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On BCK Algebras - Part I.b: An Attempt to Treat Unitarily the Algebras of Logic. New Algebras[1]

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Abstract: Since all the algebras connected to logic have, more or less explicitely, an associated order relation, it follows that they have two presentations, dual to each other. We classify these dual presentations in "left" and "right" ones and we consider that, when dealing with several algebras in the same research, it is useful to present them unitarily, either as "left" algebras or as "right" algebras. In some circumstances, this choice is essential, for instance if we want to build the ordinal sum (product) between a BL algebra and an MV algebra. We have chosen the "left" presentation and several algebras of logic have been redefined as particular cases of BCK algebras.

We introduce several new properties of algebras of logic, besides those usually existing in the literature, which generate a more refined classification, depending on the properties satisfied. In this work (Parts I-V) we make an exhaustive study of these algebras - with two bounds and with one bound - and we present classes of finite examples, in bounded case.

In this Part I, divided in two because of its length, after surveying chronologically several algebras related to logic, as residuated lattices, Hilbert algebras, MV algebras, divisible residuated lattices, BCK algebras, Wajsberg algebras, BL algebras, MTL algebras, WNM algebras, IMTL algebras, NM algebras, we propose a methodology in two steps for the simultaneous work with them (the first part of Part I).

We then apply the methodology, redefining those algebras as particular cases of reversed left-BCK algebras. We analyse among others the properties Weak Nilpotent Minimum and Double Negation of a bounded BCK(P) lattice, we introduce new corresponding algebras and we establish hierarchies (the subsequent part of Part I).

Key Words: MV algebra, Wajsberg algebra, generalized-MV algebra, generalized-Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, generalized-BL algebra, divisible BCK(P) lattice, Hilbert algebra, Hertz algebra, Heyting algebra, weak-BL algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, t-norm, pocrim

Category: ACM classification: F.4.1; **AMS classification (2000):** 03G25, 06F05, 06F35

This paper presents the subsequent part of Part I.a, namely the Sections 3, 4.

[1] C. S. Calude, G. Stefanescu, and M. Zimand (eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*

3 A unitary treatment of algebras of logic as particular cases of (bounded) reversed left-BCK algebras. New algebras.

In this section, we redefine the algebras mentioned in [Iorgulescu 2007] Section 1 unitarily (as particular cases of (bounded) reversed left-BCK algebras) and gradually (by adding conditions more and more restrictive).

Thus, we divide the involved algebras into three groups: (bounded) BCK(P) algebras (which are not lattices), (bounded) BCK(P) algebras which are lattices (and do not satisfy (div) and (prel) conditions) and (bounded) BCK(P) algebras which are lattices and satisfy (div) or/and (prel) conditions.

Other related algebras are also studied, obtained by adding some of the conditions (DN), (WNM) and also (P1), (P2), (G), (C), (chain), refered as "other conditions".

Thus, we have the hierarchies from Figure 1.

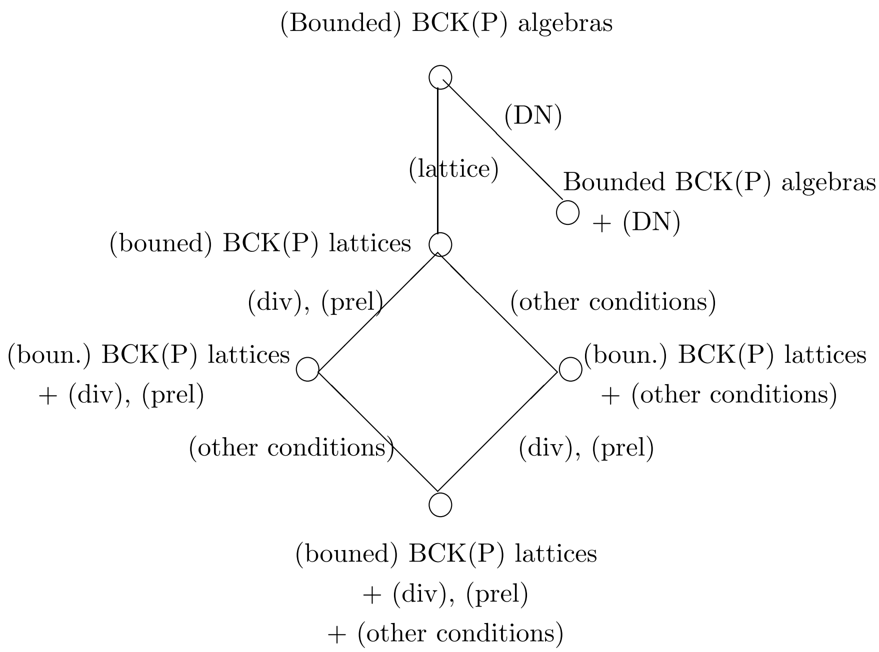

Figure 1: The subclasses of the class of (bounded) BCK(P) algebras

Consequently, the section 3 has three corresponding subsections:

In Subsection 1, we recall the definitions and the properties of (bounded) BCK(P) algebras, starting with a BCK algebra to which we add the missing bound 0 and the condition (P); we then add the condition (DN). We present a generalized-Wajsberg algebra (a generalized-MV algebra). The connection with Hilbert and Hertz algebras is made.

In Subsection 2, we add to (bounded) BCK(P) algebras the lattice condition, i.e. we obtain the (bounded) BCK(P) lattices, and we recall induced properties. We add some of the conditions (DN), (C) and (P1). We establish hierarchies.

In Subsection 3, we add to (bounded) BCK(P) lattices the conditions (div), (prel) and we present induced properties. Then we add some of the conditions (DN), (WNM), (C), (G), (chain). We make the connection with Heyting algebras, put open problems and establish hierarchies.

Note that the BCK algebras are defined as algebras with only the bound 1, while the algebras of logic are bounded (in order to be able to define a negation). Therefore, in this section, *we shall try to "generalize" all the bounded algebras of logic, i.e. to define generalized-bounded algebras as BCK algebras satisfying some additional properties.* We succeeded in some cases, but many open problems remain.

Many of the results are old, but rewritten in the unifying context of BCK algebras. Some new results are Proposition 3.52, Theorem 3.54, Proposition 3.56, Theorems 3.58, 3.59, 3.74.

3.1 (Bounded) BCK(P) algebras. The condition (DN)

First, starting from Iséki's right-BCK algebras, we show how we arrive to "reversed left-BCK algebras", that will be then simply called "BCK algebras".

We add to BCK algebras the condition (P) and the bound 0, we add in bounded case the condition (DN) and recall the properties involved in each case. Commutative BCK algebras are also recalled.

3.1.1 (Bounded) BCK(P) algebras

BCK algebras were introduced in 1966 by Iséki [Iséki 1966] as "right" algebras, using $\star$ and 0 as operations. BCK algebras do not form a variety. In 1975 (cf. [Traczyk 1979]), the important class of commutative BCK algebras was selected by Tanaka [Tanaka 1975], which turns out to be a variety. BCK algebras with condition (S) were introduced in [Iséki 1977a] and studied in [Iséki 1977b]. There exist a lot of papers on BCK algebras as right algebras and a book [Meng and Jun 1994]. There are also a few papers in the literature on BCK algebras using the dual definition, with $\rightarrow$ and 1 as operations. But an explicit connection between the dual definitions and results, the accent on the left-BCK algebras and some new results on left-BCK algebras are presented in [Iorgulescu 2003].

We shall write: "(see [Iorgulescu 2003])", when an old result, obtained for right-BCK algebras, is recalled, and it is better presented in [Iorgulescu 2003], for left-BCK algebras.

A *right-BCK algebra* [Iséki and Tanaka 1978] is a structure $\mathcal{A} = (A, \leq, \star, 0)$, where $\leq$ is a binary relation on A, $\star$ is a binary operation on A and 0 is an element of A, verifying the following axioms: for all $x, y, z \in A$,

(I-R) $(x \star y) \star (x \star z) \leq z \star y$,

(II-R) $x \star (x \star y) \leq y$,

(III-R) $x \leq x$,

(IV-R) $0 \leq x$,

(V-R) $x \leq y,\ y \leq x \Longrightarrow x = y$,

(VI-R) $x \leq y \Longleftrightarrow x \star y = 0$,

or, equivalently, (see [Grzaślewicz 1980]) is an algebra $(A, \star, 0)$ of type (2,0) satisfying the following axioms: for all $x, y, z \in A$,

(BCK-1-R) $[(x \star y) \star (x \star z)] \star (z \star y) = 0$,

(BCK-2-R) $x \star 0 = x$,

(BCK-3-R) $0 \star x = 0$,

(BCK-4-R) $x \star y = 0$ and $y \star x = 0$ imply $x = y$.

The left-BCK algebra is obtained by duality, by replacing the relation $\leq$ with the inverse relation, $\geq$, $\star$ with $\square$ and 0 with 1, as follows.

A *left-BCK algebra* is a structure $\mathcal{A} = (A, \geq, \square, 1)$, where $\geq$ is a binary relation on A, $\square$ is a binary operation on A and 1 is an element of A, verifying the axioms: for all $x, y, z \in A$,

(I-L) $(x \square y) \square (x \square z) \geq z \square y$,

(II-L) $x \square (x \square y) \geq y$,

(III-L) $x \geq x$,

(IV-L) $1 \geq x$,

(V-L) $x \geq y,\ y \geq x \Longrightarrow x = y$,

(VI-L) $x \geq y \Longleftrightarrow x \square y = 1$,

or, equivalently, is an algebra $(A, \square, 1)$ of type (2,0) verifying the axioms corresponding to (BCK-1-R) - (BCK-4-R).

