is not recursive. Then the real number $x_A = \sum_{i \geq 1} 2^{-a_i}$ is not computable [Wei00]. Let $Y := \{0,1\} \cup \{2^{-1}, 2^{-2}, \ldots\} \cup \{1+2^{-1}, 1+2^{-2}, \ldots\}$. Define $\mu(\{y\})$ for $y \in Y$ by

$$\mu(\{0\}) := \mu(\{1\}) := 1 - x_A,$$
$$\mu(\{2^{-i}\}) := \mu(\{1+2^{-i}\}) := 2^{-a_i} \quad (i \geq 1)$$

and let $\mu(B) = \sum\{\mu(\{y\}) \mid y \in B \cap Y\}$ for every Borel subset B of $\mathbb{R}$. Then $\mu((0;1)) = x_A$ and $\mu([0;1]) = 2 - x_A$, which are non-computable real numbers. We observe that for rational numbers $a < b$, $\mu((a;b))$ is $\rho_<$-computable ($\rho_<(p) = x$ iff p is a list of all $a \in, \mathbb{Q}$ such that $a < x$ [Wei00]) but computable if and only if $a \notin \{0,1\}$. It remains to show that integration $I : f \mapsto \int f \, d\mu$ is (δ_K, ρ)-computable. By Theorem 10 below it suffices to show that μ is $\delta_\mathcal{M}$-computable. By the above observation, a computable $\delta_\mathcal{M}$-name of μ can be constructed straightforwardly. $\square$

5 The Main Theorem

We can now formulate our main theorem by which a measure μ can be computed from I and vice versa such that $I(f) = \int f \, d\mu$ for all $f \in \mathcal{K}(X)$. Since our space X is σ-compact, by Theorem 1, the classical Riesz representation theorem, the operators S and T in the following theorem are well-defined.

Theorem 10 (computable Riesz representation). *Let* $\mathbf{X}$ *be a computable locally compact Hausdorff space.*

1. *The operator* $S : I \mapsto \mu$ *mapping each positive linear operator* I *to the regular Borel measure measure* μ *such that* $I(f) = \int f\,d\mu$ *is* $([\delta_K \to \rho], \delta_{\mathcal{M}})$*-computable.*
2. *The Operator* $T : \mu \mapsto I$ *mapping each regular Borel measure* μ *the the positive linear operator* I *such that* $I(f) = \int f\,d\mu$ *for* $f \in \mathcal{K}(X)$ *is* $(\delta_{\mathcal{M}}, [\delta_K \to \rho])$*-computable.*

Proof: We start with the second part. Let $\mathrm{supp}(f)\subseteq L$, L open and $\mu(L)$ finite. Let $c \in \mathbb{N}$ such that $-c < f(x) < c$ for all $x \in X$.

Let $(J_1, \ldots, J_n)$ be a partition of the interval $(-c; c)$ into disjoint intervals and let $A_i := f^{-1}[J_i] \cap L$. Then

$$\sum_i \inf(J_i)\chi_{A_i} \le f \le \sum_i \sup(J_i)\chi_{A_i}\,.$$

Since the sets A_i are measurable,

$$\sum_i \inf(J_i)\mu(A_i) \le \int_L f\,d\mu \le \sum_i \sup(J_i)\mu(A_i) \tag{11}$$

by monotonicity of integration.

Proposition 11. *For every open interval* $J\subseteq(-c; c)$ *and every* $\varepsilon > 0$ *there is some subinterval* $[a; b]\subseteq J$ *such that* $a, b \in \mathbb{Q}$, $a < b$ *and* $\mu(f^{-1}[a; b] \cap L) < \varepsilon$.

