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Bloch's Constant is Computable

Robert Rettinger
(FernUniversität Hagen, Germany
robert.rettinger@fernuni-hagen.de)

Abstract: We prove the computability of Bloch's constant by presenting the first algorithm approximating the constant up to arbitrary precision.
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1 Introduction

It is well known that any non-constant holomorphic function is open. Quantitative versions of this fact and with many respects surprising, is given by Bloch's theorem (see [Bloch 1929]), stating that for any $r > 0$ and any holomorphic function f defined on a disc $\mathbb{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ with $f'(z_0) \neq 0$ there exists a schlicht disc (see below) of radius $|f'(z_0)| \cdot r \cdot c$ inside the image $f(\mathbb{D}_r(z_0))$, where the constant c does not depend on f! Obviously c is bounded from above, thus the supremum, the so called Bloch constant β, exists. The best upper bound known for β,

$$\beta \leq \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)(1+\sqrt{3})^{1/2}}$$

found by Ahlfors and Grunsky [Ahlfors and Grunsky 1937], is at the same time conjectured to be the exact value of β. However the best lower bound known so far (quite recently found by Chen and Gauthier [Chen and Gauthier 1996]) is

$$\frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} < \beta.$$

Putting this in decimal representation gives

$$0.43321... < \beta \leq 0.47186...$$

i.e. all we know is the constant up to $4 \cdot 10^{-2}$.

In this paper we will give an algorithm to compute Bloch's constant β up to any precision in the sense that on input $\varepsilon > 0$ we can compute some rational number q with $|q - \beta| < \varepsilon$.

To define Bloch's constant and present our result more formally, we need a few more prerequisites:

As above, let $\mathbb{D}_\varepsilon(z_0)$ denote the open disc of radius ε with center z_0. To simplify notation we use $\mathbb{D}_\varepsilon := \mathbb{D}_\varepsilon(0)$ and $\mathbb{D} = \mathbb{D}_1$. Let in the sequel D be some domain, i.e. an connected and open subset of $\mathbb{C}$. A normed holomorphic function on D with $0 \in D$ is a holomorphic function f with $f(0) = 0$ and $f'(0) = 1$. The space of normed functions on D is denoted by $\mathcal{N}(D)$. Given $f \in \mathcal{N}(D)$ let $\beta_f(D, z_0)$ denote the supremum $r > 0$ so that the disc $\mathbb{D}_r(z_0)$ is covered by $f(D)$ and f^{-1} exists on $\mathbb{D}_r(z_0)$, i.e. there exists a function $f^{-1} : \mathbb{D}_r(z_0) \to D$ so that $f^{-1} \circ f$ is the identity function. These discs $\mathbb{D}_r(z_0)$ are called schlicht (with respect to D and f). Notice that in this case f^{-1} is uniquely determined and again holomorphic. Let $\beta_f(D)$ be the supremum of $\beta_f(D, z)$ for all $z \in f(D)$. Finally let $\beta(D)$ be the infimum of $\beta_f(D)$ for all $f \in \mathcal{N}(D)$. Then, by definition, we have $\beta = \beta(\mathbb{D})$.

Concerning computability of functions we use the Type-2-Turing machine, which is essentially a (classical) Turing machine where the input and output tapes can in addition contain infinite sequences of symbols. Thus such a Type-2-Turing machine (”Turing machine”, for short) computes a partial function $(\Sigma^*)^n \times (\Sigma^\omega)^m \to (\Sigma^*)^o \times (\Sigma^\omega)^p$ for suitable natural numbers m, n, o, p and a finite alphabet Σ. We will essentially use the case where either m or n and either o or p are 0. Furthermore we will assume that the symbol 0 belongs to Σ. Using representations, i.e. giving each element one or several names in Σ^ω or Σ^*, Type-2-Turing machines give natural notions of computability on a wide class of objects, such as $\mathbb{C}$, once we have fixed representations for these objects. To fix such a representation for complex numbers, let $\mathbb{Y}$ denote the set of dyadic numbers, i.e. numbers $y = 2^{-n} \cdot m$ where m and n are integers. By fixing some representation for $\mathbb{Y}$ (say by their binary representation) we get immediately the following representation for the complex numbers: A name for some $z \in \mathbb{C}$ is simply a sequence $((x_0, y_0), (x_1, y_1), ...)$ of pairs of dyadics $x_t, y_t \in \mathbb{Y}$ such that $|(x_t + iy_t) - z| < 2^{-t}$ for all $t \in \mathbb{N}$. To be more precise, a name of a complex number is an encoding of such a sequence, i.e. a word in Σ^ω. In a similar way one can also construct a representation ν of $\mathbb{C}^\omega$, which can be easily seen to be open w.r.t. the standard (product) topology on $\mathbb{C}^\omega$. For a more detailed discussion of representations see e.g. [Weihrauch 2000]. With these prerequisites, a (partial) function $f :\subseteq \mathbb{C} \to \mathbb{C}$ is called computable, iff there exists a Turing machine which for any name of an element $z \in \mathrm{dom}(f)$ computes a name for $f(z)$.

