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Computability of Topological Pressure for Sofic Shifts with Applications in Statistical Physics

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Abstract: The topological pressure of dynamical systems theory is examined from a computability theoretic point of view. It is shown that for sofic shift dynamical systems, the topological pressure is a computable function. This result is applied to a certain class of one dimensional spin systems in statistical physics. As a consequence, the specific free energy of these spin systems is computable. Finally, phase transitions of these systems are considered. It turns out that the critical temperature is recursively approximable.
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Category: F.2.1, G.1.2, J.2

1 Introduction

The topological pressure [Bowen 1975] is a quantity which belongs to one of the main concepts in the thermodynamic formalism [Ruelle 1978]. The thermodynamic formalism itself is a generalization of the concepts of statistical physics to the area of mathematical dynamical systems theory – to ergodic theory to be more concrete [Walters 1982]. The topological pressure on the other hand can be seen as a generalization of the topological entropy. The topological entropy is, besides the metric entropy, one of the main quantities in ergodic theory. This is because the topological entropy is an invariant with respect to topological conjugacy, that is if two dynamical systems are equivalent from a topological point of view, then they have the same topological entropy. The same holds for the metric entropy from a measure theoretic point of view. The topological pressure finally is related to equilibrium measures for dynamical systems.

In this paper, computability aspects of the topological pressure are investigated. Since the topological pressure is a generalization of the topological entropy, as already mentioned, the following elaboration is a continuation of [Spandl 2007] where computability aspects of the topological entropy were considered. While in [Spandl 2007], it was possible to show the computability of the topological entropy for types of shift dynamical systems far beyond the sofic shifts, the computability of the topological pressure is shown here only for sofic shifts. However, even for shifts of finite type, a subclass of the sofic shifts, the concept is applicable to a wide class of so called one dimensional spin systems, mainly investigated in theoretical statistical physics. Hence the computability aspects of the topological pressure can be transferred directly to computability aspects of these models in statistical physics. Naturally, computability theoretic aspects are of interest in that area, since there is a broad community of physicists studying these systems by Monte Carlo simulations [Landau and Binder 2000].

The paper is organized as follows. In the next section, basic notation and definitions are given. The topological pressure for general dynamical systems is introduced as well as its form for shift dynamical systems as a specialization. In Section 3, the transfer operator for shift dynamical systems is defined and the connection between the transfer operator and the topological pressure (for shift dynamical systems) is established. It turns out that for shifts of finite type, the transfer operator can be represented as a matrix of nonnegative reals. So, Perron-Frobenius theory is applicable showing that the topological pressure is the logarithm of the corresponding Perron value. This allows a computability theoretic investigation of the problem. It turns out that for shifts of finite type, the topological pressure is computable. At last, the computation of the topological pressure of a sofic shift can be reduced to the computation of the topological pressure of a shift of finite type by modifying the input function. Finally in Section 4 connections to statistical physics are drawn, more precisely to spin systems on a one dimensional lattice with arbitrary interaction (also long range interactions). It is shown that the specific free energy is computable for any kind of computable interaction function with computable modulus of convergence. Furthermore, phase transitions are examined which occur for long range interactions. It turns out that the critical temperature is recursively approximable in the sense of [Zheng and Weihrauch 2001].

2 Definition of the Topological Pressure

Let $\mathcal{A}$ denote an alphabet, that is a nonempty finite set. Then $\mathcal{A}^*$ denotes the set of all finite words over $\mathcal{A}$ and $\mathcal{A}^\omega$ the set of all infinite sequences over $\mathcal{A}$, that is $\mathcal{A}^\omega = \{f : f : \mathbb{N} \to \mathcal{A}\}$. The set of all bi-infinite sequences over $\mathcal{A}$ is denoted by $\mathcal{A}^\mathbb{Z}$. Occasionally $\mathcal{A}^\omega$ are denoted as the set of one-sided sequences, in symbols also $\mathcal{A}^\mathbb{N}$. The empty word is denoted by λ. For every $w \in \mathcal{A}^*$, $|w|$ denotes the length of w. The concatenation of words u and v of $\mathcal{A}^*$ is denoted by uv. For any word $w \in \mathcal{A}^*$ and $i, j \in \mathbb{N}$, $w_{[i,j]} := w_i \dots w_n$ is the subword of w with $n := \min(j, |w| - 1)$ if $i \le j$ and $i < |w|$, as well as $w_{[i,j]} := \lambda$ otherwise. If $p \in \mathcal{A}^\mathbb{Z}$ ($p \in \mathcal{A}^\mathbb{N}$) and $i, j \in \mathbb{Z}$ ($i, j \in \mathbb{N}$), then $p_{[i,j]} \in \mathcal{A}^*$ denotes the word $p_{[i,j]} = p_i p_{i+1} \dots p_j$ if $i \le j$ and $p_{[i,j]} = \lambda$ if $i > j$. $\mathcal{A}^\mathbb{Z}$ and $\mathcal{A}^\mathbb{N}$ are considered as metric spaces where the standard Cantor metric is assumed.

A partial function is denoted by $f :\subseteq X \to Y$, a total function by $f : X \to Y$. A (partial) function $f :\subseteq Z_1 \times \dots \times Z_k \to Z_0$ with $Z_0, Z_1 \dots Z_k \in$

$\{\mathcal{A}^*, \mathcal{A}^\omega\}$ is called *computable*, if it is computable by a Type-2 Turing machine [Weihrauch 2000]. A function $f :\subseteq X \to Y$ is called computable, if it has a computable *realization* $g :\subseteq Z_1 \to Z_0$, $Z_0, Z_1 \in \{\mathcal{A}^*, \mathcal{A}^\omega\}$, in some standard naming systems. To be more precise, $f \circ \gamma = \delta \circ g$ holds on the domain of $f \circ \gamma$ where $\gamma :\subseteq Z_1 \to X$ and $\delta :\subseteq Z_0 \to Y$ are naming systems. All concepts concerning Type-2 computability used here are in the sense of [Weihrauch 2000]. If a computable function $\mathcal{A}^\mathbb{Z} \to \mathbb{R}$ is considered, the naming system $B : \mathcal{A}^\omega \to \mathcal{A}^\mathbb{Z}$ with

$$B(x)_i := \begin{cases} x_{2i} & \text{if } i \geq 0 \\ x_{2|i|-1} & \text{if } i < 0 \end{cases}$$

for all $x \in \mathcal{A}^\omega$, $i \in \mathbb{Z}$ is used.

For the definition of the topological pressure, the approach presented in [Walters 1982] is followed. Let (X, d) be a compact metric space and $T : X \to X$ a continuous map. Then the pair (X, T) is called a (discrete-time) *dynamical system*. Furthermore consider the class $C(X)$ of all real valued, continuous functions $f : X \to \mathbb{R}$. For any $n \geq 1$, define a new metric d_n on X by $d_n(x, y) := \max_{0 \leq i \leq n-1} d(T^i(x), T^i(y))$ for all $x, y \in X$.

Definition 1. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset $F \subseteq X$ is said to *(n, ε)-span X with respect to T* if for any $x \in X$ there is some $y \in F$ such that $d_n(x, y) \leq \varepsilon$ holds.