Proof: Otherwise, there were $m \ge (\mu(L) + 1)/\varepsilon$ disjoint subintervals J' of J such that $\mu(f^{-1}[J']\cap L) \ge \varepsilon$ for each of them, and hence $\mu(L) \ge \mu(f^{-1}[J]\cap L) \ge m\varepsilon > \mu(L)$ (contradiction). □(Prop. 11)

Proposition 12. *For every* $\varepsilon > 0$ *and every* $m > 4c/\varepsilon$ *there are pairwise disjoint open intervals* $I_0, \ldots, I_m\subseteq(-c; c)$ *with rational boundaries of length* $< \varepsilon$ *such that*

$$\mu(L) - \varepsilon < \sum_{k=0}^{m} \mu(f^{-1}[I_k] \cap L) \le \mu(L)\,.$$

Proof: For $k = 0, \ldots, m$ let $x_k := -c + 2kc/m$. Then $x_0 = -c$, $x_m = c$ and $x_k - x_{k-1} = 2c/m < \varepsilon/2$ for $k = 1, \ldots, m$.

By Proposition 11, for each $k \in \{1, \ldots, m\}$ for the interval $(x_{k-1}; x_k)$ there is some interval $[a_k; b_k]$ such that $a_k, b_k \in \mathbb{Q}$, $a_k < b_k$ and $\mu(f^{-1}[a_k; b_k] \cap L) <$

ε/m. Define $b_0 := -c$, $a_{m+1} := c$ and $I_k := (b_k; a_{k+1})$ for $k = 0, \dots, m$. Then $a_{k+1} - b_k < \varepsilon$ and

$$\begin{aligned}\mu(L) - \varepsilon &= \mu(f^{-1}(-c,c) \cap L) - \varepsilon \\ &= \sum_{k=0}^{m} \mu(f^{-1}[I_k] \cap L) + \sum_{k=1}^{m} \mu(f^{-1}[a_i; b_i] \cap L) - \varepsilon \\ &< \sum_{k=0}^{m} \mu(f^{-1}[I_k] \cap L) + m\frac{\varepsilon}{m} - \varepsilon \\ &= \sum_{k=0}^{m} \mu(f^{-1}[I_k] \cap L) \quad \le \ \mu(L)\,. \end{aligned}$$

□(Prop. 12)

Proposition 13. *For $\varepsilon > 0$ and $m > 4c/\varepsilon$ let $I_0, \dots, I_m \subseteq (-c; c)$ be the intervals from Proposition 12. Let $V_k := f^{-1}[I_k] \cap L$ and let $d_0, \dots, d_m$ be rational numbers such that*

$$(\forall k)\ d_k < \mu(V_k) \quad \textit{and} \quad \sum_{k=0}^{m} \mu(V_k) - \sum_{k=0}^{m} d_k < \varepsilon\,. \tag{12}$$

Then

$$\left| \int f\, d\mu - \sum_{k=0}^{m} \ \inf(I_k)\, d_k \right| \le (\mu(L) + 4c)\,\varepsilon.$$

Proof: Let $J := (-c, c) \setminus \bigcup_k I_k$ and $B := f^{-1}[J] \cap L$. Then $\mu(B) < \varepsilon$ by Proposition 12 and

$$\begin{aligned} &\left| \int f\, d\mu - \sum_{k=0}^{m} \ \inf(I_k)\, d_k \right| \quad = \quad \left| \int_L f\, d\mu - \sum_{k=0}^{m} \ \inf(I_k)\, d_k \right| \\ &\le \left| \int_L f\, d\mu - \sum_{k=0}^{m} \ \inf(I_k)\, \mu(V_k) \right| + \left| \sum_{k=0}^{m} \ \inf(I_k)\, \mu(V_k) - \sum_{k=0}^{m} \ \inf(I_k)\, d_k \right| \\ &\le \left| \int_L f\, d\mu - \sum_{k=0}^{m} \ \inf(I_k)\, \mu(V_k) \right| + c \sum_{k=0}^{m} (\mu(V_k) - d_k) \\ &\le \left| \int_L f\, d\mu - \sum_{k=0}^{m} \ \inf(I_k)\, \mu(V_k) - \inf(J)\, \mu(B) \right| + |\inf(J)|\, \mu(B) + c\varepsilon \\ &\le \left| \sum_{k=0}^{m} (\sup(I_k) - \inf(I_k))\, \mu(V_k) + (\sup(J) - \inf(J))\, \mu(B) \right| \\ &\qquad + |\inf(J)|\, \mu(B) + c\varepsilon \quad (\text{by} \quad (11)) \\ &\le \left| \sum_{k=0}^{m} \varepsilon\, \mu(V_k) \right| + (\sup(J) - \inf(J))\, \mu(B) \quad + 2c\varepsilon \quad (\text{length}(I_k) < \varepsilon) \\ &\le \varepsilon\mu(L) + 4c\varepsilon \quad = \quad (\mu(L) + 4c)\,\varepsilon\,. \end{aligned}$$