Beside computable functions we will also need computable multifunctions $f :\subseteq M \rightrightarrows N$, where $m \in M$ can have many different values $f(m)$. Given representations of M and N we call such a multifunction computable if there exists a computable realization as in the definition of computable functions (here, given the name of some m, the realization may output any name of any element of $f(m)$). This kind of relaxing the classical definition of functions can hughly simplify things in Type-2-theory, which is mainly to the fact that elements usually have many names. Asking for functions therefore means that an algorithm computing this function must compute the same value $f(m)$ given any name of the element m. In many circumstances, it is, however, sufficient to find values with certain properties (depending on m). To give an example, we will compute in Lemma 8 below approximations to the values β_g up to some precision ε, where a name of g is given. The exact approximations can vary for different names of g and the only thing we can guarantee is that any result we produce is indeed an approximation up to error ε. If we asked to have a unique value, the computational task would be unneccessarily complicated, and in some cases even impossible.

The main idea of our algorithm is to compute for several normed functions the corresponding β-values. Following the definition, it seems that we have to take the infimum for all normed functions, which could not be done in finitely many steps. Thus the decisive step will be to restrict the space of functions which we will have to consider in our algorithm. To this end we will first (Section 2) reduce the problem to a class of functions (β-bounding functions) with certain properties, which allow to apply compactness arguments later on. In Section 3 we will show, how to compute the β-value for a single function. Finally in Section 4 we will prove that Bloch's constant is computable.

2 The Set of β-bounding Functions

One problem we have to face in our proof of the computability of Bloch's constant is that the involved functions are not bounded. By restricting the domain, however, we can circumvent this problem quite easily. Thus the following lemma will come in handy later on. The upper bound is obvious, whereas the lower bound follows from the fact $\varepsilon\beta = \beta(\mathbb{D}_\varepsilon)$ (see e.g. [Conway 1978], Corollary XII.1.7).

Lemma 1. *Let ε with $0 < \varepsilon < 1$ and a domain D with $\mathbb{D}_\varepsilon \subseteq D \subseteq \mathbb{D}$ be given. Then*

$$\varepsilon\beta \leq \beta(D) \leq \beta.$$

Our definition of β-bounding functions will be based on the following lemma. (Notice that the choice 0.48 in the below lemma is somewhat arbitrary. We could have chosen any value larger than β.)

Lemma 2. *For any $f \in \mathcal{N}(\mathbb{D})$ with $\beta_f(\mathbb{D}) \leq 0.48$ we have*

$$|f(z)| \leq c_0 \int_0^{|z|} \frac{1}{1-t^2} dt \tag{1}$$

for $c_0 = \frac{4}{\sqrt{3}} 0.48$ and all $z \in \mathbb{D}$.

Proof: According to a theorem of Ahlfors (see e.g. [Remmert 1991]), any holomorphic function f in $\mathbb{D}$ fulfills the following property: If $|f'(z)|\cdot(1-|z|^2)\geq M$ for some $z\in\mathbb{D}$ then the image $f(\mathbb{D})$ contains schlicht discs of radius $\frac{\sqrt{3}}{4}M$. Taking $M=c_0=\frac{4}{\sqrt{3}}\cdot 0.48$ and taking integrals, we see that only functions for which (1) above holds can fulfill $\beta_f(\mathbb{D})\leq 0.48$.

□

To simplify things let $c_1 := 0.48\cdot\frac{4}{\sqrt{3}}$. Furthermore, for $\varepsilon\in]0;1[$, let $\rho(\varepsilon)=\int_0^{\varepsilon}\frac{1}{1-t^2}dt$. Notice that ρ is increasing and computable.

We define the set of β-bounding functions to be the set

$$\mathcal{T}=\{f\in\mathcal{N}(\mathbb{D})||f(z)|\leq c_1\cdot\rho(\varepsilon)\text{ for all }0<\varepsilon<1,|z|\leq\varepsilon\}$$

Thus the above lemma implies the

Corollary 3. *For each $\varepsilon>0$ there exist $f\in\mathcal{T}$ so that*

$$\beta_f(\mathbb{D})\leq\beta+\varepsilon.$$

The following simple application of Cauchy's formula allows us to replace functions in $\mathcal{T}$ by bounded families of coefficients (the coefficients of the corresponding Taylor series).

Lemma 4. *Let $f\in\mathcal{T}$ and ε, $0<\varepsilon<1$, be given. Then for the Taylor series $f(z)=\sum_{i=0}^{\infty}a_iz^i$ we have*

$$|a_i|\leq\frac{c_1\cdot\rho(\varepsilon)}{\varepsilon^i}$$

Proof: By Lemma 2 we have $|f(z)|\leq c_1\cdot\rho(\varepsilon)$ for all z with $|z|=\varepsilon$. In addition, by Cauchy formula, we have

$$|a_i|=|\frac{f^{(i)}(0)}{i!}|=\frac{1}{2\pi}|\int_{|z|=\varepsilon}\frac{f(t)}{(-t)^{i+1}}dt|\leq\frac{2\pi\varepsilon}{2\pi}\sup_{|z|=\varepsilon}\frac{|f(z)|}{\varepsilon^{i+1}}$$

□

Based on Lemma 4 we define a set A_ε of sequences $(a_0,a_1,...)$ of complex coefficients which essentially is a superset of the Taylor coefficients of the functions in $\mathcal{T}$. We will identify these sequences with their functions $f(z)=\sum_{i=0}^{\infty}a_iz^i$ later on without further mentioning. (We will see below that these functions are indeed defined on the topological closure $\overline{\mathbb{D}_{\sqrt[3]{\varepsilon}}}$ of $\mathbb{D}_{\sqrt[3]{\varepsilon}}$.)