Definition 2. The *topological pressure* of (X, T) is defined as the map $P(T, .) : C(X) \to \mathbb{R} \cup \{\infty\}$, given by

$$P(T, f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, f, \varepsilon)$$

for all $f \in C(X)$ with

$$P_n(T, f, \varepsilon) := \inf\{\sum_{x \in F} \exp(\sum_{i=0}^{n-1} f(T^i(x))) : F \text{ is a } (n, \varepsilon)\text{-spanning set for } X\}.$$

Here, the natural logarithm is considered. The *topological entropy* is the pressure with the null function, that is the constant function with value zero (sometimes, in the definition of the topological entropy the logarithm of base 2 is used instead of the natural logarithm).

In the following, special classes of dynamical systems are considered: shifts over a finite alphabet $\mathcal{A}$. Let $\sigma : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ be the (continuous) shift map defined by $\sigma(x)_i := x_{i+1}$ for all $x \in \mathcal{A}^\mathbb{Z}$. Then, for a closed, shift invariant subset $X \subseteq \mathcal{A}^\mathbb{Z}$, $\sigma(X) = X$, the pair (X, σ_X) is called a *shift dynamical system* or a *shift* and X is called a *shift space*. Here, $\sigma_X : X \to X$ is the restriction of σ to X. Occasionally, the subscript X in σ_X is omitted as well as also X is called a shift. For a shift space X, $\mathcal{A}^*(X)$ denotes the set of all words in $\mathcal{A}^*$ occurring as a subword in some element in X. $\mathcal{A}^*(X)$ is called the *language of* X. The complement of the language of a shift space is a *set of forbidden words*. To be more precise, a set of forbidden words of some shift space X is any subset $\mathcal{F} \subseteq \mathcal{A}^*$ such that X is the result of deleting all elements of $\mathcal{A}^{\mathbb{Z}}$ having some word in $\mathcal{F}$ as subword. If a shift space has a finite set of forbidden words, it is called a *shift of finite type*. A shift of finite type is called M*-step*, $M \geq 0$, if there is a corresponding set of forbidden words where the maximal length of the words in this set is $M+1$.

Let X and Y be two shift spaces. A function $\Phi : X \to Y$ is called a *homomorphism* if it is continuous and commutes with the shift map: $\Phi \circ \sigma_X = \sigma_Y \circ \Phi$. A homomorphism $\Phi : X \to Y$ is called a *factor map* if it is onto. In that case, Y is called a *factor* of X. Finally, a shift is called a *sofic shift* if it is the factor of a shift of finite type.

Proposition 3. *Let (X, σ) be a shift over some alphabet $\mathcal{A}$. Then the topological pressure is given by*

$$P(f) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{u \in \mathcal{A}^n(X)} \exp(S_n(f,u))$$

with

$$S_n(f,u) := \inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\}$$

for all $n \in \mathbb{N}$, $u \in \mathcal{A}^n(X)$, where $\mathcal{A}^n(X)$ is the set of all words of length n occurring in elements of X.

This fact is standard to some extent. However, since the tools to prove the proposition are used in the course of this work, the complete proof is shown here.

Let X be a shift space and $f \in C(X)$. Then for any $k \in \mathbb{N}$ set $\mathrm{Var}_k(f) := \sup\{|f(x) - f(y)| : x, y \in X,\ d(x,y) < 2^{-k}\}$.

Lemma 4. *Let (X, σ) be a shift and $f \in C(X)$. Then,*

1. *for any $u \in \mathcal{A}^*(X)$ of length n,*

$$\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x_{[0,n-1]} = u\} \leq \inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x_{[0,n-1]} = u\} + 2\sum_{i=0}^{n-1} \mathrm{Var}_i(f)$$

   holds and

2. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{Var}_i(f) = 0$.

Proof. First, it is $\mathrm{Var}_k(f) = \sup\{|f(x) - f(y)| : x, y \in X,\ x_{[-k,k]} = y_{[-k,k]}\}$. Thus, for any $u \in \mathcal{A}^*(X)$ of length n,

$$\begin{aligned}
&\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x_{[0,n-1]} = u\} \\
&\leq \sum_{i=0}^{n-1} \sup\{f(\sigma^i(x)) : x_{[0,n-1]} = u\} \\
&\leq \sum_{i=0}^{n-1} (\inf\{f(\sigma^i(x)) : x_{[0,n-1]} = u\} + \\
&\quad \sup\{|f(x) - f(y)| : x, y \in X,\ x_{[-i,n-1-i]} = y_{[-i,n-1-i]}\}) \\
&\leq \inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x_{[0,n-1]} = u\} + 2 \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \mathrm{Var}_i(f) \\
&\leq \inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x_{[0,n-1]} = u\} + 2 \sum_{i=0}^{n-1} \mathrm{Var}_i(f)
\end{aligned}$$

holds. Here, $\lfloor x \rfloor$ for some $x \in \mathbb{R}$ is the greatest integer $n \in \mathbb{Z}$ with $n \leq x$.

Since f is continuous and X compact, f is uniformly continuous. Therefore, $\lim_{n\to\infty} \mathrm{Var}_n(f) = 0$ holds. Then, also $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{Var}_i(f) = 0$ holds. □

Lemma 5. $c_n := \log \sum_{u \in \mathcal{A}^n(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\})$ *is subadditive, that is for all* $n, m \in \mathbb{N}$, $c_{n+m} \leq c_n + c_m$ *holds.*

Therefore, the proper or improper limit $\lim\limits_{n\to\infty} \frac{c_n}{n}$ *exists and equals* $\inf_n\{\frac{c_n}{n}\}$.

Proof. For all $n, m \in \mathbb{N}$,

$$\begin{aligned}
c_{n+m} &= \log \sum_{u \in \mathcal{A}^{n+m}(X)} \exp(\sup\{\sum_{i=0}^{n+m-1} f(\sigma^i(x)) : x \in X,\ x_{[0,n+m-1]} = u\}) \\
&\leq \log \sum_{u \in \mathcal{A}^n(X)} \sum_{v \in \mathcal{A}^m(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) + \sum_{i=0}^{m-1} f(\sigma^i(y)) : \\
&\quad x_{[0,n-1]} = u,\ y_{[0,m-1]} = v\}) \\
&\leq \log \sum_{u \in \mathcal{A}^n(X)} \sum_{v \in \mathcal{A}^m(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x_{[0,n-1]} = u\} + \\
&\quad \sup\{\sum_{i=0}^{m-1} f(\sigma^i(x)) : x_{[0,m-1]} = v\})
\end{aligned}$$

$$= \log \sum_{u \in \mathcal{A}^n(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\}) +$$

$$\log \sum_{u \in \mathcal{A}^m(X)} \exp(\sup\{\sum_{i=0}^{m-1} f(\sigma^i(x)) : x \in X,\ x_{[0,m-1]} = u\})$$

$$= c_n + c_m$$

holds.

The second statement is a standard argument (see e.g., Lemma 4.1.7 in [Lind and Marcus 1995]). □

As a direct consequence of Lemma 4 and Lemma 5, it holds the following

Corollary 6. *For all $f \in C(X)$,*

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{u \in \mathcal{A}^n(X)} \exp(\inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\}) =$$

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{u \in \mathcal{A}^n(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\})$$

holds.