□(Prop. 13)

We explain how $\int f\,d\mu$ can be computed from f and μ with respect to the representations δ_K and $\delta_{\mathcal{M}}$. Suppose $\delta_K\langle w,p\rangle = f$ and $\delta_{\mathcal{M}}\langle q,r_0,s_0,r_1,s_1,\ldots\rangle = \mu$ according to (10) and Definition 9. For the open real intervals with rational end points we consider some standard notation ν'. It suffices to compute from f, μ and $n\in\mathbb{N}$ some $y_n\in\mathbb{Q}$ such that $|\int f\,d\mu - y_n| \le 2^{-n}$.

(A1) Find some open set $L\in\tau$ and its measure $\mu(L)$ such that $\mathrm{supp}(f)\subseteq L$.

(A2) Find some $c\in\mathbb{N}$ such that $\mathrm{range}(f)\subseteq(-c,c)$.

(A3) Find some $l\in\mathbb{N}$ such that $(\mu(L)+4c)2^{-l} < 2^{-n}$ and let $m := 4c\cdot 2^l + 1$.

(A4) Find rational numbers $d_0,\ldots,d_m$ and pairwise disjoint rational subintervals $I_0,\ldots,I_m$ of $(-c;c)$ of length $< 2^{-l}$ such that

$$(\forall k)\ d_k < \mu(f^{-1}[I_k]\cap L) \ \text{ and } \ \sum_{k=0}^m \mu(f^{-1}[I_k]\cap L) - \sum_{k=0}^m d_k < 2^{-l}\,.$$

(A5) Let $y_n := \sum_{j=0}^m \ \inf(I_j)\,d_j$ and $y := \lim_n y_n$.

By Proposition 13, $|\int_L f\,d\mu - y_n| \le (\mu(L)+4c)2^{-l} \le 2^{-n}$, therefore, $\int f\,d\mu = \lim_n y_n$.

We show that in this way, $\int f\,d\mu$ can actually be computed from f and μ.

(A1) Applying using (5) and (8) we can find some i such that $\overline{\nu^*(w)}\subseteq\theta(r_i)$. For $L := \theta(r_i)$, $\mathrm{supp}(f)\subseteq L$ and $\mu(L) = \rho(s_i)$.

(A2) By (5) a κ-name of $\overline{\nu^*(w)}$ can be computed.

The function $(f,K)\mapsto f[K]$ is $([\delta\to\rho],\kappa,\kappa_{\mathbb{R}})$-computable ([GW05], $\kappa_{\mathbb{R}}$ is the cover representation of the compact subsets of $\mathbb{R}$ from Definition 3), and and the multifunction $M \rightrightarrows c$ such that $c\in\mathbb{N}$ and $M\subseteq(-c;c)$ is $(\kappa_{\mathbb{R}},\nu_{\mathbb{N}})$-computable ([Wei00]). Therefore, we can compute some $c\in\mathbb{N}$ such that $f[\overline{\nu^*(w)}]\subseteq(-c;c)$. We are finished since $\mathrm{range}(f)\subseteq(\{0\}\cup f[\overline{\nu^*(w)}]\subseteq(-c;c)$.