Let $\varepsilon'>0$ be a fixed constant. Then we fix for every $\varepsilon>0$ and $i\in\mathbb{N}$ a dyadic $d_{\varepsilon,i}$ with

$$\frac{c_0\cdot\rho(\varepsilon^{1/4})}{\varepsilon^{i/4}}<d_{\varepsilon,i}\leq\frac{c_1\cdot\rho(\varepsilon^{1/4})}{\varepsilon^{i/4}}.$$

A_ε is then defined by

$$A_\varepsilon := \{(0, 1, a_2, ...)|a_i \in \mathbb{C}, |a_i| \leq d_{\varepsilon,i}\}$$

Using the representation ν of $\mathbb{C}^\omega$ defined in the introduction, one can easily verify that there exists a recursively enumerable set $B \subseteq \Sigma^* \times \mathbb{N}$ such that (1) $\nu(v\Sigma^\omega)$ intersects the open kernel of $A_{1-1/k}$ for every $(v, k) \in B$ and (2) every element of $A_{1-1/k}$ has a ν-name all of whose prefices v (times k) belong to B.

Next we will prove that the elements of A_ε do indeed define bounded, holomorphic functions on $\mathbb{D}_\varepsilon$ (and even $\overline{\mathbb{D}_{\sqrt[3]{\varepsilon}}}$):

Lemma 5. *Let ε with $0 < \varepsilon < 1$ be given. Then for each $(a_i)_{i\in\mathbb{N}} \in A_\varepsilon$ we have*

1. *$\sum_{i=0}^\infty a_i z^i$ converges absolutely on $\overline{\mathbb{D}_{\sqrt[3]{\varepsilon}}}$,*
2. *$|\sum_{i=0}^\infty a_i z^i| \leq \sqrt[3]{\varepsilon} + (c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4})/(1 - \varepsilon^{1/12})$ for all $z \in \overline{\mathbb{D}_{\sqrt[3]{\varepsilon}}}$ and*
3. *for all $f \in A_\varepsilon$, $n \in \mathbb{N}$ and $z \in \overline{\mathbb{D}_{\sqrt{\varepsilon}}}$*

$$|f^{(n)}(z)| \leq n! \cdot \sqrt[3]{\varepsilon} \cdot (\sqrt[3]{\varepsilon} + (c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4})/(1 - \varepsilon^{1/12}))/(\sqrt[3]{\varepsilon} - \sqrt{\varepsilon})^{n+1}.$$

Proof: The following inequality proves 1. and 2.

$$\begin{aligned} |\textstyle\sum_{i=0}^\infty a_i z^i| &\leq \sqrt[3]{\varepsilon} + \textstyle\sum_{i=2}^\infty \frac{(c_1+\varepsilon')\cdot\rho(\varepsilon^{1/4})}{\varepsilon^{i/4}} \sqrt[3]{\varepsilon}^i \\ &\leq \sqrt[3]{\varepsilon} + (c_1 + \varepsilon') \cdot \rho(\varepsilon^{1/4}) \textstyle\sum_{i=2}^\infty \varepsilon^{i/12} \end{aligned}$$

for all $z \in \overline{\mathbb{D}_{\sqrt[3]{\varepsilon}}}$.

By taking derivatives and using Cauchy's integral formula we finally get 3.
□

Notice that the bounds in 2. and 3. do actually not depend on the elements of A_ε. Thus we can uniformly compute the corresponding function f on $\overline{\mathbb{D}_{\sqrt{\varepsilon}}}$. This is the essence of Lemma 5 which we will use later on:

Corollary 6. *There exists a computable function $f : \mathbb{Q} \to \mathbb{Q}^3$ so that $f(\varepsilon) = (\mu, \mu', \mu'')$ for $0 < \varepsilon < 1$ implies, that μ, μ' and μ'' are upper bounds on $|g|(\overline{\mathbb{D}_{\sqrt{\varepsilon}}})$, $|g'|(\overline{\mathbb{D}_{\sqrt{\varepsilon}}})$ and $|g''|(\overline{\mathbb{D}_{\sqrt{\varepsilon}}})$, respectively, for all $g \in A_\varepsilon$, where $|g|$ denotes the function with $|g|(z) = |g(z)|$ and g', g'' denote the first and second derivative of g.*

Finally we will use the standard (product) topology on $\mathbb{C}^\omega$ and thus A_ε. Especially we have that A_ε is compact by Tychonoff's Theorem.

Before proceeding we summarize the decisive properties of A_ε by the following lemma:

Lemma 7. *Let ε with $0 < \varepsilon < 1$ be given. Then*

1. $\mathcal{T} \subseteq A_\varepsilon \subseteq A_\varepsilon$ *and*
2. A_ε *is compact.*

The second property allows us to approximate β with fixed precision in finitely many steps. The main point is to prove that we can find, for each $f \in A_\varepsilon$, approximations of $\beta_f(\mathbb{D}_\varepsilon)$ so that these approximations are actually approximations for whole neighborhoods. Afterwards we can use standard compactness arguments.

