Journal of Universal Computer Science, vol. 13, no. 11 (2007), 1501-1513
submitted: 10/9/07, accepted: 19/11/07, appeared: 28/11/07 © J.UCS

Spectral Densest Subgraph and Independence Number of a Graph[1]

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Abstract: In this paper, we study spectral versions of the densest subgraph problem and the largest independence subset problem. In the first part, we give an algorithm for identifying small subgraphs with large spectral radius. We also prove a Hoffman-type ratio bound for the order of an induced subgraph whose spectral radius is bounded from above.
Key Words: graphs, eigenvalues, densest subgraph, independence number
Category: F.4.1

1 Introduction

Spectral graph theory is the study of eigenvalues and eigenvectors of graphs. These provide relevant information regarding the structure of a graph which is especially important when we are dealing with large graphs.

The eigenvalues we will discuss in this paper are the eigenvalues λ_i of the adjacency matrix A of a graph G, indexed so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The greatest eigenvalue, λ_1, is also called the spectral radius. If G is d-regular, then it is easy to see that $\lambda_1 = d$ and also, $\lambda_2 < d$ if and only if G is connected. Note that there are also other matrices that one can associate to a graph, namely the combinatorial Laplacian and the normalized Laplacian (see [Chung 1997, Grone and Merris 1994] for more details on these matrices and their eigenvalues properties).

The spectral radius of a graph is related to the chromatic number, the independence number and the clique number of a graph, all of which are hard to compute in general(see [Haemers 1995, Hoffman 1970, Nikiforov 2001, Wilf 1986]). Bollobás and Nikiforov [Bollobás and Nikiforov 2007] have also found connections regarding the spectral radius and the number of complete subgraphs of a

[1] C. S. Calude, G. Stefanescu, and M. Zimand (eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*
[2] Research supported by an NSERC Postdoctoral Fellowship

connected graph. There are results in the literature which show that the spectral radius is connected to the spread of viruses in a network (see [Wang et al. 2003, Jamakovic et al.]). The spectral radius is also closely related to the top singular value of a directed graph's adjacency matrix, which plays a central role in Kleinberg's hubs-and-authorities algorithm for web ranking [Kleinberg 1998].

The second largest eigenvalue λ_2 of a d-regular graph contains important information regarding the expansion properties of the graph. Informally, a connected d-regular graph whose second largest eigenvalue is small (see [Cioabă 2006, Nilli 2004] for more details on how small λ_2 can be) is a very good expander. Expanders are sparse graphs with high connectivity properties which are very useful in various applications in computer science, error-correcting codes, network design among others (see the recent survey [Hoory et al. 2006] by Hoory, Linial and Wigderson for more details on expanders properties and applications).

The other eigenvalues can reflect structural properties of a graph as well. The smallest eigenvalue λ_n is closely connected to the independence number of a graph ([Haemers 1995, Hoffman 1970]) as well as the maximum cut of a graph ([Alon and Sudakov 2000]). In [Brouwer and Haemers 2005, Cioabă et al.], the third largest eigenvalue λ_3 of a connected regular graph has been studied in connection with the size of the largest matching of the graph.

The spectral radius is related to density of a connected graph G. If G has n vertices and e edges, then it is well known (see [Cvetkovic et al. 1980]) that

$$\lambda_1 \geq \frac{2e}{n} \tag{1}$$

with equality if and only if G is regular. This inequality can be slightly improved when the graph is irregular (see [Cioabă and Gregory 2007, Nikiforov a]). We discuss some basic properties of the eigenvalues in Section 2.

In Section 3, we consider the problem of finding small subgraphs with large spectral radius. Since the general problem of deciding whether there exists a subgraph having at most k vertices and spectral radius at least λ is NP-complete, we consider the following notion of approximation for the problem. We say an algorithm is a $(f(\lambda), g(k))$-approximation algorithm for the spectral radius problem if, whenever the input graph contains a set with spectral radius at least λ on at most k vertices, the algorithm returns a subgraph with spectral radius at least $f(\lambda)$ on at most $g(k)$ vertices. Our main result on this topic is a polynomial time $(\lambda/4, \Delta k^2)$-approximation algorithm for the spectral radius problem.

The key component of our algorithm is a subroutine that searches for a set with bounded size and large spectral radius near a specified starting vertex. This is related to the problem of finding a dense subgraph near a specified starting vertex, which was considered in [Andersen 2007]. The subroutine developed in this paper allows us to obtain a better approximation ratio for the spectral radius problem than we could obtain by applying the algorithm from [Andersen 2007];

by searching directly for a subgraph with large spectral radius we obtain the stated approximation ratio of $(\lambda/4, \Delta k^2)$, while searching for a dense subgraph would yield an approximation ratio of roughly $(\lambda/\log n, \Delta k^2)$.

In Section 4, we study large subgraphs with small spectral radius. We consider the following spectral generalization of the usual notion of an independent set. Recall that a subset of vertices S of a connected graph G is an independent set if the subgraph induced by S contains no edges. Given $\lambda > 0$, we call a subset T an λ-independent set if the subgraph induced by T has spectral radius at most λ. It is easy to see that an λ-independent set is a usual independent set when $\lambda \in [0, 1)$.

We prove a Hoffman-type ratio bound for the largest order of an λ-independent subset. This implies previous results from [Bilu 2006, Cardoso et al.]. We also improve a previous result from [Bollobás and Nikiforov 2007].