The *reversed* left-BCK algebra is obtained by reversing the operation $\square$, i.e. by replacing $x \square y$ by $y \to x = y \to_L x$, for all x, y. Note that we can also reverse a right-BCK algebra by reversing the operation $\star$, i.e. by replacing $x \star y$ by $y \to_R x$, for all x, y [Iorgulescu 2004a].

We need to reverse the left-BCK algebra in order to arrive to the implication $\to$ which appears in residuated lattices and in BL algebras, for examples.

Definition 3.1 A *reversed left-BCK algebra* is a structure $\mathcal{A} = (A, \geq, \to, 1)$, where $\geq$ is a binary relation on A, $\to$ is a binary operation on A and 1 is an element of A, verifying, the axioms: for all $x, y, z \in A$,

(I) $(z \to x) \to (y \to x) \geq y \to z$,

(II) $(y \to x) \to x \geq y$,

(III) $x \geq x$,
(IV) $1 \geq x$,
(V) $x \geq y,\ y \geq x \Longrightarrow x = y$,
(VI) $x \geq y \Longleftrightarrow y \rightarrow x = 1$,

or, equivalently,

Definition 3.2 (See [Grzaślewicz 1980])
A *reversed left-BCK algebra* is an algebra $(A, \rightarrow, 1)$ of type (2,0) verifying the axioms: for all $x, y, z \in A$,
(BCK-1) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,
(BCK-2) $1 \rightarrow x = x$,
(BCK-3) $x \rightarrow 1 = 1$,
(BCK-4) $y \rightarrow x = 1$ and $x \rightarrow y = 1$ imply $x = y$.

From now on, we shall deal only with reversed left-BCK algebras and thus we shall simply say "BCK algebra" instead of "reversed left-BCK algebra". We shall work with Definition 3.1. We shall freely write $x \geq y$ or $y \leq x$ in the sequel.

Proposition 3.3 *(See [Iséki and Tanaka 1978])*
The following properties hold in a BCK algebra:

$$x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z, \tag{1}$$

$$x \leq y,\ y \leq z \Rightarrow x \leq z, \tag{2}$$

$$z \rightarrow (y \rightarrow x) = y \rightarrow (z \rightarrow x), \tag{3}$$

$$z \leq y \rightarrow x \Leftrightarrow y \leq z \rightarrow x, \tag{4}$$

$$z \rightarrow x \leq (y \rightarrow z) \rightarrow (y \rightarrow x), \tag{5}$$

$$x \leq y \rightarrow x, \tag{6}$$

$$1 \rightarrow x = x, \tag{7}$$

$$x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y. \tag{8}$$

$$[(y \rightarrow x) \rightarrow x] \rightarrow x = y \rightarrow x. \tag{9}$$

Recall that "$\leq$" is a partial order relation and that $(A, \leq, 1)$ is a poset with greatest element 1.

Theorem 3.4 *[Iorgulescu 2003]*
i) Let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a structure such that:

($A^{\cdot,1}$) $(A, \leq)$ is a poset with greatest element 1;

(A2) $(A, \rightarrow, 1)$ *verifies: for all* $x, y, z \in A$*,*

(R1) $1 \rightarrow x = x$*,*

(R2) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$*;*

(A3) $x \rightarrow y = 1 \Longleftrightarrow x \leq y$*, for all* $x, y \in A$*;*

(A4) $x \leq y \Longrightarrow z \rightarrow x \leq z \rightarrow y$*, for all* $x, y, z \in A$*.*

Then, $\mathcal{A}$ *is a BCK algebra.*

ii) Conversely, every BCK algebra satisfies $(A^{\cdot,1})$ *- (A4).*

By this theorem, we have obtained the following equivalent definition of BCK algebras:

Definition 3.5 [Iorgulescu 2003]

A *BCK algebra* is a structure $\mathcal{A} = (A, \leq, \rightarrow, 1)$ such that the above $(A^{\cdot,1})$ - (A4) hold.

We have also obtained the following definitions:

Definition 3.6 [Iorgulescu 2003] (See the corresponding definitions of a t-norm, of an abelian left-monoid and of a partially ordered, abelian, integral left-monoid from Section 2)

(i) A *residuum (or an implication)* on the poset $(A, \leq)$ with greatest element 1 is a binary operation $\rightarrow$ verifying (A2), (A3), (A4) from Theorem 3.4.

(ii) The structure $(A, \leq, \rightarrow, 1)$ such that (A2) and (A3) hold is called an *abelian left-residoid*.

(iii) The structure $(A, \leq, \rightarrow, 1)$ such that (A1) - (A4) hold is called a *partially ordered, abelian, integral left-residoid* (i.e. a duplicate name for "reversed left-BCK algebra") (*integral* means that the greatest element of the poset $(A, \leq)$ is the element 1 of the abelian left-residoid).

Definition 3.7 (See [Iorgulescu 2003])

A *BCK algebra with condition (P)* (i.e. *with product*) or a *BCK(P) algebra* for short, is a BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, 1)$ satisfying the condition (P):

(P) for all $x, y \in A$, there exists $\quad x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$.

Note that any bounded linearly ordered BCK algebra is with condition (P).

Proposition 3.8 *([Iorgulescu 2003], Theorem 2.13) Let* $\mathcal{A}$ *be a BCK(P) algebra. Then, the condition (RP) holds:*

(RP) $x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$*, for all* x, y, z*.*

Proposition 3.9 *(See for example [Iorgulescu 2004a]) Let us consider the BCK(P) algebra* $\mathcal{A} = (A, \leq, \rightarrow, 1)$*. Then, for all* $x, y, z \in A$*:*

$$x \odot y \leq x,\ y\,, \tag{10}$$

$$x \odot (x \rightarrow y) \leq x,\ y\,, \tag{11}$$

$$y \leq x \rightarrow (x \odot y), \tag{12}$$

$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z), \tag{13}$$

$$(y \rightarrow z) \odot x \leq y \rightarrow (z \odot x), \tag{14}$$

$$(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z, \tag{15}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z, \tag{16}$$

$$(x \odot z) \rightarrow (y \odot z) \leq x \rightarrow (z \rightarrow y), \tag{17}$$

$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z) \leq z \rightarrow (x \rightarrow y), \tag{18}$$

$$x \leq y \quad \Rightarrow \quad x \odot z \leq y \odot z. \tag{19}$$

Proposition 3.10 *(See [Iorgulescu 2003]) Let* $\mathcal{A} = (A, \leq, \rightarrow, 1)$ *be a BCK(P) algebra, where for all* $x, y \in A$*:*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}.$$

Then the algebra $(A, \leq, \odot, 1)$ *is a partially ordered, commutative, integral left-monoid, or, equivalently, the operation* $\odot$ *is a t-norm on the poset* $(A, \leq, 1)$ *with greatest element* 1*.*

A fundamental result for this paper is the following [Iorgulescu 2003] (Theorems 2.13, 2.50, 2.55) (see [Iorgulescu 2007] Figure 1):

Theorem 3.11 *BCK(P) algebras are (categorically) equivalent to pocrims (partially ordered, commutative, residuated, integral left-monoids) (see [Iorgulescu 2007], Figure 1 and the first row in Figure 2), i.e.* **BCK(P)** $\cong$ **pocrims**.

Definition 3.12 [Iséki and Tanaka 1978]

If there is an element, 0, of a BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, 1)$, satisfying $0 \leq x$ (i.e. $0 \rightarrow x = 1$), for all $x \in A$, then 0 is called the *zero* of $\mathcal{A}$. A BCK algebra with zero is called to be *bounded* and it is denoted by: $(A, \leq, \rightarrow, 0, 1)$.

Proposition 3.13 *(See for example [Iorgulescu 2004a]) Let us consider the bounded BCK(P) algebra* $\mathcal{A} = (A, \leq, \rightarrow, 0, 1)$*. Then, for all* $x, y, z \in A$*:*

$$0 \odot x (= x \odot 0) = 0. \tag{20}$$

Let $\mathcal{A} = (A, \leq, \rightarrow, 0, 1)$ be a bounded BCK algebra.

Define, for all $x \in A$, a negation $^-$, by [Iséki and Tanaka 1978]: for all $x \in A$,

$$x^- \overset{def}{=} x \rightarrow 0. \tag{21}$$

Proposition 3.14 *[Iséki and Tanaka 1978] In a bounded BCK algebra $\mathcal{A}$, the following properties hold, for all $x, y \in A$:*

$$1^- = 0,\ 0^- = 1, \tag{22}$$

$$x \leq (x^-)^-, \tag{23}$$

$$x \rightarrow y \leq y^- \rightarrow x^-, \tag{24}$$

$$x \leq y \Rightarrow y^- \leq x^-, \tag{25}$$

$$y \rightarrow x^- = x \rightarrow y^-, \tag{26}$$

$$((x^-)^-)^- = x^-. \tag{27}$$

Proposition 3.15 *Let $\mathcal{A}$ be a bounded BCK(P) algebra. Then,*

$$x \rightarrow y^- = (y \odot x)^-, \tag{28}$$

$$x \odot x^- = 0. \tag{29}$$

Proof. (28) is proved in [Iorgulescu 2003]. We prove (29): $x \odot x^- = 0 \Leftrightarrow x \odot x^- \leq 0 \overset{(RP)}{\Leftrightarrow} x \leq x^- \rightarrow 0 = (x^-)^-$, which is true. □

In a BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, 1)$ we define, for all $x, y \in A$
(see [Iséki and Tanaka 1978]):

$$x \vee y \overset{def}{=} (x \rightarrow y) \rightarrow y. \tag{30}$$

Proposition 3.16 *(See [Iséki and Tanaka 1978]) Let $\mathcal{A}$ be a bounded BCK algebra. Then, for all $x \in A$:*

$$0 \vee x = x, \tag{31}$$

$$x \vee 0 = (x^-)^-. \tag{32}$$

Definition 3.17 If $x \vee y = y \vee x$, for all $x, y \in A$, then the BCK algebra $\mathcal{A}$ is called to be *commutative* (see [Iséki and Tanaka 1978]) or, better, $\vee$-*commutative* [Iorgulescu 2003].