3 The $\boldsymbol{\beta}$-value of a single Function

In this section we show how to approximate the size of discs in the image of a given holomorphic β-bounding function.

Assume that a holomorphic function $f : \mathbb{D} \to \mathbb{D}_\mu$, $\mu > 0$, is given and we want to compute the supremum of the radius of schlicht discs in the image $f(\mathbb{D})$. Actually, we have to cope with unbounded functions f but switching to $\mathbb{D}_\varepsilon$ for suitable $\varepsilon > 0$ changes the situation to uniformly bounded families of functions. We will see below that approximating the image $f(\mathbb{D}_\varepsilon)$ can be done quite straightforwardly. However, we are faced with the need of quite expensive injectivity tests, if we proceed naively. To circumvent this problem we will use homotopic methods, which avoids injectivity tests.

Lemma 8. *There exists a computable multifunction F_β which computes for given rational $\varepsilon \in (0;1)$ and $f \in A_\varepsilon$ a pair $(\gamma, v) \in \mathbb{Q} \times \Sigma^*$ so that $f \in \nu(v\Sigma^\omega)$ and*

$$\beta_g(\mathbb{D}_\varepsilon) - (1-\varepsilon) < \gamma < \beta_g(\mathbb{D}_{\sqrt{\varepsilon}}) + 1 - \varepsilon$$

for all $g \in \nu(v\Sigma^\omega)$.

Proof: Let ε with $0 < \varepsilon < 1$ and $f \in A_\varepsilon$ be given and μ' and μ'' be upper bounds on $\sup_{z \in \mathbb{D}_{\sqrt{\varepsilon}}}(|f'(z)|)$ and $\sup_{z \in \mathbb{D}_{\sqrt{\varepsilon}}}(|f''(z)|)$, respectively, which can be computed by Corollary 6. W.l.o.g. we can assume that $\varepsilon > 3/4$ and $\mu', \mu'' > 1$.

The main idea of the algorithm given below is as follows: Assume we want to compute $\beta_f(\mathbb{D}_\varepsilon, f(z))$ for some $z \in \mathbb{D}_\varepsilon$ with $f'(z) \neq 0$. To do so we use the fact that a branch g of f^{-1} exists locally at $f(z)$ so that $g(f(z)) = z$. Thus there exists some small $d > 0$ so that g exists on $\mathbb{D}_d(f(z))$. Once we have found such a d we can try to enlarge $\mathbb{D}_d(f(z))$ by finding small discs D_y for every boundary point $y \in \partial\mathbb{D}_d(f(z))$, so that some branch of f^{-1} exists on these discs which coincide with g on $\mathbb{D}_d(f(z))$. This guarantees that a branch h of f^{-1} exists on the union of the discs D_y with $\mathbb{D}_d(f(z))$, so that still $h(f(z)) = z$. Replacing g by h and $\mathbb{D}_d(f(z))$ by the largest disc in the union of the D_y and $\mathbb{D}_d(f(z))$, we can proceed in the same way untill we find some boundary point y so that f^{-1} cannot be extended to some neighborhood of y. In this case we know that we have found the largest schlicht disc around $f(z)$ for this branch of f^{-1}. Repeating this procedure for all $z \in \mathbb{D}_\varepsilon$ with $f'(z) \neq 0$ will finally determine $\beta_f(\mathbb{D}_\varepsilon)$.

The implementation of this idea has to cope with several problems: We have to take into account that we can compute things only with finite precision. Therefore we will consider the domain $\mathbb{D}_{\sqrt{\varepsilon}}$ rather than $\mathbb{D}_\varepsilon$ to avoid sharp decisions whether some point belongs to $\mathbb{D}_\varepsilon$ or not. Furthermore we can neither find the discs for all $z \in \mathbb{D}_\varepsilon$ nor consider all boundary points of the considered discs above. The solution to this problem is to apply the above idea on grids rather than the whole plane. To this end we will use the following simple observation:

Let $z \in \mathbb{D}_{\sqrt{\varepsilon}}$ and $\delta > 0$ with $d(z, \partial\mathbb{D}_{\sqrt{\varepsilon}}) > \delta$ be given. Then

$$|f'(z)| - \mu'' \cdot \delta \leq |f'(z')| \leq |f'(z)| + \mu'' \cdot \delta$$

for all $z' \in \mathbb{D}_\delta(z)$. Furthermore if $|f'(z) - f'(z')| < |f'(z)|$ for all $z' \in \mathbb{D}_\delta(z)$ then f maps the disc $\mathbb{D}_\delta(z)$ one-to-one onto some neighborhood of $f(z)$ (see e.g. [Conway 1978], Lemma XII.1.3]). Applying Koebe's 1/4 Theorem (see e.g. [Henrici 1986]) we thus get