Finally, the proof of Proposition 3 is given.

*Proof.*Proof of Proposition 3. First, for all $n, k \geq 1$, $x \in X$,

$$\{y : d_n(x,y) \leq 2^{-k}\} = \{y : y_{[-k+1,k+n-2]} = x_{[-k+1,k+n-2]}\}$$

holds. This gives

$$\inf\{\sum_{x \in F} \exp(\sum_{i=0}^{n-1} f(\sigma^i(x))) : F \text{ is a } (n, 2^{-k})\text{-spanning set for } X\} =$$

$$\inf\{\sum_{u \in \mathcal{A}^{2k+n-2}(X)} \exp(\sum_{i=0}^{n-1} f(\sigma^i(x))) : x \in X,\ x_{[-k+1,k+n-2]} = u\}$$

Therefore,

$$\inf\{\sum_{x \in F} \exp(\sum_{i=0}^{n-1} f(\sigma^i(x))) : F \text{ is a } (n, 2^{-k})\text{-spanning set for } X\}$$

$$\geq \sum_{u \in \mathcal{A}^{2k+n-2}(X)} \exp(\inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x \in X,\ x_{[-k+1,k+n-2]} = u\})$$

$$\geq \sum_{u \in \mathcal{A}^n(X)} \exp(\inf\{\sum_{i=0}^{n-1} f(\sigma^i(x))) : x \in X,\ x_{[0,n-1]} = u\})$$

and, on the other hand,

$$\begin{aligned}
&\inf\{\sum_{x\in F}\exp(\sum_{i=0}^{n-1} f(\sigma^i(x))) : F \text{ is a } (n,2^{-k})\text{-spanning set for } X\}\\
&\leq \sum_{u\in\mathcal{A}^{2k+n-2}(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x\in X,\ x_{[-k+1,k+n-2]}=u\})\\
&\leq \frac{|\mathcal{A}^{2k+n-2}(X)|}{|\mathcal{A}^n(X)|}\sum_{u\in\mathcal{A}^n(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x))) : x\in X,\ x_{[0,n-1]}=u\})\\
&\leq |\mathcal{A}^{2k-2}(X)|\sum_{u\in\mathcal{A}^n(X)} \exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x))) : x\in X,\ x_{[0,n-1]}=u\})
\end{aligned}$$

holds. Here, the inequality $|\mathcal{A}^{2k+n-2}(X)| \leq |\mathcal{A}^{2k-2}(X)|\cdot|\mathcal{A}^n(X)|$ is used. So, the estimation

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log\sum_{u\in\mathcal{A}^n(X)}\exp(\inf\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x\in X,\ x_{[0,n-1]}=u\}) \quad \leq P(f)\leq\\
&\limsup_{n\to\infty}\frac{1}{n}\log\sum_{u\in\mathcal{A}^n(X)}\exp(\sup\{\sum_{i=0}^{n-1} f(\sigma^i(x)) : x\in X,\ x_{[0,n-1]}=u\})
\end{aligned}$$

is derived. The assertion follows now with Corollary 6. □

3 Properties of the Topological Pressure

In this section, the transfer operator is introduced and its relation to the topological pressure is presented. This gives directly a method for computing the topological pressure.

Now, one-sided shifts are considered. Let $\mathcal{A}$ be an alphabet. A *one-sided shift* over $\mathcal{A}$ is a subset $X^+ \subseteq \mathcal{A}^{\mathbb{N}}$ such that there is a shift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ with $X^+ = \{x \in \mathcal{A}^{\mathbb{N}} : \exists y \in X\ x = y_{[0,\infty)}\}$. In X^+, there also is a shift map $\sigma : X^+ \to X^+$ given by $\sigma(x)_i := x_{i+1}$. The one-sided shift map is continuous, but not injective and therefore no homeomorphism. Furthermore, X^+ is closed in the Cantor topology of $\mathcal{A}^{\mathbb{N}}$ and is shift invariant, that is $\sigma(X^+) = X^+$. So, the pair (X^+,σ) forms a dynamical system. On the class of all continuous functions over X^+, $C(X^+)$, the topological pressure of (X^+,σ) is defined analogously to the two-sided case.

On the one-sided shifts, for any continuous function $\varphi \in C(X^+)$ the so called transfer operator can be defined [Bowen 1975, Ruelle 1978].

Definition 7. Let $\varphi \in C(X^+)$ be given. The *transfer operator* with respect to the one-sided shift (X^+, σ), $\mathcal{L}_\varphi : C(X^+) \to C(X^+)$, is given by

$$(\mathcal{L}_\varphi f)(x) := \sum_{y \in \sigma^{-1}(x)} e^{\varphi(y)} f(y)$$

for all $f \in C(X^+)$.

Definition 8. Let $n \in \mathbb{N}$ and X^+ be a one-sided shift space. The subclass $C_n(X^+) \subseteq C(X^+)$ of the class of all continuous functions over X^+ with finite domain of dependence of length $n+1$ is defined as follows. If $f \in C_n(X^+)$, then the value of $f(x)$ for some $x \in X^+$ depends only on $x_{[0,n]}$. In other words, $f(x) = f(y)$ for all $x, y \in X^+$ with $x_{[0,n]} = y_{[0,n]}$.

Proposition 9. *Let X^+ be a one-sided M-step shift of finite type for some $M \in \mathbb{N}$. Then for any $n \geq M$ and $\varphi \in C_n(X^+)$, $\mathcal{L}_\varphi f \in C_{n-1}(X^+)$ for all $f \in C_{n-1}(X^+)$ and $\mathcal{L}_\varphi f \in C_{m-1}(X^+)$ for all $f \in C_m(X^+)$ with $m \geq n$.*

Proof. Let $\varphi \in C_n(X^+)$, $n \geq M$ and $m \geq n-1$. Consider some function $f \in C_m(X^+)$. Define a function $f_{m+1} : \mathcal{A}^{m+1} \to \mathbb{R}$ by $f_{m+1}(u) := f(ux)$ for some $x \in X^+$ such that $ux \in X^+$ if $u \in \mathcal{A}^{m+1}(X)$ and $f_{m+1}(u) := 0$ if $u \notin \mathcal{A}^{m+1}(X)$. Additionally, define $\varphi_{n+1} : \mathcal{A}^{n+1} \to \mathbb{R}$ analogously for φ. Next let $\chi_{X^+} : \mathcal{A}^{\mathbb{N}} \to \{0,1\}$ be the characteristic function of X^+ and $\chi_{\mathcal{A}^*(X)} : \mathcal{A}^* \to \{0,1\}$ the characteristic function of $\mathcal{A}^*(X)$. Then, since X^+ is a shift of finite type, according to Theorem 2.1.8 in [Lind and Marcus 1995], $\chi_{X^+}(uvx) = \chi_{\mathcal{A}^*(X)}(uv)$ for all $u, v \in \mathcal{A}^*(X)$, $x \in X^+$ such that $uvx \in X^+$ and $|v| \geq M$. Then for all $u \in \mathcal{A}^{m+1}(X)$, $x \in X^+$ such that $ux \in X^+$,

$$\begin{aligned}
(\mathcal{L}_\varphi f)(ux) &= \sum_{y \in \sigma^{-1}(ux)} e^{\varphi(y)} f(y) \\
&= \sum_{a \in \mathcal{A}} \chi_{X^+}(aux) e^{\varphi_{n+1}(au_{[0,n-1]})} f_{m+1}(au_{[0,m-1]}) \\
&= \sum_{a \in \mathcal{A}} \chi_{\mathcal{A}^*(X)}(au_{[0,M-1]}) e^{\varphi_{n+1}(au_{[0,n-1]})} f_{m+1}(au_{[0,m-1]})
\end{aligned}$$

is independent of x. Furthermore, $\mathcal{L}_\varphi f \in C_{m-1}(X^+)$ for all $m \geq n$ and $\mathcal{L}_\varphi f \in C_{n-1}(X^+)$ for $m = n-1$. $\square$

So, for the eigenvalue problem of the transfer operator, the following corollary is a direct consequence.