Lemma 3.18 *[Iséki and Tanaka 1978] A BCK algebra is $\vee$-commutative iff it is a semilattice with respect to $\vee$ (under $\leq$).*

Theorem 3.19 *Let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a $\vee$-commutative BCK algebra. Let $x, y, z \in A$ such that $x \geq z$, $y \geq z$. Then,*

(1) there exists $x \wedge y$ and

$$x \wedge y = [(x \rightarrow z) \vee (y \rightarrow z)] \rightarrow z, \tag{33}$$

(2) $(x \rightarrow y) \vee (y \rightarrow x) = 1$, i.e. the equality from (prel) is satisfied.

Note that the part (1) of the above Theorem is just ([Lin 2002], Theorem 1), rewritten for reversed left-BCK algebras, while the part (2) was inspired to me from the proof of part (1).

Corollary 3.20 *Let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a $\vee$-commutative BCK(P) algebra. Then,*

(1) $(A, \leq)$ is a lattice, where for any $x, y \in A$,
$x \vee y = (y \rightarrow x) \rightarrow x$, i.e. condition (C) is satisfied,
$x \wedge y = ([x \rightarrow (x \odot y)] \vee [y \rightarrow (x \odot y)]) \rightarrow (x \odot y)$.

(2) for any $x, y \in A$, $(x \rightarrow y) \vee (y \rightarrow x) = 1$, i.e. (prel) is satisfied.

Note that the part (1) of the above Corollary is just ([Lin 2002], Corollary 1), rewritten for reversed left-BCK algebras, while the part (2) follows by above (1) and by (2) of Theorem 3.19.

Example Consider the example of $\vee$-commutative BCK algebra $\mathcal{A}_K = (A, \leq, \rightarrow, 1)$ from [Komori 1978], with $A = (a, b, c, 1)$ and $b, c < a < 1$, b and c being incomparable, and the following table of $\rightarrow$:

$\rightarrow$	a	b	c	1
a	1	a	a	1
b	1	1	a	1
c	1	a	1	1
1	a	b	c	1

Note that A is not a lattice and thefore $\mathcal{A}_K$ is not with condition (P), by above Corollary; indeed, for example $a \odot a = \min\{z \mid a \leq a \rightarrow z\} = \min\{a, b, c, 1\}$ does not exist. Thus, $\mathcal{A}_K$ is not a $\vee$-commutative BCK(P) algebra, and therefore (prel) is not satisfied.

Corollary 3.21 *(See [Iséki and Tanaka 1978]) Let $\mathcal{A}$ be a bounded, $\vee$-commutative BCK algebra. Then, $\mathcal{A}$ is with condition (DN) (and hence it is with condition (P), by Theorem 3.32).*

In a bounded, $\vee$-commutative BCK algebra $\mathcal{A}$, define, for all $x, y \in A$ (see [Iséki and Tanaka 1978]):

$$x \wedge y \stackrel{def}{=} (x^- \vee y^-)^-. \tag{34}$$

Proposition 3.22 *(See [Iséki and Tanaka 1978]) If a BCK algebra is bounded and $\vee$-commutative, then it is a lattice with respect to $\vee$, $\wedge$ (under $\leq$).*

Note that a $\vee$-commutative BCK algebra can be a lattice without being bounded.

The bounded, $\vee$-commutative BCK algebra (or BCK(P) algebra) is $\equiv$ (an equivalent definition of) Wajsberg algebra [Mundici 1986], [Font et al. 1984] (see [Iorgulescu 2003]).

We are defining now a "generalized-Wajsberg algebra" and a "generalized-MV algebra", following [Iorgulescu 2007] Definition 1.4:

Definition 3.23

A *generalized-Wajsberg algebra* (*generalized-MV algebra*) is an X algebra, which is an ordered structure with greatest element 1, such that bounded X algebras are termwise equivalent to Wajsberg algebras (MV algebras, respectively).

Remarks 3.24

(1) Note that there exist several generalized-Wajsberg algebras, such as:
- the "$\vee$-commutative BCK algebra" (since bounded $\vee$-commutative BCK algebras are termwise equivalent to Wajsberg algebras),
- the "$\vee$-commutative BCK(P) algebra" (since bounded $\vee$-commutative BCK(P) algebras are termwise equivalent to Wajsberg algebras),
- the "Wajsberg hoop" [Ferreirim 1992], [Blok and Ferreirim 2000], [Blok and Ferreirim 1993], [Agliano et al.] (since bounded Wajsberg hoops are termwise equivalent to Wajsberg algebras).

(2) Remark that the class of $\vee$-commutative BCK algebras strictly contains the class of $\vee$-commutative BCK(P) algebras, which contains the class of Wajsberg hoops.

(3) *In the rest of the paper we choose to consider as "generalized-Wajsberg algebra" the "$\vee$-commutative BCK(P) algebra".*

(4) *Consequently, in the rest of the paper also we choose to consider as "generalized-MV algebra" the "$\vee$-commutative pocrim"* (since bounded $\vee$-commutative pocrims are termwise equivalent to MV algebras).

(5) For these choices, we have: $\mathbf{W^g} \cong \mathbf{MV^g}$, $(\mathbf{W^g})^\mathbf{b} \equiv \mathbf{W}$, $(\mathbf{MV^g})^\mathbf{b} \equiv \mathbf{MV}$.

(6) The motivation of this choice is given by the stronger proprieties of $\vee$-commutative BCK(P) algebras, as follow from above Theorem 3.19, Corollary 3.20 and example.

Definition 3.25 [Iséki and Tanaka 1978]

Let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a BCK algebra.

(1) $\mathcal{A}$ is *positive implicative* if the following condition (PImpl) holds:
(PImpl) $z \to (y \to x) = (z \to y) \to (z \to x)$, for all $x, y, z \in A$.

(2) $\mathcal{A}$ is *implicative* if the following condition (Impl) holds:
(Impl) $(x \to y) \to x = x$, for all $x, y \in A$.

Proposition 3.26 *[Iséki and Tanaka 1978]*

(1) A BCK algebra $\mathcal{A}$ is positive implicative if and only if the following condition holds:

$$y \to (y \to x) = y \to x.$$

(2) In a commutative BCK algebra, conditions (PImpl) and (Impl) are equivalent.

(3) An implicative BCK algebra is commutative and positive implicative.

(4) Any bounded implicative BCK algebra is a Boolean algebra.

(5) Any positive implicative BCK(P) algebra satisfies condition (G) (i.e. $x \odot x = x$).

Remarks 3.27

(1) Positive implicative BCK algebra $\equiv$ (is an equivalent definition of) **Hilbert algebra**, by the properties of positive implicative BCK algebras [Iséki and Tanaka 1978] and Hilbert algebras [Diego 1966].

(2) Positive implicative BCK(P) algebra $\equiv$ **Hertz algebra** (Hertz algebras [Buşneag 1993], [Porta 1963], [Figallo et al. 2005] are Hilbert algebras which are also meet semilattices with respect to the natural order $x \leq y \Longleftrightarrow x \to y = 1$, verifying (H) $x \to ((y \to (x \wedge y)) = 1$, as D. Buşneag has announced us in a personal note). Indeed, if $\mathcal{A}$ is a positive implicative BCK algebra with condition (P), then it satisfies condition (G), hence, by Proposition 3.10, it is a meet semilattice with $\wedge = \odot$ and satisfies (H) by (12).

(3) Implicative BCK algebra $\equiv$ **implication algebra** (implication algebras were introduced in 1967, by Abbott [Abbott 1967a], [Abbott 1967b], as Hilbert algebras satisfying above condition (Impl)). Indeed, apply Proposition 3.26(3) and above remark (1).

(4) Bounded implicative BCK algebra $\cong$ (i.e. is term equivalent to) Boolean algebra, by Proposition 3.26(4). Consequently, by [Iorgulescu 2007] Definition 1.4, *the "implicative BCK algebra" ($\equiv$ implication algebra) is a generalized-Boolean algebra.*

3.1.2 Bounded BCK(P) algebras with condition (DN)

Definition 3.28 [Iorgulescu 2003] If a bounded BCK algebra $\mathcal{A} = (A, \leq, \to, 0, 1)$ verifies, for every $x \in A$:

$$(x^-)^- = x,$$

then we shall say that $\mathcal{A}$ is *with condition (DN) (Double Negation)* or $\mathcal{A}$ is *involutive* or is a $BCK_{(DN)}$ algebra, for short.

Open problem 3.29 Find a generalized-$BCK_{(DN)}$ algebra.

Lemma 3.30 *(See [Iséki and Tanaka 1978]) Let $\mathcal{A}$ be a $BCK_{(DN)}$ algebra. Then, for all $x, y \in A$:*

$$x \leq y \Leftrightarrow y^- \leq x^-, \tag{35}$$

$$x \to y = y^- \to x^-, \tag{36}$$

$$y^- \to x = x^- \to y. \tag{37}$$

Remark 3.31 The property (36) of $\to$ is called "the contrapositive symmetry with respect to the strong negation" in [Fodor 1995].