Lemma 9. *Let $z \in \mathbb{D}_{\sqrt{\varepsilon}}$ with $f'(z) \neq 0$ and $\delta < |f'(z)|/\mu''$ with $\mathbb{D}_\delta(z) \subseteq \mathbb{D}_{\sqrt{\varepsilon}}$ be given. Then a neighborhood of z is mapped one-to-one onto $\mathbb{D}_{|f'(z)|\cdot\delta/4}(f(z))$ by f.*

The formal implementation of the above idea now looks as follows, where we proceed in 4 steps:

Step 1: Find finite sequences D_0, ..., D_n and E_0, ..., E_m, $m, n \in \mathbb{N}$, of discs with complex dyadic center and dyadic radius and furthermore $\Delta > 0$ with $\Delta < (\sqrt{\varepsilon} - \varepsilon)/2$ so that

- $\bigcup_{i=0}^n D_i \subsetneq \bigcup_{i=0}^m E_i \subseteq \mathbb{D}_{\sqrt{\varepsilon}}$,
- for all $z \in \mathbb{D}_{(\varepsilon+\sqrt{\varepsilon})/2}$ with $|f'(z)| < (1-\varepsilon)/10$ there exist some i with $z \in D_i$,
- $|f'(z)| < (1-\varepsilon)/9$ for all $z \in \bigcup_{j=0}^m E_j$ and
- for all $z \in \mathbb{D}_{(\varepsilon+\sqrt{\varepsilon})/2} \setminus \bigcup_{j=0}^m E_j$ we have $d(z, \bigcup_{i=0}^n D_i) > \Delta$.

This can be done by approximate computations on a grid. To be more precise take $G = (\frac{1-\varepsilon}{900\mu''}\mathbb{Z} + i\frac{1-\varepsilon}{900\mu''}\mathbb{Z}) \cap \mathbb{D}_{(\sqrt{\varepsilon}+\varepsilon)/2}$ and compute for each grid point $z \in G$ the value $f'(z)$ up to an error of at most $\frac{1-\varepsilon}{900}$ (we denote this value by $\hat{f}(z)$ for the time beeing). Then label each grid point z by "d", "e" or "n" according to the following rules: if $|\hat{f}(z)| < \frac{1-\varepsilon}{10} + 2 \cdot \frac{1-\varepsilon}{900}$ label z by "d". If otherwise

$|\hat{f}(z)| < \frac{1-\varepsilon}{9} - 2 \cdot \frac{1-\varepsilon}{900}$ label z by "e". In all other cases label z by "n". Finally fix D_1,...,D_n to be the sets $\mathbb{D}_{\frac{1-\varepsilon}{900\mu''}}(z)$ of those grid points labeled by "d". Similarly let E_1,...,E_m be the sets $\mathbb{D}_{\frac{1-\varepsilon}{900\mu''}}(z)$ of those grid points labeled by "e" or "d". Choose Δ to be $\frac{1-\varepsilon}{900\mu''}$. (Notice that by the remarks above $|f'|$ can vary on fixed E_i or D_i by at most $\frac{1-\varepsilon}{900}$ and furthermore we have $\frac{1-\varepsilon}{900\mu''} < (\sqrt{\varepsilon} - \varepsilon)/2$.)

In preparation to the next step let ρ be defined by $\rho = \min\{(\sqrt{\varepsilon} - \varepsilon)/3, (1/4) \cdot (1-\varepsilon)^2/(20^2 \cdot \mu'')\}$.

Step 2: Compute a grid $G_D \subseteq \frac{\rho \cdot (1-\varepsilon)}{16 \cdot \mu'}\mathbb{Z} + i\frac{\rho \cdot (1-\varepsilon)}{16 \cdot \mu'}\mathbb{Z}$ so that for all $z \in \frac{\rho \cdot (1-\varepsilon)}{16 \cdot \mu'}\mathbb{Z} + i\frac{\rho \cdot (1-\varepsilon)}{16 \cdot \mu'}\mathbb{Z}$

- if $d(z, \mathbb{D}_\varepsilon \setminus \bigcup_{j=0}^m E_j) < \min\{\Delta/4, (1-\varepsilon)/(4\mu')\}$ then z belongs to G_D and
- if $z \in G_D$ then $d(z, \mathbb{D}_\varepsilon \setminus \bigcup_{j=0}^m E_j) < \min\{\Delta/2, (1-\varepsilon)/(2\mu')\}$.

Following our main idea, this domain grid G_D will give the z's (grid nodes) for which we approximate the radius of a disc D with center $f(z)$ on which a branch g of f^{-1} with $g(f(z)) = z$ exists.

Similarly we define an image grid G_I by $G_I = \frac{\rho \cdot (1-\varepsilon)}{16}\mathbb{Z} + i\frac{\rho \cdot (1-\varepsilon)}{16}\mathbb{Z}$.

We will see below that for all $z \in \mathbb{D}_{(\varepsilon+\sqrt{\varepsilon})/2} \setminus \bigcup_{i=0}^n D_i$ the disc $\mathbb{D}_\rho(f(z))$ is schlicht (and a neighborhood of z is mapped one to one onto this disc by f). Thus we can compute the branch of f^{-1} on $\mathbb{D}_{\rho/2}(f(z))$ with $f^{-1}(f(z)) = z$ for example by computing its Taylor series by samples (see e.g. [Müller 1993]), where the samples can be computed by evaluations of f at different points.