Corollary 10. *Let X^+ be a one-sided M-step shift of finite type for some $M \in \mathbb{N}$ and $\varphi \in C_n(X^+)$ for some $n \geq M$. Let $f \in C(X^+)$ be an eigenfunction of the transfer operator $\mathcal{L}_\varphi$. Then either $f \in C_{n-1}(X^+)$ or $f \notin C_m(X^+)$ for all $m \in \mathbb{N}$.*

The functions in $C_{n-1}(X^+)$ can be interpreted as vectors in $\mathbb{R}^{|\mathcal{A}|^n}$. Then the transfer operator can be written as an $|\mathcal{A}|^n$ by $|\mathcal{A}|^n$ *transfer matrix* $T = (T_{u,v})$ with $(\mathcal{L}_\varphi f_n)(v) = \sum_{u\in\mathcal{A}^n} f_n(u)T_{u,v}$. The transfer matrix has the explicit form $T_{u,v} = \delta_{v_{[0,n-2]},u_{[1,n-1]}}\chi_{\mathcal{A}^*(X)}(uv_{n-1})e^{\varphi_{n+1}(uv_{n-1})}$, where $\delta_{.}$ is Kronecker's delta.

So, the eigenvalue problem of the transfer operator is in part reduced to the eigenvalue problem of $Tf_n = \lambda f_n$ of the transfer matrix T. Since T is a non-negative matrix, the Perron-Frobenius theory is applicable [Gantmacher 1959, Seneta 1981]. In the following, it will be shown that the transfer matrix completely determines the topological pressure of (X,σ) if X is a shift of finite type.

Definition 11. Let $n \in \mathbb{N}$ and X be a two-sided shift space. The subclass $C_n(X) \subseteq C(X)$ of the class of all continuous functions over X with finite domain of dependence of length $2n+1$ is defined as follows. If $f \in C_n(X)$, then the value of $f(x)$ for any $x \in X$ depends only on $x_{[-n,n]}$. In other words, $f(x) = f(y)$ for all $x, y \in X$ with $x_{[-n,n]} = y_{[-n,n]}$.

Lemma 12. *Let X^+ be a one-sided shift space and $\varphi \in C(X^+)$. Then for $m \geq 1$, the m-th iterate of the transfer operator $\mathcal{L}_\varphi : C(X^+) \to C(X^+)$, $\mathcal{L}_\varphi^m$, is given by*

$$(\mathcal{L}_\varphi^m f)(x) = \sum_{y\in\sigma^{-m}(x)} \exp(\sum_{i=0}^{m-1} \varphi(\sigma^i(y)))f(y). \tag{1}$$

Furthermore, let X^+ be M-step and $\varphi \in C_n(X^+)$ for some $n \geq M$, $n \geq 1$. Then for $m \geq n$, the m-th iterate of the transfer matrix T, corresponding to $\mathcal{L}_\varphi$ has the form

$$T_{v,u}^m = \sum_{w\in\mathcal{A}^{m-n}} \chi_{\mathcal{A}^*(X)}(vwu)\exp(\sum_{i=0}^{m-1} \varphi_{n+1}((vwu)_{[i,i+n]})) \tag{2}$$

Proof. Equation (1) is easily seen by induction over m. Then if $\varphi \in C_n(X^+)$, $n \geq M$, $m \geq n$ and $f \in C_{n-1}(X^+)$, analogously to the proof of Proposition 9 it can be shown that for all $u \in \mathcal{A}^n(X)$,

$$\begin{aligned}(\mathcal{L}_\varphi^m f)_n(u) &= \sum_{v\in\mathcal{A}^m} \chi_{\mathcal{A}^*(X)}(vu)\exp(\sum_{i=0}^{m-1} \varphi_{n+1}((vu)_{[i,i+n]}))f_n(v_{[0,n-1]}) \\ &= \sum_{v\in\mathcal{A}^n}\sum_{w\in\mathcal{A}^{m-n}} \chi_{\mathcal{A}^*(X)}(vwu)\exp(\sum_{i=0}^{m-1} \varphi_{n+1}((vwu)_{[i,i+n]}))f_n(v)\end{aligned}$$

holds. Hence, $T_{v,u}^m = \sum_{w\in\mathcal{A}^{m-n}} \chi_{\mathcal{A}^*(X)}(vwu)\exp(\sum_{i=0}^{m-1} \varphi_{n+1}((vwu)_{[i,i+n]}))$ follows. □

Theorem 13 is a generalization of Theorem B in [Gurevich 1984] if the corresponding transfer matrix is not irreducible.

Theorem 13. *Let X be an M-step shift of finite type, $n \geq M$ and $\varphi \in C_n(X)$. Then the topological pressure of φ, $P(\varphi)$, is given by $P(\varphi) = \log \lambda$ where λ is the Perron value of the transfer matrix corresponding to $\mathcal{L}_{\varphi^+}$. Here, $\varphi^+ \in C_n(X^+)$ is some function with $\varphi^+(x) = \varphi(y)$ for all $x \in X^+$ and some $y \in X$ with $x = y_{[0,\infty)}$.*

Proof. Since X is an M-step shift of finite type and $\varphi^+ \in C_n(X^+)$ with $n \geq M$, consider the eigenvalue problem of the corresponding transfer matrix T. First assume that T is irreducible. Then there is an eigenfunction $\psi \in C_{n-1}(X^+)$ of $\mathcal{L}_{\varphi^+}$ corresponding to the Perron vector of T, with eigenvalue $\lambda > 0$ corresponding to the Perron value of T such that ψ is strictly positive: $\max(\psi) > 0$ and $\min(\psi) > 0$.