The following fundamental result is proved in [Iorgulescu 2003] (Theorems 2.24 and 2.26):

Theorem 3.32 *Let $\mathcal{A} = (A, \leq, \to, 0, 1)$ be a $BCK_{(DN)}$ algebra. Then $\mathcal{A}$ is with condition (P) and, for all $x, y \in A$, we have:*

$$x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to z\} = (x \to y^-)^-, \tag{38}$$

$$x \to y = (x \odot y^-)^-. \tag{39}$$

Theorem 3.33 *Let $\mathcal{A} = (A, \leq, \to, 0, 1)$ be a $BCK_{(DN)}$ algebra. Then, the condition (P2) from [Iorgulescu 2007] Definition 2.11 is satisfied.*

Proof. By preceeding Theorem, $\mathcal{A}$ is with condition (P) and $x \odot y = (x \to y^-)^-$.

Then, (P2) becomes:
$(z^-)^- \odot [(x \odot z) \to (y \odot z)] \leq x \to y \overset{comm. of \odot}{\Longleftrightarrow} [(x \odot z) \to (y \odot z)] \odot z \leq x \to y \overset{(RP)}{\Longleftrightarrow}$ $[(x \odot z) \to (y \odot z)] \leq z \to (x \to y)$, which is true by (17) and (3). $\Box$

3.2 (Bounded) BCK(P) algebras with lattice condition. Other conditions

In this subsection, we add the lattice condition to a (bounded) BCK(P) algebra, i.e. we obtain a (bounded) BCK(P) lattice and recall the induced properties. We also analyse (bounded) BCK(P) lattices with conditions (DN), (C), (P1). We establish hierarchies.

3.2.1 (Bounded) BCK(P) algebras with lattice condition, i.e. (bounded) BCK(P) lattices

Definition 3.34

(1) Let $\mathcal{A} = (A, \leq, \rightarrow, 1)$ be a BCK algebra or a BCK(P) algebra. If the poset $(A, \leq)$ is a lattice, then we shall say that $\mathcal{A}$ is a *BCK lattice* or a *BCK(P) lattice*, respectively.

(1') If the BCK algebra (or the BCK(P) algebra) is bounded, we shall say that the BCK lattice (or the BCK(P) lattice, respectively) is *bounded*.

Note that BCK(P) lattices are called "BCK-lattices with condition (S)" in [Idziak 1984]; note that it should be (P) instead of (S) in that paper.

A BCK lattice (or a BCK(P) lattice) $\mathcal{A} = (A, \leq, \rightarrow, 1)$ will be denoted:

$$\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1),$$

where $x \geq y \Leftrightarrow x \wedge y = y \Leftrightarrow x \vee y = x$, for all $x, y \in A$.

Denote by **BCK(P)-L** and **BCK(P)-L**^{b} the class of BCK(P) lattices and of bounded BCK(P) lattices, respectively.

We recall another fundamental result for this paper, which follows by Theorem 3.11:

Theorem 3.35 *(Bounded) BCK(P) lattices are (categorically) equivalent to (bounded) residuated lattices, i.e. BCK(P)-L $\cong$ R-L (see the second row in [Iorgulescu 2007] Figure 2) and BCK(P)-L$^{\mathbf{b}}$ $\cong$ R-L$^{\mathbf{b}}$.*

Lemma 3.36 *(See [Lin 2002], Lemma 2) Let $(A, \wedge, \vee, \rightarrow, 1)$ be a BCK lattice. Then, for any $x, y, z \in A$, we have:*

(i) $z \rightarrow (x \wedge y) \leq (z \rightarrow x) \wedge (z \rightarrow y)$;

(ii) $z \rightarrow (x \vee y) \geq (z \rightarrow x) \vee (z \rightarrow y)$;

(iii) $(x \wedge y) \rightarrow z \geq (x \rightarrow z) \vee (y \rightarrow z)$.

Proof.

(i): Since $x \wedge y \leq x, y$, then $z \rightarrow (x \wedge y) \leq z \rightarrow x,\ z \rightarrow y$, i.e. $z \rightarrow (x \wedge y)$ is a lower bound of $\{z \rightarrow x, z \rightarrow y\}$. Hence, $z \rightarrow (x \wedge y) \leq (z \rightarrow x) \wedge (z \rightarrow y)$.

(ii): Since $x, y \leq x \vee y$, then $z \rightarrow x,\ z \rightarrow y \leq z \rightarrow (x \vee y)$, i.e. $z \rightarrow (x \vee y)$ is an upper bound of $\{z \rightarrow x, z \rightarrow y\}$. Hence, $(z \rightarrow x) \vee (z \rightarrow y) \leq z \rightarrow (x \vee y)$.

(iii): Since $x \wedge y \leq x,\ y$, then $(x \wedge y) \rightarrow z \geq x \rightarrow z,\ y \rightarrow z$, i.e. $(x \wedge y) \rightarrow z$ is an upper bound of $\{x \rightarrow z, y \rightarrow z\}$. Hence, $(x \rightarrow z) \vee (y \rightarrow z) \leq (x \wedge y) \rightarrow z$. $\Box$

Proposition 3.37 *[Kowalski and Ono 2001], [Turunen 1999] Let $\mathcal{A}$ be a BCK(P) lattice. Then the following properties hold, for all $x, y, z \in A$, $Y, Z \subseteq A$:*

$$\text{if } \vee Z \text{ exists, then } x \odot \vee Z = \vee\{x \odot z \mid z \in Z\}, \tag{40}$$

$$g \vee (h \odot k) \geq (g \vee h) \odot (g \vee k), \tag{41}$$

$$\textit{if } \vee Z \textit{ exists, then } (\vee Z) \to x = \wedge\{z \to x \mid z \in Z\}, \tag{42}$$

$$\textit{if } \wedge Z \textit{ exists, then } x \to (\wedge Z) = \wedge\{x \to z \mid z \in Z\}, \tag{43}$$

$$\textit{if } \wedge Y \textit{ exists, then } \vee \{y \to x \mid y \in Y\} \leq (\wedge Y) \to x, \tag{44}$$

$$\textit{if } \vee Y \textit{ exists, then } \vee \{x \to y \mid y \in Y\} \leq x \to (\vee Y), \tag{45}$$

$$y \to z = \max\{x \mid x \odot y \leq z\}. \tag{46}$$

Remark 3.38 There is an example of complete BCK(P) lattice satisfying:
1) $\vee_{i\in\Gamma}(y_i \to x) < (\wedge_{i\in\Gamma} y_i) \to x$,
2) $\vee_{i\in\Gamma}(x \to y_i) < x \to (\vee_{i\in\Gamma} y_i)$ (see [Turunen 1999], page 156, Exercise 15).

Proposition 3.39 *(See [Iorgulescu 2003]) Let $\mathcal{A}$ be a BCK(P) lattice. Then we have:*

$$x \odot y \leq x \odot (x \to y) \leq x \wedge y. \tag{47}$$

$$x \to (x \wedge y) = x \to y, \tag{48}$$

Proposition 3.40 *(See [Iorgulescu 2003]) In a bounded BCK(P) lattice we have the properties:*

$$(x \vee y)^- = x^- \wedge y^-, \tag{49}$$

$$x^- \vee y^- \leq (x \wedge y)^-. \tag{50}$$

Hence, we have the hierarchies from Figure 2.

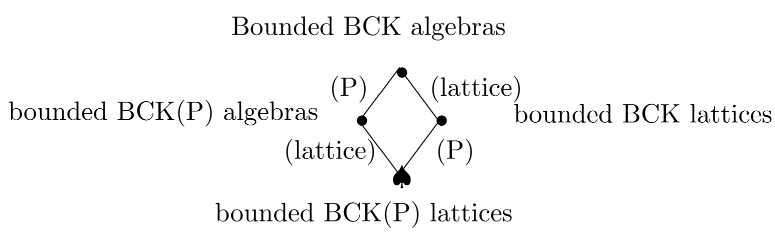

Figure 2: Classes of bounded BCK algebras

3.2.2 (Bounded) BCK(P) lattices with conditions (DN), (WNM), (P1), (C), (G), (chain)

We study now (bounded) BCK(P) lattices with some conditions and make the connection with (div), (prel) conditions and with Heyting algebras.

Definition 3.41 Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ be a bounded BCK(P) lattice.

(1) We say that $\mathcal{A}$ is *with (DN) condition* or a *BCK(P)$_{(DN)}$ lattice*, for short, if the associated bounded BCK(P) algebra is with (DN) condition (Definition 3.28).

(2) We say that $\mathcal{A}$ is a $_{(WNM)}$ *BCK(P) lattice* if it satisfies (WNM) condition.

(3) We say that $\mathcal{A}$ is a $_{(WNM)}$*BCK(P)$_{(DN)}$ lattice* if it verifies both (DN) and (WNM) conditions.

Open problem 3.42 Find a generalized-$_{(WNM)}$BCK(P) lattice.

Let **BCK(P)-L**$_{(DN)}$, $_{(WNM)}$**BCK(P)-L** , $_{(WNM)}$**BCK(P)-L**$_{(DN)}$ denote the class of BCK(P)$_{(DN)}$ lattices, $_{(WNM)}$BCK(P) lattices, $_{(WNM)}$BCK(P)$_{(DN)}$ lattices, respectively.

The following result follows by Theorem 3.35:

Theorem 3.43 *The bounded BCK(P) lattices with (DN) condition (with (WNM) condition) are termwise equivalent (are categorically equivalent) to bounded residuated lattices with (DN) condition, also named "Girard monoids" [Höhle 1995] (with (WNM) condition).*

Thus, **BCK(P)-L**$_{(DN)} \cong$ **R-L**$_{(DN)}$ and $_{(WNM)}$**BCK(P)-L**$_{(DN)} \cong$ $_{(WNM)}$**R-L**$_{(DN)}$.