(A3) Numbers l and m can be computed from $\mu(L)$ and c.

(A4) By Proposition 12, there are pairwise disjoint rational subintervals $I_0,\ldots,I_m$ of $(-c;c)$ of length $< 2^{-l}$ such that $\mu(L) - 2^{-l} < \sum_{k=0}^m \mu(f^{-1}[I_k]\cap L) \le \mu(L)$. Then there are rational numbers $d_k < \mu(f^{-1}[I_k]\cap L)$ such that

$$\mu(L) - 2^{-l} < \sum_{k=0}^m d_k < \sum_{k=0}^m \mu(f^{-1}[I_k]\cap L) \le \mu(L)\,.$$

For these intervals and numbers, $\sum_{k=0}^m \mu(f^{-1}[I_k]\cap L) - \sum_{k=0}^m d_k < 2^{-l}$. For finding such rational intervals I_k and rational numbers d_k it suffices to find intervals and numbers such that $\mu(L) - 2^{-l} < \sum_{k=0}^m d_k$ and $d_k < \mu(f^{-1}[I_k]\cap L)$ for all $k = 0,1,\ldots,m$. This search can be done computably in the input data, if we know that the set $\{(x,b)\mid x<b\}$ is $(\rho,\nu_{\mathbb{Q}})$-r.e. (this is well-known) and that the set $\{(a,I,L')\mid a<\mu(f^{-1}[I]\cap L')\}$ is $(\nu_{\mathbb{Q}},\nu',\theta)$-r.e. in the input data.

The function $(f,I)\mapsto f^{-1}[I]$ is computable [GW05]. By (3) and (A1), $(f,I,L')\mapsto f^{-1}[I]\cap L'$ is computable. Since the measure μ is regular, for rational a and open U, $a<\mu(U)$ iff $a<\mu(K)$ and $K\subseteq U$ for some compact set K.

By Lemma 5, $K \subseteq U$ iff $K \subseteq V \subseteq \overline{V} \subseteq U$ for some $V \in \beta^*$. Therefore, $a < \mu(U)$ iff $a < \mu(V)$ and $\overline{V} \subseteq U$ for some $V \in \beta^*$. Since $a < \mu(V)$ for $a \in \mathbb{Q}$ and $V \in \beta^*$ is r.e. in the input data (Definition 9) and $\overline{V} \subseteq U$ for $V \in \beta^*$ and $U \in \tau$ is r.e. by (5,8), $a < \mu(U)$ is r.e. in the input data. Therefore, $a < \mu(f^{-1}[I] \cap L')\}$ is $(\nu_{\mathbb{Q}}, \nu', \theta)$-r.e. in the input data.

(A5) From the numbers d_j and intervals I_j the sequence $(y_n)_n$ and its limit $\int f \, d\mu$ can be computed.

Therefore, $(\mu, f) \mapsto \int f \, d\mu$ is $(\delta_{\mathcal{M}}, \delta_K, \rho)$-computable, and hence by (2), the operator $T : \mu \mapsto I$ is $(\delta_{\mathcal{M}}, [\delta_K \to \rho])$-computable.

By the classical Riesz Theorem, Theorem 1, for every positive linear functional I there is some measure μ such that $I(f) = \int f \, d\mu$. Therefore, by the above result every positive linear operator $I : \mathcal{K}(X) \to \mathbb{R}$ is in the range of $[\delta_K \to \rho]$. This proves Lemma 7.

Finally we show that the operator $S : I \mapsto \mu$ is computable. Suppose $[\delta_K \to \rho](t) = I$. We must compute the sequences $q, r_0, s_0, r_1, s_1, \ldots \in \Sigma^\omega$ from Definition 9 for a name of the measure μ.

First we show how to compute a sequence q which must be a list of all $\langle u, v \rangle$ such that $\nu_{\mathbb{Q}}(u) < \mu(\nu^*(v))$.