The eigenvalue problem directly gives

$$\sum_{v \in \mathcal{A}^n} T^m_{v,u} \psi_n(v) = \lambda^m \psi_n(u)$$

for all $m \geq 1$ and hence

$$\sum_{v,u \in \mathcal{A}^n} T^m_{v,u} \psi_n(v) = \lambda^m \sum_{u \in \mathcal{A}^n} \psi_n(u). \tag{3}$$

Set $\psi^+ := \max(\psi) > 0$ and $\psi^- := \min(\psi) > 0$. Then according to the previous lemma,

$$\begin{aligned}
&\lambda^m \psi^- |\mathcal{A}^n(X)| \\
&\leq \psi^+ \sum_{v \in \mathcal{A}^n(X)} \sum_{w \in \mathcal{A}^m(X)} \chi_{\mathcal{A}^*(X)}(vw) \exp(\sup\{\sum_{i=0}^{m-1} \varphi^+(\sigma^i(x)) : x_{[0,n+m-1]} = vw\}) \\
&\leq \psi^+ |\mathcal{A}^n(X)| \sum_{w \in \mathcal{A}^m(X)} \exp(\sup\{\sum_{i=0}^{m-1} \varphi^+(\sigma^i(x)) : x_{[0,m-1]} = w\})
\end{aligned}$$

holds, and on the other hand

$$\begin{aligned}
&\lambda^m \psi^+ |\mathcal{A}^n(X)| \\
&\geq \psi^- \sum_{v \in \mathcal{A}^n(X)} \sum_{w \in \mathcal{A}^m(X)} \chi_{\mathcal{A}^*(X)}(vw) \exp(\inf\{\sum_{i=0}^{m-1} \varphi^+(\sigma^i(x)) : x_{[0,n+m-1]} = vw\}) \\
&\geq \psi^- \sum_{w \in \mathcal{A}^m(X)} \exp(\inf\{\sum_{i=0}^{m-1} \varphi^+(\sigma^i(x)) : x_{[0,m-1]} = w\}).
\end{aligned}$$

By definition of φ^+, one has

$$\sup\{\sum_{i=0}^{m-1} \varphi^+(\sigma^i(x)) : x_{[0,m-1]} = w\} \leq \sup\{\sum_{i=0}^{m-1} \varphi(\sigma^i(x)) : x_{[0,m-1]} = w\}$$

and

$$\inf\{\sum_{i=0}^{m-1}\varphi^+(\sigma^i(x)) : x_{[0,m-1]} = w\} \geq \inf\{\sum_{i=0}^{m-1}\varphi(\sigma^i(x)) : x_{[0,m-1]} = w\}$$

for all $w \in \mathcal{A}^m(X)$, $m \in \mathbb{N}$.

So,

$$P(\varphi) \leq \lim_{m\to\infty}\frac{1}{m}\log\lambda^m \leq P(\varphi) + \lim_{m\to\infty}\frac{2}{m}\sum_{i=0}^{m-1}\mathrm{Var}_i(\varphi)$$

follows by Lemma 4 and finally $P(\varphi) = \log\lambda$.

Eventually assume that the transfer matrix T is not irreducible. Then T can be decomposed in $K > 0$ irreducible components each having an eigenfunction $\psi_i \in C_{n_i-1}(X^+)$ with eigenvalue λ_i corresponding to the Perron vectors and Perron values of the submatrices. Then Equation (3) has to be replaced by

$$\sum_{i=1}^{K}\sum_{v,u\in\mathcal{A}^{n_i}} T^m_{v,u}\psi_{n_i}(v) = \sum_{i=1}^{K}\lambda_i^m \sum_{u\in\mathcal{A}^{n_i}}\psi_{n_i}(u)$$

and the further analysis is done as above. In that case, the estimation

$$P(\varphi) \leq \lim_{m\to\infty}\frac{1}{m}\log\sum_{i=1}^{K}\lambda_i^m \leq P(\varphi)$$

is derived. So, $P(\varphi) = \log\lambda$ follows where $\lambda = \max_i \lambda_i$ is the Perron value of the transfer matrix T. □

In order to generalize this result to sofic shifts, some more tools are needed.

Lemma 14. *Let X and Y be shift spaces and $\Phi : X \to Y$ a homomorphism. If $\varphi \in C_n(Y)$ holds for some function $\varphi : Y \to \mathbb{R}$, then there exists some $m \geq n$ with $\varphi \circ \Phi \in C_m(X)$.*

Proof. This is a consequence of Theorem 6.2.9 in [Lind and Marcus 1995]. □

Lemma 15. *Let X be a sofic shift. Then there is some shift of finite type Y, a factor map $\Phi : Y \to X$ and a constant $C \geq 1$ such that $|\Phi^{-1}(\{x\})| \leq C$ holds for all $x \in X$.*

Proof. See Theorem 3.3.2 and Example 8.1.6 in [Lind and Marcus 1995]. □

It turns out that the pressure of a sofic shift X can be calculated via some shift of finite type Y such that X is a factor of Y.

Theorem 16. *Let X be a sofic shift and $\varphi \in C_n(X)$. Furthermore let Y be a shift of finite type and $\Phi : Y \to X$ a factor map according to Lemma 15. Then the topological pressure of φ with respect to X, $P_X(\varphi)$, is given by $P_X(\varphi) = P_Y(\varphi\circ\Phi)$ where $P_Y(.)$ is the topological pressure with respect to Y.*

Proof. For $\varphi \in C_n(X)$ one has

$$\begin{aligned}
P_X(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\mathcal{A}^n(X)} \exp(\inf\{\sum_{i=0}^{n-1} \varphi(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\}) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\mathcal{A}^n(X)} \exp(\inf\{\sum_{i=0}^{n-1} \varphi(\sigma^i(\Phi(y))) : y \in Y,\ \Phi(y)_{[0,n-1]} = u\}) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\mathcal{A}^n(X)} \exp(\inf\{\sum_{i=0}^{n-1} (\varphi\circ\Phi)(\sigma^i(y)) : y \in Y, \\
&\quad \Phi(y)_{[0,n-1]} = u\}) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\mathcal{A}^n(X)} \inf\{\exp(\inf\{\sum_{i=0}^{n-1} (\varphi\circ\Phi)(\sigma^i(y)) : y \in Y, \\
&\quad y_{[0,n-1]} = v\}) :\ v \in \mathcal{A}^n(Y),\ \exists\, y \in Y\ y_{[0,n-1]} = v \wedge \Phi(y)_{[0,n-1]} = u\}.
\end{aligned}$$

Therefore, the following bounds can be derived. First,

$$\begin{aligned}
P_X(\varphi) &\le \lim_{n\to\infty} \frac{1}{n} \log \sum_{v\in\mathcal{A}^n(Y)} \exp(\inf\{\sum_{i=0}^{n-1} (\varphi\circ\Phi)(\sigma^i(y)) : y \in Y,\ y_{[0,n-1]} = v\}) \\
&= P_Y(\varphi\circ\Phi)
\end{aligned}$$

and second

$$\begin{aligned}
P_X(\varphi) &\ge \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\mathcal{A}^n(X)} \frac{1}{N(u)} \sum_{v\in\mathcal{A}^n(Y)} \delta(u,v) \exp(\inf\{\sum_{i=0}^{n-1} (\varphi\circ\Phi)(\sigma^i(y)) : \\
&\quad y \in Y,\ y_{[0,n-1]} = v\} - Var(\varphi\circ\Phi)) \\
&\ge \lim_{n\to\infty} \frac{1}{n} \log(\frac{1}{C} \exp(-Var(\varphi\circ\Phi)) \sum_{v\in\mathcal{A}^n(Y)} \exp(\inf\{\sum_{i=0}^{n-1} (\varphi\circ\Phi)(\sigma^i(y)) : \\
&\quad y \in Y,\ y_{[0,n-1]} = v\})) \\
&= P_Y(\varphi\circ\Phi).
\end{aligned}$$