Proposition 3.44 *(See [Iorgulescu 2003]) Let $\mathcal{A} = (A, \leq, \rightarrow, 0, 1)$ be a BCK(P)$_{(DN)}$ lattice. Then we have:*

$$(x \wedge y)^- = x^- \vee y^-, \tag{51}$$

$$x \wedge y = (x^- \vee y^-)^-. \tag{52}$$

Proposition 3.45 *Let $\mathcal{A}$ be a BCK(P)$_{(DN)}$ lattice which satisfies the condition (P1):*

(P1) for all $x \in A$, $\quad x \wedge x^- = 0$.

Then $\mathcal{A}$ is a Boolean algebra.

Proof. If $x \wedge x^- = 0$, it follows that we also have:
$x \vee x^- = (x^-)^- \vee x^- = (x^- \wedge x)^- = 0^- = 1$, by Proposition 3.44, hence $\mathcal{A}$ is a Boolean algebra. □

Hence we have the hierarchy from Figure 3.

Definition 3.46 [Iorgulescu 2003] We say that a BCK lattice (or a BCK(P) lattice) $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1)$ is *with condition (C)* if the following condition (C) holds:

(C) for all $x, y \in A$, $\; x \vee y = (x \rightarrow y) \rightarrow y$.

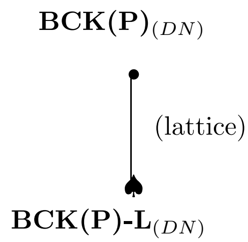

Figure 3: Classes of bounded BCK(P) algebras, with (DN)

Remark 3.47 "BCK(P) lattice with condition (C)" $\equiv$ (is an equivalent definition of) "$\vee$-commutative BCK(P) algebra", by Corollary 3.20. Hence, we have obtained four generalizations of Wajsberg algebras: the $\vee$-commutative BCK algebra, the $\vee$-commutative BCK(P) algebra, the BCK(P) lattice with condition (C) and the Wajsberg hoop.

The following result can be considered as a generalization of the result saying that "a generalized-Wajsberg algebra (Remarks 3.24) satisfies the divisibility condition, (div)" (see next paper, Part II [Iorgulescu 2008]):

Theorem 3.48 *(See [Lin 2002], Theorem 4) Let* $(A, \wedge, \vee, \rightarrow, 1)$ *be a* $\vee$*-commutative BCK lattice (i.e. a BCK lattice with (C)). Then, for any* $x, y, z \in A$*,*

$$z \rightarrow x = z \rightarrow y \iff x \wedge z = y \wedge z.$$

Corollary 3.49 *[Iorgulescu 2003] Let* $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ *be a bounded BCK(P) lattice with* **(C)** *condition. Then* $\mathcal{A}$ *is with (DN) condition.*

Theorem 3.50 *[Iorgulescu 2003] The bounded BCK(P) lattice with (C) condition is an equivalent definition ($\equiv$) of Wajsberg algebra.*

Definition 3.51

(1) We say that a bounded BCK(P) lattice $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is of *Product type* if it satisfies the conditions (P1) and (P2) from [Iorgulescu 2007] Definition 2.11.

(2) We say that a BCK(P) lattice $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1)$ is of *Gödel type* if it satisfies the condition (G) from [Iorgulescu 2007] Definition 2.12.

Proposition 3.52 *Let* $\mathcal{A}$ *be a BCK(P) lattice of Gödel type. Then, for all* $x, y \in A$*,* $x \odot y = x \wedge y$*.*

Proof. (Bart Van Gasse): $x \odot y \le x \wedge y = (x \wedge y) \odot (x \wedge y) \le x \odot y$. $\Box$

Conversely, we have the following

Proposition 3.53 *Let $\mathcal{A}$ be a BCK(P) lattice such that for all $x, y \in A$, $x \odot y = x \wedge y$. Then, $\mathcal{A}$ is of Gödel type.*

Proof. For all $x \in A$, we have $x \odot x = x \wedge x = x$, hence (G) holds. □

Theorem 3.54 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1)$ be a BCK(P) lattice of Gödel type. Then $\mathcal{A}$ verifies (div) condition.*

Proof. By Proposition 3.52, $x \odot y = x \wedge y$, for all x, y, hence $x \odot (x \rightarrow y) = x \wedge (x \rightarrow y)$. We must prove that

$$x \wedge y = x \wedge (x \rightarrow y). \tag{53}$$

Since by (6), $y \leq x \rightarrow y$, then

$$x \wedge y \leq x \wedge (x \rightarrow y). \tag{54}$$

Since by (48), $x \rightarrow y \leq x \rightarrow (x \wedge y)$, then by (RP), we obtain

$$x \odot (x \rightarrow y) = x \wedge (x \rightarrow y) \leq x \wedge y. \tag{55}$$

By (54) and (55) it follows (53). □

Corollary 3.55 *Let $\mathcal{A}$ be a positive implicative BCK(P) lattice. Then, $\mathcal{A}$ verifies (div) condition.*

Proof. By Proposition 3.26(5), $\mathcal{A}$ is of Gödel type; then apply Theorem 3.54. □

Proposition 3.56 *Let $\mathcal{A}$ be a bounded BCK(P) lattice of Gödel type. Then $\mathcal{A}$ verifies (P1) condition.*

Proof. By Proposition 3.52, we have $x \odot y = x \wedge y$ and by Proposition 3.15, we have $x \odot x^- = 0$. Then, $x \wedge x^- = x \odot x^- = 0$. □

Corollary 3.57 *Let $\mathcal{A}$ be a bounded BCK(P)$_{(DN)}$ lattice of Gödel type. Then $\mathcal{A}$ is a Boolean algebra.*

Proof. By Propositions 3.45, 3.56. □

Theorem 3.58 *Let $\mathcal{A}$ be a bounded BCK(P) lattice of Gödel type. Then $\mathcal{A}$ verifies (WNM) condition (i.e. it is a $_{(WNM)}$BCK(P) lattice) .*

Proof. By Proposition 3.52, $x \odot y = x \wedge y$, for all $x, y \in A$, hence:
$(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = (x \odot y)^- \vee [(x \odot y) \rightarrow (x \odot y)] = (x \odot y)^- \vee 1 = 1$, by (III), (VI). □

Theorem 3.59 *Let $\mathcal{A}$ be a bounded BCK(P) lattice verifying the following:*
"for each $x, y \in A$, such that $x \odot y \neq 0$, we have $x \odot y = x \wedge y$".
Then, $\mathcal{A}$ is a $_{(WNM)}$BCK(P) lattice.

Proof. For all $x, y \in A$, there are two cases:

Case (1): $x \odot y = 0$; then (WNM) condition is satisfied:
$(x \odot y)^- \vee [(x \wedge y) \to (x \odot y)] = 0^- \vee [(x \wedge y) \to 0] = 1 \vee [(x \wedge y) \to 0] = 1$.

Case (2): $x \odot y \neq 0$; then, by hypothesis, $x \odot y = x \wedge y$; then (WNM) condition is satisfied:
$(x \odot y)^- \vee [(x \wedge y) \to (x \odot y)] = (x \odot y)^- \vee [(x \odot y) \to (x \odot y)] = (x \odot y)^- \vee 1 = 1$.
□

Theorem 3.60 *Let $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ be a BCK(P) lattice. Then,*

$$(chain) \implies (prel),$$

where:
(chain) for all $x, y \in A$, $x \leq y$ or $y \leq x$.

Proof. Let $x, y \in A$; then either $x \leq y$ or $y \leq x$, i.e. either $x \to y = 1$ or $y \to x = 1$, hence $(x \to y) \vee (y \to x) = 1$ and thus (prel) holds. □

Remark 3.61 If $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ is a BCK(P) lattice, then condition (G) implies condition (div) (Theorem 3.54) and condition (chain) implies condition (prel) (Theorem 3.60).

Open problems 3.62

(1) In a BCK(P) lattice (residuated lattice), find a condition, say (C_p), which verifies: (prel) + $(C_p) \iff$ (chain), and such that (prel) and (C_p) are independent, i.e. there exists a BCK(P) lattice verifying (prel) and not verifying (C_p) and there exists a BCK(P) lattice verifying (C_p) and not verifying (prel).

(2) In a BCK(P) lattice, find a condition, say (C_G), which verifies: (div) + $(C_G) \iff$ (G), and such that (div) and (C_G) are independent.

3.3 (Bounded) BCK(P) lattices with (div), (prel). Other conditions

In this subsection, we study (bounded) BCK(P) lattices with some of the conditions (div), (prel), but also (DN), (C), (WNM), (G), (chain). Thus, we can find which properties of Hájek(P) algebras come from the two conditions (div) and (prel).

We introduce divisible (bounded) BCK(P) lattices and (generalized-) weak-Hájek(P) algebras.

We make the connection with Heyting algebras, put open problems and establish hierarchies.

3.3.1 (Bounded) BCK(P) lattices with (div), (prel)

Let us introduce the following classes of (bounded) BCK(P) lattices; some are old.

Definition 3.63

a) A *divisible BCK(P) lattice* is a BCK(P) lattice verifying (div) condition.

a') A *bounded divisible BCK(P) lattice or a divisible bounded BCK(P) lattice* is a bounded BCK(P) lattice verifying (div) condition [Iorgulescu 2003].

b) A *weak-Hájek(P) algebra* is a bounded BCK(P) lattice verifying (prel) condition.

b') A *generalized-weak-Hájek(P) algebra or weak-generalized-Hájek(P) algebra* (standard generalization) is a BCK(P) lattice verifying (prel) condition.

c) A *Hájek(P) algebra* is a bounded BCK(P) lattice verifying (div) and (prel) conditions [Iorgulescu 2003].

c') A *generalized-Hájek(P) algebra* (standard generalization) is a BCK(P) lattice verifying (div) and (prel) conditions.