We proceed as follows:

Step 3: For all $z \in G_D$ do

Step 3a: Compute some $y \in G_I$ so that $|f(z) - y| \leq \frac{\rho \cdot (1-\varepsilon)}{8}$, set $G := \{y\}$ and $d(z) = \frac{\rho \cdot (1-\varepsilon)}{8}$

Step 3b: For each point $y' \in G_I \setminus G$ with $|y - y'| < d(z)$ do

Step i: Compute a sequence $y_0 = y$, y_1, ..., $y_n = y'$ for suitable $n \in \mathbb{N}$ and $y_i \in G_I$, so that $|y_{i+1} - y_i| \leq \sqrt{2} \cdot \frac{\rho \cdot (1-\varepsilon)}{16}$ and $y_i \in G$ for all $i = 0, ..., n-1$.

Step ii: Compute for $i = 0, ..., n-1$ the branches g_i of f^{-1} on $\mathbb{D}_{\rho/2}(y_i)$ so that $z \in g_0(\mathbb{D}_{\rho/2}(y_0))$ and $g_{i+1}(y_i) = g_i(y_i)$ for all $i = 0, ..., n-2$.

Step iii: If $d(g_{n-1}(y_n), \mathbb{D}_\varepsilon \setminus \bigcup_{j=0}^m E_j) < \Delta/4$ so $G := G \cup \{y'\}$
if $d(g_{n-1}(y_n), \mathbb{D}_\varepsilon \setminus \bigcup_{j=0}^m E_j)) > \Delta/2$ proceed with the next z
otherwise set $G := G \cup \{y'\}$ or proceed with the next z

Step 3c: increase $d(z)$ by $\frac{\rho \cdot (1-\varepsilon)}{16}$ and proceed with step 3b

Notice that Step 3 will end after a finite number of steps because $f(\mathbb{D}_{\sqrt{\varepsilon}})$ is bounded.

Step 4 Set $d = \max_{z \in G_D} d(z)$

Notice that the above construction does only use finitely many symbols, say t, of an input $w \in \Sigma^\omega$ (a ν-name of f). Thus the construction works identically for all $g \in \nu(v\Sigma^*$, where v is the prefix of length t of w.

Thus it remains to show that d is a suitable approximation of $\beta_f(\mathbb{D}_\varepsilon)$, i.e. that $|d - \beta_f(\mathbb{D}_\varepsilon)| < 1 - \varepsilon$, because then (d, v) is a correct value for $F_\beta(f)$.

Before we will show that d is a suitable approximation to $\beta_f(\mathbb{D}_\varepsilon)$ we will first give the proof that Step 3 can be realized as described, i.e. that f^{-1} does indeed locally exist.

Claim 1 *For each $z \in \mathbb{C}$ with $d(z, \mathbb{D}_\varepsilon \setminus \bigcup_{j=1}^m E_j) < \Delta$ we have that $\mathbb{D}_\rho(f(z))$ is schlicht in $f(\mathbb{D}_{\sqrt{\varepsilon}})$. Furthermore a neighborhood of z is mapped one to one onto $\mathbb{D}_\rho(f(z))$.*

Proof: Every $z \in \mathbb{D}_{(\varepsilon+\sqrt{\varepsilon})/2}$ with $d(z, \mathbb{D}_\varepsilon \setminus \bigcup_{j=1}^m E_j) < \Delta$ belongs to $\mathbb{D}_{(\varepsilon+\sqrt{\varepsilon})/2} \setminus \bigcup_{i=0}^n D_i$ and fulfills therefore $|f'(z)| \geq (1-\varepsilon)/10$. The statement therefore follows from Lemma 9 and the definition of ρ above.

$\diamond$

In Step 3 we essentially use the Kreiskettenverfahren to continue f^{-1} along a path in G. This step very much follows the main idea given in the introduction of this proof: For given $y = f(z)$ we try to find large $d(z)$ so that a branch g of f^{-1} with $g(f(z)) = z$ exists on $\mathbb{D}_{d(z)}(y)$. By Claim 1 such a branch of f^{-1} exists for the first value of $d(z)$ given by Step 3 b i. Assume now that we have already computed some $d(z)$ so that g exists on $\mathbb{D}_{d(z)}(y)$. In Step 3 b ii we then essentially try to find for each grid point y' in $\mathbb{D}_{d(z)+\frac{\rho\cdot(1-\varepsilon)}{16}}(y)$ a continuation of g to $\mathbb{D}_{d(z)} \cup \mathbb{D}_{\rho/2}(y')$. (Actually we update $d(z)$ first and test then for $d(z)$ instead of $d(z) + \frac{\rho\cdot(1-\varepsilon)}{16}$, but to simplify things we ignore this for the moment.) If the continuation to all such discs is possible, we can continue g also to $\mathbb{D}_{d(z)+\frac{\rho\cdot(1-\varepsilon)}{16}}(y)$: For any pairs of such discs $\mathbb{D}_{\rho/2}(y')$ and $\mathbb{D}_{\rho/2}(y'')$ with $\mathbb{D}_{\rho/2}(y') \cap \mathbb{D}_{\rho/2}(y'') \neq \emptyset$ we have that $\mathbb{D}_{d(z)}(y) \cap \mathbb{D}_{\rho/2}(y') \cap \mathbb{D}_{\rho/2}(y'') \neq \emptyset$ is a subset of the connected domain $\mathbb{D}_{\rho/2}(y') \cap \mathbb{D}_{\rho/2}(y'')$. Thus the continuation of g to the first and second disc coincide on $\mathbb{D}_{\rho/2}(y') \cap \mathbb{D}_{\rho/2}(y'')$ by the identity theorem (see e.g. [Ahlfors 1966]).