By Definition 8.2, $\mu(\nu^*(v)) = \sup\{\mu(K) \mid K \text{ compact}, \;\; K \subseteq \nu^*(v)\}$. Suppose, $K \subseteq \nu^*(v)$. By Lemma 5.1 there is some $W \in \beta^*$ such that $K \subseteq W \subseteq \overline{W} \subseteq \nu^*(v)$. Therefore,

$$a < \mu(\nu^*(v)) \iff (\exists w) \, \overline{\nu^*(w)} \subseteq \nu^*(v) \;\text{ and }\; a < \mu(\overline{\nu^*(w)}) \, .$$

By Lemma 5.2 and (4,5) there is a computable function g such that for $f := \delta_K \langle v, g(w, v) \rangle$,

$$\chi_{\overline{\nu^*(w)}} \prec f \prec \chi_{\nu^*(v)} \;\text{ if }\; \overline{\nu^*(w)} \subseteq \nu^*(v) \, ,$$

therefore, by integration and monotonicity of the integral

$$\mu\overline{\nu^*(w)} \leq \int f \, d\mu \leq \mu\nu^*(v) \;\text{ if }\; \overline{\nu^*(w)} \subseteq \nu^*(v) \, .$$

Since $\int f \, d\mu = I(\delta_K \langle v, g(w, v) \rangle$,

$$a < \mu(\nu^*(v)) \iff (\exists w) a < I(\delta_K \langle v, g(w, v) \rangle) \, .$$

Therefore, from a $[\delta_K \to \rho]$-name of I we can compute a list of all $\langle u, v \rangle$ such that $\nu_{\mathbb{Q}}(u) < \mu(\nu^*(v))$.

It remains to compute a sequence $(r_i, s_i)_i$ such that 3. and 4. in Definition 9 hold true.

Proposition 14. *By means of I from every $W \in \beta^*$ we can compute sets $Y, Z \in \beta^*$ and a number $m \in \mathbb{N}$ such that $\overline{W} \subseteq Y \subseteq \overline{Y} \subseteq Z$ and $\mu(Z) - \mu(\overline{Y}) \leq m$.*

Proof: By (5), $W \to \overline{W}$ is (ν^*, κ)-computable. By the definition of the representation κ, Definition 3.4, from $\overline{W}$ we can find some $U \in \beta^*$ such that $\overline{W} \subseteq U$. Applying Lemma 5.1 twice we can compute $Y, Z \in \beta^*$ such that $\overline{W} \subseteq Y \subseteq \overline{Y} \subseteq Z \subseteq \overline{Z} \subseteq U$. By Lemma 5.2 we can compute functions $f_1, f_2 \in \mathcal{K}(\mathbb{R})$ such that $\overline{W} \prec f_1 \prec Y \subseteq \overline{Y} \subseteq Z \subseteq \overline{Z} \prec f_2 \prec Y \subseteq U$, therefore, $f_1 \le \chi_{\overline{Y}} \le \chi_Z \le f_2$. By integration, $I(f_1) \le \mu(\overline{Y}) \le \mu(Z) \le I(f_2)$. Determine m such that $m > I(f_2) - I(f_1)$. $\square$(Prop. 14)

Proposition 15. *By means of I for all $W, U \in \beta^*$ and $a \in \mathbb{Q}$ such that $\overline{W} \subseteq U$ and $\mu(U) - \mu(\overline{W}) < a$ we can compute sets $Y, Z \in \beta^*$ such that $\overline{W} \subseteq Y \subseteq \overline{Y} \subseteq Z \subseteq \overline{Z} \subseteq U$ and $\mu(Z) - \mu(\overline{Y}) < a/2$.*