Let **divBCK(P)-L** and **divBCK(P)-L**^{b} denote the class of divisible BCK(P) lattices and bounded divisible BCK(P) lattices, respectively.

Let **weak-Ha(P)** and **weak-Ha(P)**^{g} denote the class of weak-Hájek(P) algebras and weak-generalized-Hájek(P) algebras, respectively.

Let **Ha(P)** and **Ha(P)**^{g} denote the class of Hájek(P) algebras and of generalized-Hájek(P) algebras, respectively. We have, by Theorem 3.35, the following:

Theorem 3.64

div*BCK(P)-L* $\cong$ **div***R-L*, **div***BCK(P)-L*$^{\mathbf{b}}$ $\cong$ **div***R-L*$^{\mathbf{b}}$ *[Iorgulescu 2003].*

weak-Ha(P) $\cong$ *weak-BL* $=$ **MTL**, *weak-Ha*$(\mathbf{P})^{\mathbf{g}}$ $\cong$ *weak-BL*$^{\mathbf{g}}$ $=$ $\mathbf{MTL^g}$.

$\mathbf{Ha}(\mathbf{P}) \cong \mathbf{BL}$ *[Iorgulescu 2003], hence* $\mathbf{Ha}(\mathbf{P})^{\mathbf{g}} \cong \mathbf{BL^g}$ *(see the third row in [Iorgulescu 2007] Figure 2).*

The conclusion from the next Proposition is well known for BL algebras; we point out that it follows by condition (div), in a more general case.

Proposition 3.65 *Let* $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1)$ *be a divisible BCK(P) lattice. Then,* $a \wedge (\vee_{i\in I} b_i) = \vee_{i\in I}(a \wedge b_i)$, *whenever the arbitrary unions exist.*

Proof. $a \wedge (\vee_{i\in I} b_i) = (\vee_{i\in I} b_i) \odot [(\vee_{j\in I} b_j) \rightarrow a] = \vee_{i\in I}[b_i \odot (\vee_{j\in I} b_j \rightarrow a)]$, by (div) and (40). But, for any $i \in I$: $b_i \leq \vee_{j\in I} b_j$; then, $\vee_{j\in I} b_j \rightarrow a \leq b_i \rightarrow a$, by (1); hence $b_i \odot (\vee_{j\in I} b_j \rightarrow a) \leq b_i \odot (b_i \rightarrow a) = b_i \wedge a$, by (19); it follows that $\vee_{i\in I}[b_i \odot (\vee_{j\in I} b_j \rightarrow a)] \leq \vee_{i\in I}(a \wedge b_i)$. Thus, $a \wedge (\vee_{i\in I} b_i) \leq \vee_{i\in I}(a \wedge b_i)$. The other inequality is obvious. □

By this proposition we immediately obtain:

Corollary 3.66 *Let $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 1)$ be a divisible BCK(P) lattice. Then, $\mathcal{L}(\mathcal{A}) = (A, \wedge, \vee, 1)$ is a distributive lattice.*

□

The conclusion from the next Proposition is well known for BL algebras and MTL algebras; we point out that it follows by condition (prel).

Proposition 3.67 *Let $\mathcal{A}$ be a weak-Hájek(P) algebra. Then, for all $x, y \in A$,*

$$(x \wedge y)^- = x^- \vee y^-.$$

Proof. By (50), we have the inequality $x^- \vee y^- \leq (x \wedge y)^-$. It remains to prove the converse inequality:

$$(x \wedge y)^- \leq x^- \vee y^-. \tag{56}$$

Indeed, we have: $x \rightarrow y \overset{(48)}{=} x \rightarrow (x \wedge y) \overset{(24)}{\leq} (x \wedge y)^- \rightarrow x^-$ and $x \odot y = y \odot x$. Hence, by (RP), $(x \rightarrow y) \odot (x \wedge y)^- \leq x^-$. Similarly, $(y \rightarrow x) \odot (x \wedge y)^- \leq y^-$. It follows that:

$$(x \wedge y)^- = 1 \odot (x \wedge y)^- \overset{(prel)}{=} [(x \rightarrow y) \vee (y \rightarrow x)] \odot (x \wedge y)^- \overset{(40)}{=}$$

$$[(x \wedge y)^- \odot (x \rightarrow y)] \vee [(x \wedge y)^- \odot (y \rightarrow x)] \leq x^- \vee y^-,$$

i.e. (56) holds. □

The result from the next Proposition is well known for BL and MTL algebras:

Proposition 3.68 *Let $\mathcal{A}$ be a weak-Hájek(P) algebra. Then, the condition $(C_\vee)$ holds, where:*
$(C_\vee)$ for all $x, y \in A$, $\quad x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$.

We shall prove that (prel) implies $(C_\vee)$ on a BCK(P) lattice (i.e. in one bound case) in next paper Part II [Iorgulescu 2008].

3.3.2 (Bounded) BCK(P) lattices with (div), (prel) and other conditions

Here we shall add to (bounded) divisible BCK(P) lattices, to (generalized) weak-Hájek(P) algebras or to (generalized) Hájek(P) algebras some other conditions: (DN), (WNM), (G), (C) etc.

Definition 3.69 Let Y denote bounded divisible BCK(P) lattice or weak-Hájek(P) algebra or Hájek(P) algebra.

(1) We say that Y is *with (DN) condition* or a $Y_{(DN)}$ *algebra* for short, if the associated bounded BCK(P) lattice is *with (DN) condition*, i.e. is a $\mathrm{BCK(P)}_{(DN)}$ lattice.

(2 We say that Y is *with (WNM) condition* or a $_{(WNM)}Y$ *algebra* for short, if the associated bounded BCK(P) lattice is with (WNM) condition, i.e. it is a $_{(WNM)}$BCK(P) lattice.

(3) We say that Y is a $_{(WNM)}\mathrm{Y}_{(DN)}$ algebra, if the associated bounded BCK(P) lattice is with (WNM) and (DN) conditions, i.e. it is a $_{(WNM)}\mathrm{BCK(P)}_{(DN)}$ lattice.

Let $\mathbf{Y}_{(DN)}$, $_{(WNM)}\mathbf{Y}$ and $_{(WNM)}\mathbf{Y}_{(DN)}$ denote the class of $\mathrm{Y}_{(DN)}$ algebras, $_{(WNM)}$Y algebras and $_{(WNM)}\mathrm{Y}_{(DN)}$ algebras, respectively.

Proposition 3.70 *Let $\mathcal{A} = (A, \wedge, \vee, \to, 0, 1)$ be a bounded BCK(P) lattice. Then, we have the equivalence:*

$$(div) + (DN) \Longleftrightarrow (C).$$

Proof.

$\Longrightarrow$: By Proposition 3.44 and Theorem 3.32, $x \vee y = (x^-)^- \vee (y^-)^- = (x^- \wedge y^-)^- = [x^- \odot (x^- \to y^-)]^- \overset{(36)}{=} [x^- \odot (y \to x)]^- = (y \to x) \to x$.

$\Longleftarrow$: By Corollary 3.49, (C) $\Longrightarrow$ (DN). It remains to prove that (C) $\Longrightarrow$ (div). By Theorem 3.32, $x \wedge y = (x^- \vee y^-)^- = [(y^- \to x^-) \to x^-]^- \overset{(36)}{=} [(x \to y) \to x^-]^- = (x \to y) \odot x = x \odot (x \to y)$.

□

Corollary 3.71 *Any divisible bounded BCK(P) lattice satisfies (DN) condition iff it is a Wajsberg algebra.*

Proof. Since "a bounded BCK(P) lattice satisfying (C) condition" is an equivalent definition of Wajsberg algebra, by Theorem 3.50. □

The above Corollary says: $\mathbf{divBCK(P)\text{-}L}_{(DN)}$ = **divBCK(P)-L** + (DN) $\equiv$ **W**.

Recall [Hájek 1998] that a Hájek(P) algebra is a Wajsberg algebra iff it is with (DN) condition. We then write: $\mathbf{Ha(P)}_{(DN)}$ = **Ha(P)** + (DN) $\equiv$ **W**.

By Theorem 3.54, the BCK(P) lattices of Gödel type are the divisible BCK(P) lattices of Gödel type.

Recall now the following:

Definition 3.72 ([Boicescu et al. 1991], pages 202, 203) Let $\mathcal{L} = (L, \wedge, \vee)$ be a lattice.

(i) For every $y, z \in L$, the *relative pseudocomplement of y with respect to z*, provided it exists, is the greatest element x such that $x \wedge y \leq z$; it is denoted by $y \Rightarrow z$ (i.e. $y \Rightarrow z \overset{notation}{=} \max\{x \mid x \wedge y \leq z\}$).

(ii) $\mathcal{L}$ is said to be *relatively pseudocomplemented* provided the relative pseudocomplement $y \Rightarrow z$ exists for every $y, z \in L$.

(iii) A *Heyting algebra* is a relatively pseudocomplemented lattice with 0, i.e. a bounded one.

Note that by the above Definition, relatively pseudocomplemented lattices and Heyting algebras are placed in the 4-th column of the Mendeleev-type table [Iorgulescu 2007].

If $\mathcal{L}$ is a relatively pseudocomplemented lattice, then $\Rightarrow$ can be viewed as a binary operation on L and there exists the greatest element, 1, of the lattice: $1 = x \Rightarrow x$, for all $x \in L$. Consequently, we have the following equivalent definition, by (PR)=(RP), with $\odot = \wedge$:

Definition 3.73

(1) A *relatively pseudocomplemented lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee, \Rightarrow, 1)$, where $(L, \wedge, \vee, 1)$ is a lattice with greatest element and the binary operation $\Rightarrow$ on L verifies:

(PR) for all $x, y, z \in L$, $\quad x \leq y \Rightarrow z$ if and only if $x \wedge y \leq z$.