As we actually update $d(z)$ before testing, we get $d(z) \leq \beta_f(\mathbb{D}_{(\sqrt{\varepsilon}+\varepsilon)/2}) + \frac{\rho\cdot(1-\varepsilon)}{16}$ and thus

$$d \leq \beta_f(\mathbb{D}_{(\sqrt{\varepsilon}+\varepsilon)/2}) + \frac{\rho \cdot (1-\varepsilon)}{16} < \beta_f(\mathbb{D}_{\sqrt{\varepsilon}}) + (1-\varepsilon)$$

To show that also $\beta_f(\mathbb{D}_\varepsilon) - (1-\varepsilon) < d$ holds, let $\mathbb{D}_r(f(z))$ be some schlicht disc in $f(\mathbb{D}_\varepsilon)$ with $r > (1-\varepsilon)/4$ so that a neighborhood $N \subseteq \mathbb{D}_\varepsilon$ of $z \in \mathbb{D}_\varepsilon$ is mapped one to one onto $\mathbb{D}_r(f(z))$. Furthermore let g denote the branch of f^{-1} on $D_r(f(z))$ with $g(f(z)) = z$.

According to the following claim we have $z \notin \bigcup_{j=0}^m E_j$:

Claim 2 *Let $z \in \mathbb{D}_\varepsilon$ and $U \subseteq \mathbb{D}_\varepsilon$ be a simply connected neighborhood of z, which maps one to one onto $\mathbb{D}_{(1-\varepsilon)/4}(f(z))$ via f. Then $z \notin \bigcup_{j=0}^m E_j$.*

Proof: Assume to the contrary that $z \in \bigcup_{j=0}^m E_j$. Then by choice of the sequence E_0, ..., E_m we have $|f'(z)| < (1-\varepsilon)/9$ and thus $|(f^{-1})'(f(z))| > 9/(1-\varepsilon)$. By applying Landau's theorem we thus get that $f^{-1}(\mathbb{D}_{(1-\varepsilon)/4}(f(z)))$ contains a disc of radius $((1-\varepsilon)/8) \cdot (9/(1-\varepsilon)) > 1$ in contradiction to the definition of U.

$\diamond$

Thus, and because of the definition of the grid G_D, there exists some $z' \in G_D$ so that $|z - z'| \leq \sqrt{2} \cdot \frac{\rho\cdot(1-\varepsilon)}{16\cdot\mu'}$. In the computation of $d(z')$ (in Step 3) then first some $y \in G_I$ is chosen, so that $|f(z') - y| \leq \frac{\rho\cdot(1-\varepsilon)}{8}$ and thus $|f(z) - y| \leq |f(z) - f(z')| + \frac{\rho\cdot(1-\varepsilon)}{8} \leq \frac{\rho\cdot(1-\varepsilon)}{4}$. This means that there exists a schlicht disc $\mathbb{D}_{r-\frac{\rho\cdot(1-\varepsilon)}{4}}(y)$ so that a neighborhood of z' maps bijectively onto this disc via f. Notice that this then also holds for $\mathbb{D}_{r-\frac{1-\varepsilon}{4}}(y) \subseteq \mathbb{D}_{r-\frac{\rho\cdot(1-\varepsilon)}{4}}(y)$.

We will now show that $d(z') \geq r-(1-\varepsilon)/2$ which means that indeed $\beta_f(\mathbb{D}_\varepsilon) - (1-\varepsilon) < d$. Assume otherwise that there exists some $\hat{y} \in G_I \cap \mathbb{D}_{r-\frac{1-\varepsilon}{2}}(y)$ which is never added to G during Step 3. W.l.o.g. we can assume that all points in $\mathbb{D}_{|y-\hat{y}|}(y) \cap G_I$ are added to G at some time. Thus there exists a sequence $y_0 = y$, y_1, ..., $y_n = \hat{y}$ and the corresponding g_0, ..., g_{n-1} computed by Step 3 b i and 3 b ii. As $\hat{y}$ is not added to G we have $d(g_{n-1}(\hat{y}), \mathbb{D}_\varepsilon \setminus \bigcup_{j=0}^m E_j) \geq \Delta/4$. Furthermore as g_{n-1} coincide with g on $\mathbb{D}_{\rho/2}(y_{n-1})$ we have $g_{n-1}(\hat{y}) \in \mathbb{D}_\varepsilon$ (notice, that $\mathbb{D}_r(f(z) \subseteq f(\mathbb{D}_\varepsilon)$ and $g_{n-1}(\hat{y}) \in \bigcup_{j=0}^m E_j$. But then by Claim 2 above there cannot exist a schlicht disc of radius $(1-\varepsilon)/4$ around $\hat{y}$ in contradiction to the fact that $\mathbb{D}_{r-\frac{1-\varepsilon}{4}}(y)$ is schlicht and $\hat{y} \in \mathbb{D}_{r-\frac{1-\varepsilon}{2}}(y)$, i.e. $\mathbb{D}_{(1-\varepsilon)/4}(\hat{y}) \subseteq \mathbb{D}_{r-\frac{1-\varepsilon}{4}}(y)$.