Here $\delta(u,v)$ is given by

$$\delta(u,v) := \begin{cases} 1 & \text{if } \exists\, y \in Y\ y_{[0,n-1]} = v \wedge \Phi(y)_{[0,n-1]} = u \\ 0 & \text{otherwise} \end{cases}$$

and $N(u) := \sum_{v\in\mathcal{A}^n(Y)} \delta(u,v)$ for all $u \in \mathcal{A}^n(X)$, $v \in \mathcal{A}^n(Y)$. According to Lemma 15, $N(u) \le C$ holds for some constant $C \ge 1$ for all $n \in \mathbb{N}$ and $u \in \mathcal{A}^n(X)$. Furthermore, $Var(f) := \sup\{|f(x) - f(y)| : x, y \in Y\}$ is finite for $f \in C_n(Y)$. $\square$

Proposition 17. *Let $\varphi \in C(X)$ and X be a shift space. Then there are functions $\varphi_n^-, \varphi_n^+ \in C_n(X)$ for all $n \in \mathbb{N}$ with $P(\varphi_n^-) \leq P(\varphi_{n+1}^-) \leq P(\varphi) \leq P(\varphi_{n+1}^+) \leq P(\varphi_n^+)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} P(\varphi_n^-) = \lim_{n\to\infty} P(\varphi_n^+) = P(\varphi)$.*

Proof. Define $\varphi_n^-, \varphi_n^+ \in C_n(X)$ by $\varphi_n^-(x) := \inf\{\varphi(y) : y \in X,\ y_{[-n,n]} = x_{[-n,n]}\}$ and $\varphi_n^+(x) := \sup\{\varphi(y) : y \in X,\ y_{[-n,n]} = x_{[-n,n]}\}$. Since φ is continuous and X compact, both functions are well defined. For all $n \in \mathbb{N}$, $\varphi_n^- \leq \varphi \leq \varphi_n^+$ holds, as well as $\varphi_n^- \leq \varphi_{n+1}^-$ and $\varphi_{n+1}^+ \leq \varphi_n^+$. Since the topological pressure is monotone (see [Walters 1982], Theorem 9.7(ii)), it holds $P(\varphi_n^-) \leq P(\varphi_{n+1}^-) \leq P(\varphi) \leq P(\varphi_{n+1}^+) \leq P(\varphi_n^+)$ for all $n \in \mathbb{N}$.

Next, since φ is continuous, that is for any $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ for all $x, y \in X$ with $x_{[-n,n]} = y_{[-n,n]}$, for any $\varepsilon > 0$ there is some $n \in \mathbb{N}$ with $||\varphi_n^- - \varphi|| < \varepsilon$. Here, $||.||$ denotes the supremum norm. So, $\lim_{n\to\infty} ||\varphi^- - \varphi|| = 0$. The same holds for φ^+ instead of φ^-. Since $|P(\psi) - P(\varphi)| \leq ||\psi - \varphi||$ (see [Walters 1982], Theorem 9.7(iv)), $\lim_{n\to\infty} P(\varphi_n^-) = \lim_{n\to\infty} P(\varphi_n^+) = P(\varphi)$ follows. $\square$

The section is closed now with the following computability result:

Theorem 18. *Let X be a sofic shift. Then the topological pressure $P : C(X) \to \mathbb{R}$ is a computable function when $C(X)$ is represented by some effective standard naming system.*

Proof. Let Y be a shift of finite type according to Lemma 15. Then, by Theorem 16 and Lemma 14, the computation of the pressure with respect to X can be computed via the pressure with respect to Y by modifying the input function. Theorem 6.2.9 in [Lind and Marcus 1995] guarantees, that the corresponding factor map $\Phi : Y \to X$ has a finite description and hence is computable. Furthermore, since Y has a description by a finite set of words, the characteristic function $\chi_{\mathcal{A}^*(Y)}$ of $\mathcal{A}^*(Y)$ also is computable. Then, according to Lemma 5.2.6 in [Weihrauch 2000], there is a computable function assigning each $u \in \mathcal{A}^{2n+1}(Y)$ for any $n \in \mathbb{N}$ and a name of $\varphi \in C(Y)$ the value $\sup\{\varphi(y) : y \in Y,\ y_{[-n,n]} = u\}$ since $\{y : y \in Y,\ y_{[-n,n]} = u\}$ is compact. The same holds for $\inf\{\varphi(y) : y \in Y,\ y_{[-n,n]} = u\}$ instead of $\sup\{\varphi(y) : y \in Y,\ y_{[-n,n]} = u\}$.

Let Y be M-step, $\varphi \in C(Y)$ given and $\varphi_n^-, \varphi_n^+ \in C_n(Y)$ for all $n \in \mathbb{N}$ according to Proposition 17. By Theorem 13, if $n \geq M$, $P(\varphi_n^-) = \log \lambda_n^-$ and $P(\varphi_n^+) = \log \lambda_n^+$ where λ_n^-, λ_n^+ are the Perron values of the transfer matrices corresponding to $\mathcal{L}_{\varphi_n^-}$ and $\mathcal{L}_{\varphi_n^+}$. λ_n^- and λ_n^+ are computable. To see this, first observe that there exists a computable function assigning to a standard name of φ a standard name of the corresponding transfer matrix. This is clear since Y has a finite description and due to the definition of the matrix. Next, Equation (2) gives $T_{v,u}^n > 0$ iff $\chi_{\mathcal{A}^*(Y)}(vu) = 1$. Hence there is an algorithm computing the irreducible components of the transfer matrix, as shown in [Spandl 2007].

Furthermore, the characteristic polynomial of a matrix is computable with respect to a name of the matrix as input. According to the computable version of the fundamental theorem of algebra [Weihrauch 2000], a list of the roots of the characteristic polynomial is computable. Since the Perron value is the maximum of the absolute values of these roots and since the absolute value of a real as well as the maximum of a finite set of reals is computable, also the Perron value of each irreducible component of the transfer matrix is computable. The pressure finally is determined by the maximum of the Perron values of the irreducible components, which is clearly computable. Therefore, the pressures $P(\varphi_n^-)$ and $P(\varphi_n^-)$ are computable uniformly in φ and n for $n \geq M$. Then by Proposition 17, the assertion follows directly. □

Remark. The above theorem holds even uniformly in X. For naming systems of shift spaces, especially for sofic shifts spaces, see [Spandl 2007].

If no restriction on the type of shift is made, the above theorem does not hold in the following sense: There is no Type-2 machine computing a name of the value of the topological pressure of some continuous function over some shift space, where the input is a name of that shift space and a name of the function. In [Spandl 2007] it was shown that a corresponding machine does not exist computing the topological entropy. Since the topological entropy is the topological pressure for the null function, there is no such machine computing the topological pressure.