(1') A **Heyting algebra** is a duplicate name for bounded relatively pseudocomplemented lattice.

Note that by this equivalent Definition, relatively pseudocomplemented lattices and Heyting algebras are placed in the 3-rd column of the Mendeleev-type table.

Hence we have the following

Theorem 3.74

(1) The divisible BCK(P) lattices of Gödel type are (categorically) equivalent to relatively pseudocomplemented lattices.

(1') The bounded divisible BCK(P) lattices of Gödel type are (categorically) equivalent to Heyting algebras.

Proof. By ([Iorgulescu 2003], Theorem 3.29) and by Theorem 3.54. □

By above Theorem 3.74 and by Corollary 3.55, we obtain the following:

Corollary 3.75

(1) Hilbert(P) lattices (i.e. Hilbert algebras verifying conditions (P) and (lattice)) $\equiv$ *divisible BCK(P) lattices of Gödel type* $\cong$ *relatively pseudocomplemented lattices.*

(1') Bounded Hilbert(P) lattices $\equiv$ *bounded divisible BCK(P) lattices of Gödel type* $\cong$ *Heyting algebras.*

Hence, by using Remarks 3.27, we obtain the hierarchies from Figure 4.

We shall give examples of finite Heyting algebras, of finite proper bounded divisible BCK(P) lattices and of bounded divisible BCK(P) lattices which satisfy

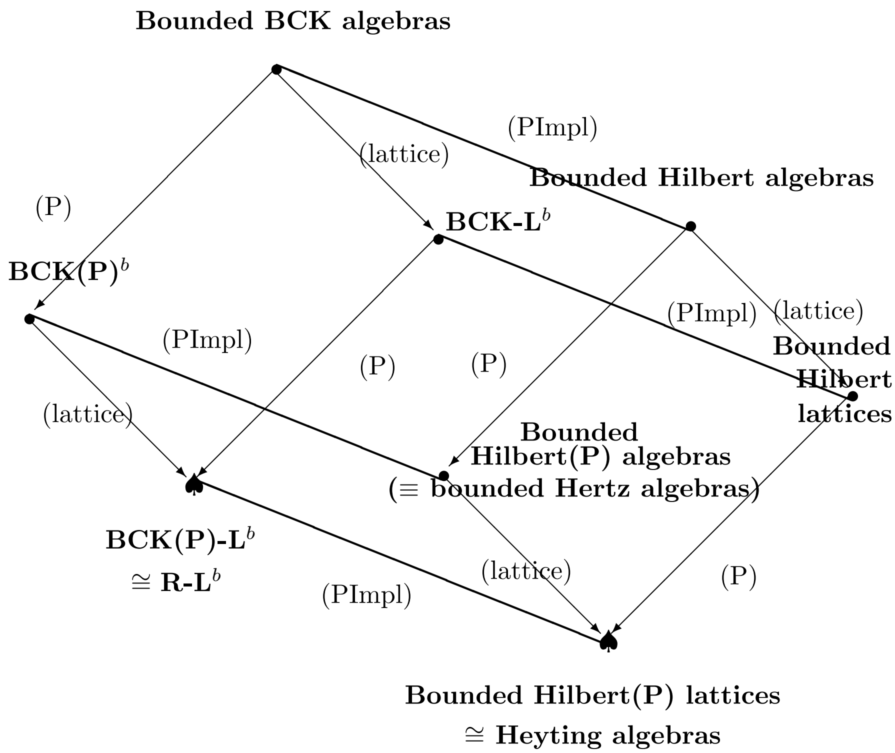

Figure 4: Connections between bounded BCK algebras, bounded Hilbert algebras and Heyting algebras

(WNM) and are not Heyting algebras, in Part III [Iorgulescu a]. We shall give examples of finite bounded Hilbert lattices, in Part V [Iorgulescu b].

Definition 3.76

(1) Let *Product*$_{BCK}$ *algebra* be a Hájek(P) algebra of Product type.

(2) Let *Gödel*$_{BCK}$ *algebra* be a Hájek(P) algebra of Gödel type.

Hence we have: **Product**$_{BCK}$ ≅ **Product**, **Gödel**$_{BCK}$ ≅ **Gödel**.

Open problem 3.77 Note that we didn't find other examples of bounded BCK(P) lattices of Product type than Hájek(P) algebras of Product type (= Product$_{BCK}$ algebras). It remains an open problem to find examples or to prove that there cannot exist.

Open problem 3.78 Find a generalized-Product$_{BCK}$ algebra.

Consequently, the class of Hájek(P) algebras contains the classes of Wajsberg, Product$_{BCK}$ and Gödel$_{BCK}$ algebras. It remains to prove that the class of generalized-Hájek(P) algebras (standard generalization) contains the class of generalized-Wajsberg algebras (Remarks 3.24), which is done later, in Part II [Iorgulescu 2008].

Recall that the Gödel algebras (BL algebras + (G)) are the Heyting algebras verifying (prel) condition.

We shall present examples of Hájek(P) algebras verifying (WNM) and not verifying (G) condition in Part III [Iorgulescu a].

Open problem 3.79 Study the class of those Hájek(P) algebras which verify (P2) condition ($\cong$ SSBL algebras).

Definition 3.80

(1) We say that a Hájek(P) algebra is *with (C) condition* if the associated bounded BCK(P) lattice is with (C) condition (Definition 3.46).

(1') We say that a generalized-Hájek(P) algebra is *with (C) condition* if the associated BCK(P) lattice is with (C) condition (Definition 3.46).

Obviously, a Hájek(P) algebra with (C) condition is an equivalent definition ($\equiv$) of Wajsberg algebra, by Theorem 3.50, and a generalized-Hájek(P) algebra with (C) condition is an equivalent definition of generalized-Wajsberg algebra, by Remark 3.47.

Theorem 3.81 *[Iorgulescu 2003] A Hájek(P) algebra is with* **(C)** *condition iff it is with (DN) condition.*

Proposition 3.82 *Every Wajsberg (MV) algebra satisfies (P2) condition from [Iorgulescu 2007] Definition 2.11.*

Proof. By Theorem 3.33. $\square$

Proposition 3.83 *A Wajsberg algebra $\mathcal{A} = (A, \rightarrow, ^-, 1)$ which satisfies (P1) condition (from the definition of Product algebra) is a Boolean algebra, where: (P1) for every $x \in A$,* $\quad x \wedge x^- = 0$.

Proof. By Proposition 3.45. $\square$

Summarizing, we have the generalizations and the particular cases of Hájek(P) algebras (BL algebras) from Figure 5. By adding the (DN) condition, we obtain the hierarchy from Figure 6.

3.3.3 The conditions (DN) and (WNM): some hierarchies

Since the conditions (DN) and (WNM) seem to be the most important, for the moment, the classes of examples of finite bounded BCK(P) lattices, that we shall

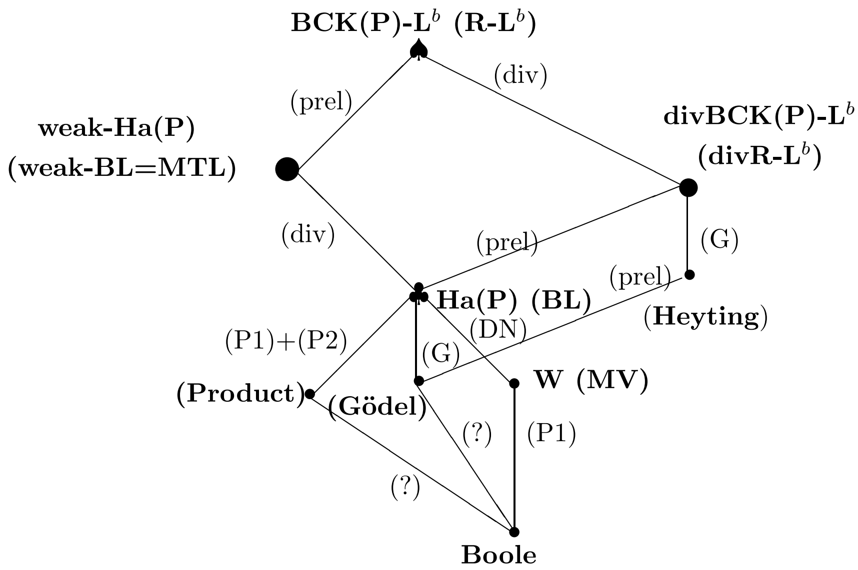

Figure 5: Generalizations and particular cases of Hájek(P) algebras (BL algebras), where ^{b} means "bounded" and (?) means "not-known minimal condition"

present in Parts III-V [Iorgulescu a] - [Iorgulescu b], will be classified following these conditions. Thus, trying to draw the hierarchies between bounded BCK(P) lattices, we obtain four planes (maps):
- the plane $\mathcal{P}^b$ of the hierarchies of subclasses of the class of bounded BCK(P) lattices;
- the plane ${}_{(WNM)}\mathcal{P}$ of the hierarchies of subclasses of the class of bounded BCK(P) lattices with condition (WNM);
- the plane $\mathcal{P}_{(DN)}$ of the hierarchies of subclasses of the class of bounded BCK(P) lattices with condition (DN) ;
- the plane ${}_{(WNM)}\mathcal{P}_{(DN)}$ of the hierarchies of subclasses of the class of bounded BCK(P) lattices with conditions (WNM) and (DN).