Proof: By Lemma 5 we can compute sets $Y_1, Y_2, Z_1, Z_2 \in \beta^*$ and functions $f_1, f_2 \in \mathcal{K}(\mathbb{R})$ such that

$$\overline{W} \prec f_1 \prec Y_1 \subseteq \overline{Y}_1 \subseteq Z_1 \subseteq \overline{Z}_1 \prec f_2 \prec Y_2 \subseteq \overline{Y}_2 \subseteq Z_2 \subseteq \overline{Z}_2 \prec f_3 \prec U\,,$$

hence

$$\chi_{\overline{W}} \le f_1 \le \chi_{\overline{Y}_1} \le \chi_{Z_1} \le f_2 \le \chi_{\overline{Y}_2} \le \chi_{Z_2} \le f_3 \le \chi_U\,.$$

By integration we obtain

$$\mu(\overline{W}) \le I(f_1) \le \mu(\overline{Y}_1) \le \mu(Z_1) \le I(f_2) \le \mu(\overline{Y}_2) \le \mu(Z_2) \le I(f_3) \le \mu(U)\,.$$

Since $\mu(U) - \mu(\overline{W}) < a$, $I(f_2) - I(f_1) < a/2$ or $I(f_3) - I(f_2) < a/2$. Since we know also upper bounds of the $\mathrm{supp}(f_i)$ we know also δ_K-names of these functions. Therefore, $I(f_1), I(f_2)$ and $I(f_3)$ can be computed by means of the given I, and we can computably verify one of the cases "$I(f_2) - I(f_1) < a/2$" and "$I(f_3) - I(f_2) < a/2$". In the first case choose $Y := Y_1$ and $Z := Z_1$, in the second case $Y := Y_2$ and $Z := Z_2$. $\square$(Prop. 15)

Proposition 16. *By means of I for every $W \in \beta^*$ we can compute an open set $V \in \tau$ and a real number x such that $\overline{W} \subseteq V$ and $\mu(V) = x$.*

Proof: By Proposition 14 from W we compute sets $Y_0, Z_0 \in \beta^*$ and a number $m \in \mathbb{N}$ such that

$$\overline{W} \subseteq Y_0 \subseteq \overline{Y}_0 \subseteq Z_0 \subseteq \overline{Z}_0 \quad\text{and}\quad \mu(Z_0) - \mu(\overline{Y}_0) < m\,.$$

By Proposition 15 for $j = 1, 2, \ldots$ we compute $Y_j, Z_j \in \beta^*$ such that

$$\overline{Y}_{j-1} \subseteq Y_j \subseteq \overline{Y}_j \subseteq Z_j \subseteq \overline{Z}_j \subseteq Z_{j-1} \quad\text{and}\quad \mu(Z_j) - \mu(\overline{Y}_j) < m \cdot 2^{-j}$$

and by Lemma 5 simultaneously functions $f_0, f_1, \ldots \in \mathcal{K}(\mathbb{R})$ such that

$$\overline{Y}_j \prec f_j \prec Z_j\,.$$

Then $\mu(\overline{Y}_j) \leq I(f_j) \leq \mu(Z_j)$. and for all $i > j$, $|I(f_j) - I(f_i)| \leq m \cdot 2^{-j}$. Therefore, $x := \lim_j I(f_j)$ can be computed by means of I. Let $V := \bigcup_j Y_J$. Then V can be computed and $\mu(V) = \sup_j \mu(Y_j) = \sup_j \mu(\overline{Y}_j) = \lim I(f_j) = x$.
□(Prop. 16).

Now it is easy to compute sequences r_i, s_i for the name of μ according to Definition 9. Let w_i be the ith element of Σ^* in a natural numbering. For computing r_i and s_i apply Proposition 16 to $W := \nu^*(w_i)$ and find r_i such that $V = \theta(r_i)$ and s_i such that $x = \rho(s_i)$. □

References

[Bau74] Bauer, H.:. *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*. Walter de Gruyter, Berlin, 1974.

[BY06] Brattka, V., Yoshikawa, A.: Towards computability of elliptic boundary value problems in variational formulation. *Journal of Complexity*, 22(6):858–880, 2006.