4 Applications to Statistical Physics

First consider an example. Let the shift space X be $\{-1,1\}^{\mathbb{Z}}$, the full shift over two symbols. The function $\varphi \in C(X)$ for which the topological pressure will be determined has the form

$$\varphi(x) = \sum_{i=1}^{\infty} a_i x_0 (x_i + x_{-i}) + b x_0. \tag{4}$$

Furthermore, $(a_n)_n$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} a_n$ exists and $b \in \mathbb{R}$.

Then, for all $n \in \mathbb{N}$, $u \in \{-1,1\}^n$,

$$\begin{aligned} S_n(\varphi, u) &= \inf\{\sum_{i=0}^{n-1} \varphi(\sigma^i(x)) : x \in X,\ x_{[0,n-1]} = u\} \\ &= \inf\{\sum_{i=0}^{n-1} (\sum_{j=1}^{\infty} a_j u_i (x_{i+j} + x_{i-j}) + b u_i) : x \in X,\ x_{[0,n-1]} = u\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} u_i\Big(\sum_{j=1}^{n-1-i} a_j u_{i+j} + \sum_{j=1}^{i} a_j u_{i-j} + b\Big) + \\
&\quad \inf\Big\{\sum_{i=0}^{n-1} u_i\Big(\sum_{j=n-i}^{\infty} a_j x_{i+j} + \sum_{j=i+1}^{\infty} a_j x_{i-j}\Big) : x_i \in \{-1,1\}\ \forall i \geq n,\ i < 0\Big\} \\
&= \Phi_n(u) + d_n(u)
\end{aligned}
$$

with the so called potential term

$$
\begin{aligned}
\Phi_n(u) &:= \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} a_{j-i} u_i u_j + \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} a_{i-j} u_i u_j + b \sum_{i=0}^{n-1} u_i && (5) \\
&= \sum_{\substack{i,j=0 \\ i \neq j}}^{n-1} a_{|i-j|} u_i u_j + b \sum_{i=0}^{n-1} u_i && (6)
\end{aligned}
$$

and a correction term given by

$$
d_n(u) := \inf\Big\{\sum_{i=0}^{n-1} u_i\Big(\sum_{j=n}^{\infty} a_{j-i} x_j + \sum_{j=1}^{\infty} a_{i+j} x_{-j}\Big) : x_i \in \{-1,1\}\ \forall i \geq n,\ i < 0\Big\}.
$$

The correction term can be estimated by

$$
|d_n(u)| \leq 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{\infty} a_j = \sum_{i=0}^{n-1} c_i
$$

with $c_n := 2\sum_{i=n+1}^{\infty} a_i$. Since $\lim_{n\to\infty} c_n = 0$, also $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} c_i = 0$ holds. Therefore,

$$
\begin{aligned}
P(\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\{-1,1\}^n} \exp(\Phi_n(u) + d_n(u)) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\{-1,1\}^n} \exp\Big(\Phi_n(u) + \sum_{i=0}^{n-1} c_i\Big) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\{-1,1\}^n} \exp(\Phi_n(u))
\end{aligned}
$$

On the other hand,

$$
P(\varphi) \geq \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\{-1,1\}^n} \exp(\Phi_n(u) - |d_n(u)|)
$$

holds, which finally gives

$$
P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{u\in\{-1,1\}^n} \exp(\Phi_n(u)). \tag{7}
$$

Now the connection to statistical mechanics can be drawn. For more details on the concepts of statistical mechanics see [Ellis 1985, Ruelle 1969]. Consider the model of a ferromagnet in one dimension. Let Λ be a finite interval of the lattice $\mathbb{Z}$. On each site of the finite lattice Λ, a magnetic dipole is placed. The magnetic moment of each dipole is assumed to have two configurations: it can point up (value 1) or down (value -1). So the considered state space is $S = \{-1, 1\}$. The whole magnet can be described by a configuration $s \in S^\Lambda$ where $s_i \in S$ gives the magnetic moment of the dipole at site $i \in \Lambda$. The *Hamiltonian* of the system, that is the interaction energy, is now given by

$$H_{\Lambda,B}(s) = -\frac{1}{2} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} J(|i-j|) s_i s_j - B \sum_{i \in \Lambda} s_i, \tag{8}$$

where $J : \mathbb{N} \to [0, \infty)$ is the dipole-dipole interaction function depending only on the distance of the two dipoles and $B \in \mathbb{R}$ is the external magnetic field. In thermodynamic equilibrium at temperature $T > 0$, a specific state s of the magnet has probability

$$\pi_{\Lambda,\beta,B}(s) = \frac{1}{Z_{\Lambda,\beta,B}} e^{-\beta H_{\Lambda,B}(s)},$$

where $\beta = 1/T$ is the inverse temperature and $Z_{\Lambda,\beta,B}$ is the normalization factor given by

$$Z_{\Lambda,\beta,B} = \sum_{s \in S^\Lambda} e^{-\beta H_{\Lambda,B}(s)},$$

which is called the *partition function*. $\pi_{\Lambda,\beta,B}$ defines a probability measure on $(S^\Lambda, \mathcal{B})$, where $\mathcal{B}$ is the set of all subsets of S^Λ. $\pi_{\Lambda,\beta,B}$ is called a *Gibbs state* or an *equilibrium state*. Closely related to the partition function is the *free energy*, given by

$$F_\Lambda(\beta, B) = -\frac{1}{\beta} \log Z_{\Lambda,\beta,B}. \tag{9}$$

for all $\beta > 0$, $B \in \mathbb{R}$. The free energy is the fundamental quantity of the system because is allows the determination of all physical quantities of the system which are of interest.

Now consider the limiting behavior as Λ tends to $\mathbb{Z}$, called the *thermodynamic limit*. It will be denoted by $\Lambda \uparrow \mathbb{Z}$. Since the Hamiltonian, and also some other quantities, becomes undefined in the thermodynamic limit, only quantities per site can be investigated. The *specific free energy* of the infinite magnet is defined by

$$f(\beta, B) = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} F_\Lambda(\beta, B). \tag{10}$$

If the interaction J is summable, it can be shown that the limit exists (see [Ellis 1985], Appendix D.1). The specific free energy will be crucial for the development of phase transitions, the main theme in the rest of this paper. But first let's look at $f(\beta, B)$ from the viewpoint of computability theory.