$\square$

4 The Main Theorem

The proof of our main theorem can now be simply done by covering A_ε by neighborhoods given by the algorithm of the previous section.

Theorem 10. *Bloch's constant is computable.*

Proof: We have to show that β can be approximated to arbitrary precision. Let thus the precision $d > 0$ be given.

We will try to find a finite sequence f_0,...,f_n of functions in A_ε for some $1 > \varepsilon \geq 1 - d/2$ so that with $F_\beta(f_i) = (\gamma_i, v_i)$ we have $\bigcup_{i=0}^{n} \nu(v_i \Sigma^\omega)$ covers A_ε. By the definition of F_β (Lemma 8) we can then approximate β by the minimum of the γ_i.

According to our remarks on the representations ν of A_ε the set B is r.e. Let now k_0 be some integer with $k_0 \geq 2/d$ and $\varepsilon := 1 - 1/k_0$. Then we can enumerate all $w \in \Sigma^*$ so that $(w, k_0) \in B$. Let w_0, w_1, ... be this enumeration. Furthermore let M_β be a Turing machine computing F_β. To simplify things we will still say that M_β computes on input $w \in \Sigma^\omega$ a pair (γ, v) although only the names of these are computed. Furthermore the pairs computed by M_β on $w_i 0^\omega$ are denoted by (γ_i, v_i), for short.

We will now proceed in steps maintaining a variable β', which holds the currently best approximation to β, and a list L=U_0, U_1,... of open sets in $\mathbb{C}^\omega$. At the beginning let β' be some known upper bound on β and L be an empty list. In step i we then compute (γ_i, v_i) by simulating M_β on $w_i 0^\omega$. We add the neighborhood $\nu(v_i \Sigma^\omega)$ to L and choose the new value of β' to be the minimum of γ_i and the old value of β'. Finally we test whether the elements in L cover A_ε. If this is the case then $|\beta' - \beta| < d$ and we can stop. Otherwise we continue with step $i + 1$.

The correctness of this construction follows immediately by Lemma 8: In this case for each $f \in A_\varepsilon$ there exists some $U_i \in L$ with $f \in U_i$ and thus

$$\beta' \leq \beta_f(\mathbb{D}_{\sqrt{\varepsilon}}) + (1 - \varepsilon)$$

and thus

$$\beta' \leq \inf_{f \in A_\varepsilon} \beta_f(\mathbb{D}_{\sqrt{\varepsilon}}) + (1 - \varepsilon) \leq \inf_{f \in A_\varepsilon} \beta_f(\mathbb{D}) + (1 - \varepsilon) < \beta + d.$$

On the other hand we have for each i:

$$\beta' \geq \beta_{\nu(w_i 0^\omega)}(\mathbb{D}_\varepsilon) - (1 - \varepsilon) \geq \beta(\mathbb{D}_\varepsilon) - (1 - \varepsilon) \geq \varepsilon \cdot \beta - (1 - \varepsilon) > \beta - d.$$

Thus it remains to show, that the construction stops at some step. To this end notice that the sequence $(\nu(v_i \Sigma^\omega))_{i \in \mathbb{N}}$ is an open covering of A_ε: By what we have said about B every function $f \in A_\varepsilon$ has some ν-name w so that for any prefix w' of w the pair (k_0, w') belongs to B. Thus M_β must stop on some $w' 0^\omega$ without having used any symbol outside w'. Therefore for every $f \in A_\varepsilon$ there exists some v_i so that $f \in \nu(v_i \Sigma^\omega)$. By Lemma 7 (2) there must therefore exist a finite subsequence v_{i_0},...,v_{i_l} of $(v_i)_{i \in \mathbb{N}}$ so that $\bigcup_{j=0}^{l} \nu(v_{i_j} \Sigma^\omega)$ covers already A_ε. This means that with $m = \max(i_0, ..., i_l)$ our construction must stop at step m or before.

□

5 Discussion

In this paper we have given the first algorithm to approximate Bloch's constant up to arbitrary precision. We can furthermore adopt this algorithm to other constants of this type as for example Landau's constant.

The most interesting open problem is, whether the conjecture on β given in the introduction holds. If so, the constant can clearly be computed in polynomial time. To this end, our algorithm can present in addition holomorphic functions which are very near the optimum concerning β-values, thus giving possibly new insights on the kind of functions involved.

Concerning our algorithm the main intriguing problem is to improve the complexity bound, which, by a more careful analysis can be seen to be roughly double exponential, to an acceptable running time. Here the main obstacle for improving the time complexity of our algorithm is that even the normed functions can explode when reaching the boundary $\partial\mathbb{D}$, thus evaluation can be quite expensive.

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