[Coh80] Cohn, D.L.: *Measure Theory*. Birkhäuser, Boston, 1980.

[Con90] Conway, J.B. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 1990.

[Eng89] Engelking, R.: *General Topology*, volume 6 of *Sigma series in pure mathematics*. Heldermann, Berlin, 1989.

[GP65] Goffman, C., Pedrick, G.: *First Course in Functional Analysis*. Prentice-Hall, Englewood Cliffs, 1965.

[GSW07] Grubba, T., Schröder, M., Weihrauch, K.: Computable metrization. *Mathematical Logic Quarterly*, 53(4–5):381–395, 2007.

[GW05] Grubba, T., Weihrauch, K.: A computable version of Dini's theorem for topological spaces. In Pinar Yolum, Tunga Güngör, Fikret Gürgen, and Can Özturan, editors, *Computer and Information Sciences - ISCIS 2005*, volume 3733 of *Lecture Notes in Computer Science*, pages 927–936, Berlin, 2005. Springer. 20th International Symposium, ISCIS, Istanbul, Turkey, October 2005.

[GX07] Grubba, T., Xu, Y.: Computability on subsets of locally compact spaces. In Jin-Yi Cai, S.Barry Cooper and Hong Zhu, editors, *Theory and Applications of Models of Computation 2007*, volume 4484 of Lecture Notes in Computer Science , pp.100–114, Berlin, 2007, Springer. 4th International Conference, TAMC 2007, Shanghai,China, May 2007.

[Her99] Hertling, P.: A real number structure that is effectively categorical. *Mathematical Logic Quarterly*, 45(2):147–182, 1999.

[KW85] Kreitz, C., Weihrauch, K.: Theory of representations. *Theoretical Computer Science*, 38:35–53, 1985.

[LW07] Lu, H., Weihrauch, K.: Computable Riesz representation for the dual of $C[0;1]$. *Mathematical Logic Quarterly*, 53(4–5):415–430, 2007.

[Mül99] Müller, N.T.: Computability on random variables. *Theoretical Computer Science*, 219:287–299, 1999.

[Ped89] Pedersen, G.K.: *Analysis Now*, volume 118 of *Graduate Texts in Mathematics*. Springer, New York, 1989.

[Rud74] Rudin, W.: *Real and Complex Analysis*. McGraw-Hill, New York, 2nd edition, 1974.

[Sch02] Schröder, M.: Extended admissibility. *Theoretical Computer Science*, 284(2):519–538, 2002.

[Sch03] Schröder, M. Admissible representations for continuous computations. Informatik Berichte 299, FernUniversität Hagen, Hagen, April 2003. Dissertation.

[Sch07] Schröder, M. Admissible representations for probability measures. *Mathematical Logic Quarterly*, 53(4–5):431–445, 2007.

[SS06] Schröder, M., Simpson, A.: Representing probability measures using probabilistic processes. *Journal of Complexity*, 22(6):768–782, 2006.

[WD05] Wu, Y., Ding, D.: Computability of measurable sets via effective metrics. *Mathematical Logic Quarterly*, 51(6):543–559, 2005.

[WD06] Wu, Y., Ding, D.: Computability of measurable sets via effective topologies. *Archive for Mathematical Logic*, 45(3):365–379, 2006.

[Wei99] Weihrauch, K.: Computability on the probability measures on the Borel sets of the unit interval. *Theoretical Computer Science*, 219:421–437, 1999.

[Wei00] Weihrauch, K.: *Computable Analysis*. Springer, Berlin, 2000.

[Wei08] Weihrauch, K.: Multi-functions on multi-represented sets are closed under flowchart programming. This special issue of JUCS.

[Wu05] Wu, Y.: Computable measure theory. PhD dissertation, in preparation, 2005.

[WW06] Wu, Y., Weihrauch, K.: A computable version of the Daniell-Stone theorem on integration and linear functionals. *Theoretical Computer Science*, 359(1–3):28–42, 2006.