The connection between the specific free energy and the topological pressure in the above example is now evident (see also the treatment in [Mayer 1991]). Just compare the Equations (5) and (8) as well as the Equations (7) and (9), (10). For the second comparison note that, since the Hamiltonian is translationally invariant, the limit $\Lambda \to \mathbb{Z}$ can be replaced by the limit $\Lambda \to \mathbb{N}$ and $\Lambda \subseteq \mathbb{N}$. Therefore:

$$f(\beta, B) = -\frac{1}{\beta} P(\varphi)$$

where φ is according to Equation (4) with $a_i = \frac{\beta}{2} J(i)$ for all $i \geq 1$ and $b = \beta B$. According to Theorem 18, there is the

Theorem 19. *Let $J : \mathbb{N} \to [0, \infty)$ be a summable and computable interaction function with a computable modulus of convergence. Then the specific free energy $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ corresponding to the Hamiltonian (8) is a computable function.*

The concept of phase transitions is now introduced via the so called spontaneous magnetization. The *magnetic moment* of the system is defined by

$$M_\Lambda(\beta, B) = \sum_{s \in S^\Lambda} \sigma_\Lambda(s) \pi_{\Lambda,\beta,B}(s)$$

where $\sigma_\Lambda(s) = \sum_{i \in \Lambda} s_i$ is the total magnetic moment of the configuration $s \in S^\Lambda$. The *magnetization* is the magnetic moment per site in the infinite volume limit, defined by

$$m(\beta, B) = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} M_\Lambda(\beta, B).$$

Finally, the *magnetic susceptibility* is defined by

$$\chi(\beta, B) = \frac{\partial m(\beta, B)}{\partial B}.$$

If the interaction J is summable, the magnetization exists for all $\beta > 0$, $B \in \mathbb{R}$ and the magnetic susceptibility exists for all $\beta > 0$, $B \neq 0$ (see [Ellis 1985], Theorem IV.5.1, IV.5.2, IV.5.3 and Lemma V.7.4). Furthermore, the specific free energy, the magnetization and the magnetic susceptibility have the following properties:

(a.1) $f(\beta, B)$ is a concave and even function in B and two times continuously differentiable in B for $B \neq 0$.

(a.2) $m(\beta, B) = -\frac{\partial f(\beta,B)}{\partial B}$ for all $\beta > 0$, $B \neq 0$.

(a.3) $0 \leq m(\beta, B) \leq 1$ for all $\beta > 0$, $B \geq 0$.

(a.4) For $\beta > 0$ fixed, $m(\beta, B)$ is an increasing and concave function in $B \geq 0$ and for $B \geq 0$ fixed, $m(\beta, B)$ is an increasing function of $\beta > 0$.

Note that, according to the Items (a.1) to (a.4), $f(\beta, B)$ is decreasing in B for all $B \geq 0$, $m(\beta, B)$ is an odd function in B for all $B \in \mathbb{R}$ and $\chi(\beta, B)$ is an even, nonnegative function in B for all $\beta > 0$, $B \neq 0$.

According to the Items (a.1) and (a.2), continuity of the magnetization may break down for certain values of β only for $B = 0$. Then there are still the following properties

(b.1) The limits $m^+(\beta) := \lim_{B \to 0^+} m(\beta, B)$ and $m^-(\beta) := \lim_{B \to 0^-} m(\beta, B)$ exist for all $\beta > 0$.

(b.2) $m^+(\beta) = \frac{\partial f(\beta,0)}{\partial B^+}$ and $m^-(\beta) = \frac{\partial f(\beta,0)}{\partial B^-}$.

(b.3) $m^+(\beta) \geq m(\beta, 0) \geq 0$ for all $\beta > 0$ and m^+ is an increasing function.

(b.4) $m^-(\beta) = -m^+(\beta)$ for all $\beta > 0$.

Now let $\beta_c := \sup\{\beta > 0 : m^+(\beta) = 0\}$. First consider the case that $\beta_c = \infty$. Then for all $\beta > 0$, the magnetization $m(\beta, B)$ is continuous for all $B \in \mathbb{R}$. Second consider the case that β_c is finite. Then by the above items, only for $0 \leq \beta \leq \beta_c$, $m(\beta, B)$ is continuous for all $B \in \mathbb{R}$. Continuity fails for $\beta > \beta_c$ at $B = 0$ and $m^+(\beta) > 0$ follows. Then it is said that the systems shows a *spontaneous magnetization* at inverse temperature β_c and a *phase transition* occurs.

If the interaction function $J : \mathbb{N}\backslash\{0\} \to [0, \infty)$ has the form $J(n) = n^{-\alpha}$, $\alpha > 1$ is was shown that $\beta_c < \infty$ iff $\alpha \leq 2$ [Dyson 1969, Fröhlich and Spencer 1982].

It was already shown that the specific free energy is computable if J is summable and computable. The final question is now which of the above defined quantities are computable as well. Especially, is β_c computable, if it is finite? To answer these questions, some more tools are needed.

Lemma 20. *Let $f : (0, \infty) \to \mathbb{R}$ be a differentiable and computable function. If f is increasing and concave, then also the derivative $f' : (0, \infty) \to \mathbb{R}$ is computable. The same holds if f is increasing and convex, decreasing and concave or decreasing and convex instead of increasing and concave.*

Note that, if $f : (0, \infty) \to \mathbb{R}$ is differentiable and concave or convex, $f' : (0, \infty) \to \mathbb{R}$ is continuous.

Proof. Let $x \in (0, \infty)$ be given. Then there are numbers $x^+, x^- \in \mathbb{Q} \cap (0, \infty)$ with $x^- < x < x^+$ and x^+, x^- are computable uniformly in x. Consider now the sequences $(a_i^+)_i$ and $(a_i^-)_i$ of real numbers, defined by

$$a_i^+ := \frac{f(x) - f(x - (x - x^-)/(i+1))}{(x - x^-)/(i+1)}$$

and

$$a_i^- := \frac{f(x+(x^+-x)/(i+1)) - f(x)}{(x^+-x)/(i+1)}$$

for all $i \in \mathbb{N}$. Then $(a_i^+)_i$ is computable and decreasing, $(a_i^-)_i$ is computable and increasing. Furthermore $\lim_{i\to\infty} a_i^+ = \lim_{i\to\infty} a_i^- = f'(x)$. Hence $f'(x)$ is computable. The other cases are shown similarly. □

Now it is not too hard to show the following properties:

Proposition 21. *Let $J : \mathbb{N} \to [0,\infty)$ be a summable and computable interaction function with a computable modulus of convergence. Then the following properties hold.*

1. *The functions $m, \chi : (0,\infty) \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$ are computable.*
2. *The function $m^+ : (0,\infty) \to [0,1]$ defined in Item (b.1) is right-computable.*
3. *There exists a computable sequence $(b_n)_n$ of rational numbers with* $\lim_{n\to\infty} = \beta_c$.

Proof. Item 1 follows directly with Theorem 19, Lemma 20, the Items (a.1) to (a.4) and the definition of χ. Item 2 is a direct consequence of Item 1 and the fact that m is increasing in B for $B \geq 0$. Item 3: This is a direct consequence of Item 2. Fix some $\varepsilon > 0$. Then, by Item 2 and exhaustive search, there exists some computable, increasing and converging sequence $(b_n^{(\varepsilon)})_n$ of rational numbers with limit $b^{(\varepsilon)} := \lim_{n\to\infty} b_n^{(\varepsilon)} \geq \beta_c$ and $m^+(b^{(\varepsilon)}) = \varepsilon$. Now let $b : \mathbb{N}^2 \to \mathbb{R}$ be defined by $b(n,m) := b_m^{(2^{-n})}$. It is clear that b is a computable function and $\lim_{n\to\infty} b(n,n) = \beta_c$ holds. Hence, set $b_n := b(n,n)$ for all $n \in \mathbb{N}$. □

The above proposition shows that the *inverse critical temperature* β_c is only recursively approximable in the sense of [Zheng and Weihrauch 2001]. Indeed, it seems that, without any further information about the system, the inverse critical temperature is not computable. However, this result is in accordance with typical computer simulations of such systems since data obtained by Monte Carlo simulations also lack of any error bound.

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