We give in Figure 7 the spatial vue of the four planes $\mathcal{P}^b$, ${}_{(WNM)}\mathcal{P}$, $\mathcal{P}_{(DN)}$ and ${}_{(WNM)}\mathcal{P}_{(DN)}$.

By "cutting" with "vertical planes", we obtain different hierarchies, as for example that in the following Figure 8.

4 Conclusions

Surveying chronologically some algebras of logic, we have noticed that their definitions are given in rather different terms, making hard to see the connections between them and to connect them. For example, we cannot make the ordinal

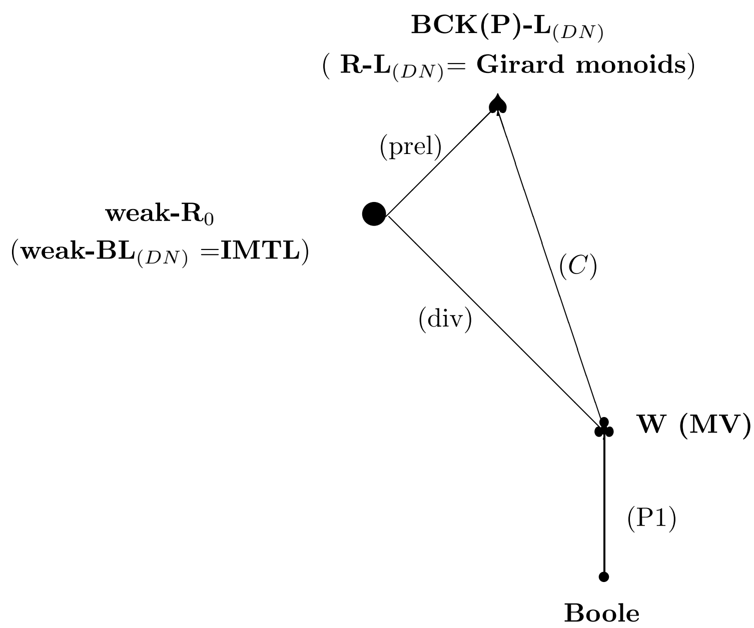

Figure 6: Generalizations and particular cases of Wajsberg algebras (MV algebras)

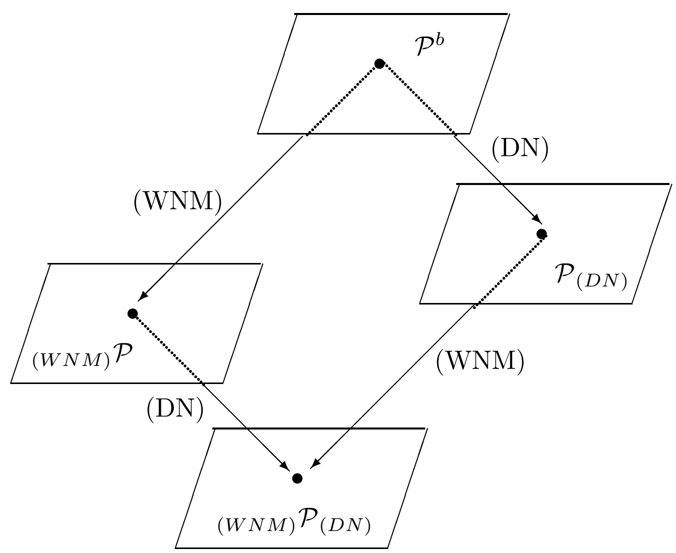

Figure 7: Spatial vue of the four planes $\mathcal{P}^b$, ${}_{(WNM)}\mathcal{P}$, $\mathcal{P}_{(DN)}$, ${}_{(WNM)}\mathcal{P}_{(DN)}$

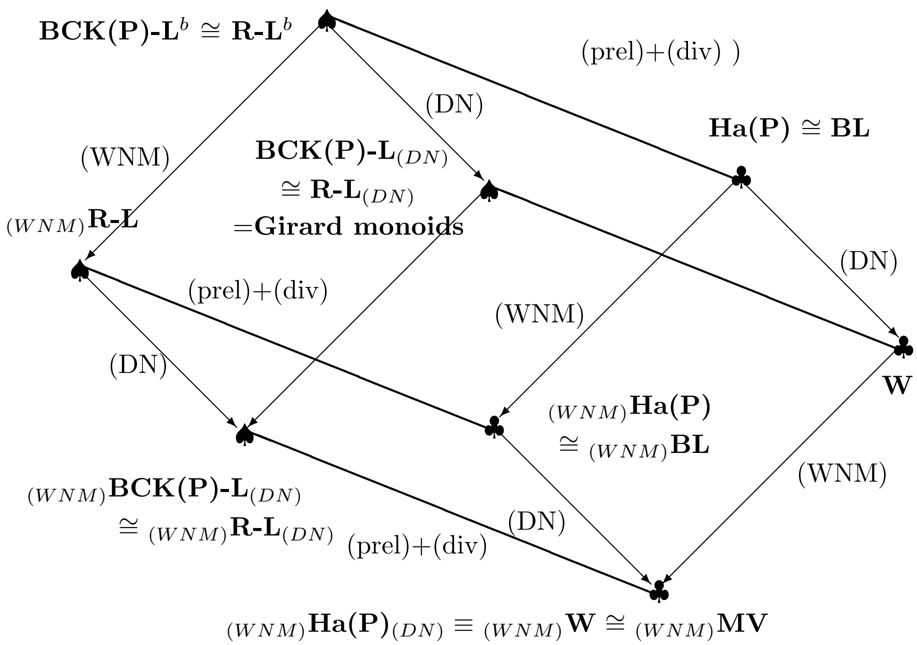

Figure 8: Vertical sections through bounded residuated lattices and BL algebras

sum (product) between a BL algebra and an MV algebra, with their initial definitions. We have proposed a methodology in two steps to bring their definitions to a "common denominator" and we have chosen to work with "left-algebras" and with $(\rightarrow, 1)$ as principal primitive operations, i.e. we have chosen to reconsider all these algebras as particular cases of (bounded) reversed left-BCK algebras, in order to be closer to the logic and to be able to make the connections with Hilbert algebras.

We then have redefined the involved algebras **gradually** (by adding more and more properties) as particular cases of (bounded) reversed left-BCK algebras, but with new names (as Wajsberg algebra redefines MV algebra). For example, the new name for BL algebra, redefined as a particular case of bounded reversed left-BCK algebra, is Hájek(P) algebra. Thus, one can find now in this paper the properties of weak-Hájek(P) algebras (MTL algebras), bounded divisible BCK(P) lattices (bounded divisible residuated lattices) and Hájek(P) algebras (BL algebras) - for examples - divided into three groups: those coming from the fact that they are bounded BCK(P) algebras, those coming from the fact that they are lattices (bounded BCK(P) lattices) and finally those coming from (prel) or/and (div) conditions, thus understanding better their common properties and the differences.

We have presented some new properties and hence new algebras. We have studied the properties (conditions) (DN), (WNM), but also (P1), (P2), (C), (G), (chain). Some open problems were identified.

Note that in this Part I, we have "generalized" many of the bounded algebras involved in the study, finding generalized versions for the others remaining an open problem. We can make the following resuming table of the correspondence "generalized-bounded algebra"-"bounded algebra".

"X algebra" (alg. only with 1) $\hookrightarrow$ generalized-"Y algebra"	bounded "X algebra" $\hookleftarrow$ "Y algebra" (bounded alg.)
BCK algebra $\hookrightarrow$	bounded BCK algebra
BCK(P) algebra $\hookrightarrow$ $\cong$ pocrim $\hookrightarrow$	bounded BCK(P) algebra $\cong$ bounded pocrim
pos. implic. BCK algebra $\hookrightarrow$ $\equiv$ Hilbert algebra $\hookrightarrow$	bounded pos. implic. BCK algebra $\equiv$ bounded Hilbert algebra
BCK(P) lattice $\hookrightarrow$ $\cong$ residuated lattice $\hookrightarrow$	bounded BCK(P) lattice $\cong$ bounded residuated lattice
divisible BCK(P) lattice $\hookrightarrow$ $\cong$ divisible residuated lattice $\hookrightarrow$	bounded divisible BCK(P) lattice $\cong$ bounded div. residuated lattice
div.BCK(P) lat. Gödel type $\hookrightarrow$ $\cong$ rel.pseudocompl. lattice $\hookrightarrow$	bounded div.BCK(P) lat. Gödel type $\cong$ bounded rel. pseudocompl. lattice $=$ Heyting algebra
BCK(P) lattice+(prel) $\cong$ residuated lattice + (prel)	$\hookleftarrow$ weak-Hájek(P) algebra $\cong$ $\hookleftarrow$ weak-BL algebra = MTL algebra
BCK(P) lattice+(prel)+(div) $\cong$ **res. lattice+(prel)+(div)** basic hoop	$\hookleftarrow$ Hájek(P) algebra $\cong$ $\hookleftarrow$ BL algebra
$\vee$-commutative BCK algebra $\vee$**-comm. BCK(P) algebra** $\equiv$ BCK(P) lattice + (C) Wajsberg hoop	$\hookleftarrow$ Wajsberg algebra
$\vee$-commutative pocrim	$\hookleftarrow$ MV algebra
implicative BCK algebra $\equiv$ implication algebra	$\hookleftarrow$ Boolean algebra

We believe that a comprehensive "Mendeleev-type" table of all algebras related to logic (or a set (an atlas) of many small "Mendeleev-type" tables, built on collections of such algebras, after the model from [Iorgulescu 2007] Figure 2), together with a comprehensive "map" of the hierarchies of all these algebras (or a set of many small "maps" of hierarchies corresponding to those collections of algebras) will give a more clear and precise view of the domain (mathematical logic and algebraic logic), very useful especially for the beginners in the field.

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