2 Preliminaries

Let A be the adjacency matrix of an undirected, connected graph, with $A_{i,j} = 1$ if $i \sim j$, and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. The *spectral radius* $\lambda_1(A)$ is the largest eigenvalue of A. The corresponding eigenvector is nonnegative.

It is well known (see [Cvetkovic et al. 1980]) that

$$\lambda_1 = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{x^t A x}{x^t x} \tag{2}$$

Definition 1. For a subset of vertices $S \subseteq V$, we define A_S to be the adjacency matrix restricted to the induced subgraph on S,

$$A_S(x,y) = \begin{cases} A(x,y) \text{ if } x \in S \text{ and } y \in S \\ 0 \text{ otherwise.} \end{cases}$$

The spectral radius of the induced subgraph on S is $\lambda_1(A_S)$, which we will sometimes write as $\lambda_1(S)$. It follows easily from (2) that if $S \subseteq T$, then $\lambda_1(A_S) \leq \lambda_1(A_T)$.

The *support* of a vector x is the set of vertices on which x is nonzero. The one-step neighborhood $N_1(S)$ of a set S is the set of vertices that are in S or reachable within 1 hop from S.

3 Finding a small subgraph with large spectral radius

The (k, λ)-spectral radius problem is the problem of deciding, given a graph G and values for k and λ, whether there exists a subgraph of G on at most k

vertices whose spectral radius is at least λ. This problem is NP-complete by a reduction from k-clique, which follows from the fact that a graph on k vertices has spectral radius at least $k-1$ if and only if it is a k-clique.

In this section, we present an approximation algorithm for the spectral radius problem. We say an algorithm is a $(f(\lambda), g(k))$-approximation algorithm for the spectral radius problem if, whenever the input graph contains a set with spectral radius at least λ on at most k vertices, the algorithm returns a subgraph with spectral radius at least $f(\lambda)$ on at most $g(k)$ vertices. The following theorem states that our algorithm, which will be described later in this section, is a $(\lambda/4, \Delta k^2)$-approximation algorithm for the spectral radius problem.

Theorem 2. *There is an algorithm* `SpectralRadius(k)` *that takes as input a graph and an integer k. The algorithm runs in time $O(n\Delta k^2)$, where Δ is the maximum degree of the graph. If there exists a subgraph S on at most k vertices whose spectral radius is λ, the algorithm outputs a subgraph S' satisfying $\lambda_1(S') \geq \lambda/4$ and $|S'| \leq \Delta k^2$.*

The proof of this theorem is given at the end of this section. First, we will present a key subroutine that we will use in the proof— an algorithm that searches for a set with bounded size and large spectral radius near a specified starting vertex. We will prove Theorem 2 by applying this subroutine at many different starting vertices, and taking the best answer obtained.

3.1 Finding a subgraph with large spectral radius near a given vertex

In this section we develop a subroutine `FindSubgraph(v,k)` that searches for a set with bounded size and large spectral radius near a specified starting vertex. The input to the subroutine is a graph, along with a *starting vertex* v and a *target size* k. The set returned by the algorithm always has size at most at most Δk^2. We will prove that whenever there exists a subgraph S with at most k vertices and spectral radius λ, there is at least one starting vertex $v \in S$ for which the set returned by `FindSubgraph(v,k)` has spectral radius at least $\lambda_1(S)/4$.

The subroutine `FindSubgraph(v,k)` is similar to the algorithm analyzed by Andersen in [Andersen 2007] for finding a dense subgraph near a specified starting vertex. Finding a dense subgraph is closely related to finding a subgraph with large spectral radius, since the spectral radius of a graph is always within a factor of $\log n$ of the maximum density of a subgraph contained inside it (see [Kannan and Vinay 1999]). We remark that one can obtain an $(f(\lambda), \Delta k^2)$-approximation algorithm for the spectral radius problem by using the algorithm from [Andersen 2007] to find a small dense subgraph, and this method yields an approximation ratio of roughly $f(\lambda) = \lambda/\log n$. Using the subroutine we develop here, which directly finds a subgraph with large spectral radius rather than a dense subgraph, we obtain a constant-factor approximation guarantee $f(\lambda) = \lambda/4$.

The main computation performed by $\texttt{FindSubgraph}(\texttt{v}, \texttt{k})$ is to generate a sequence of vectors, $x_0, \ldots, x_T$, where the initial vector x_0 is the indicator vector for the starting vertex v. At each step, we multiply the current vector by the adjacency matrix A, as in the power method, and then perform a pruning step that sets to zero each entry of the vector whose contribution to the norm of the vector is below a certain threshold. This pruning step ensures that the support of the vector is small at each step.

Definition 3. Given a vector x, we define $\mathrm{prune}(\epsilon, x)$ to be the vector obtained by setting to zero any entry of x whose value is at most $\epsilon\|x\|$,

$$[\mathrm{prune}(\epsilon, x)](u) = \begin{cases} x(u) & \text{if } x(u) > \epsilon\|x\|, \\ 0 & \text{otherwise.} \end{cases}$$

We now state the subroutine $\texttt{FindSubgraph}(\texttt{v}, \texttt{k})$. At each step, we compute the growth rate $g_t = \|x_t A\|/\|x_t\|$, and consider the set $S_t = N_1(\mathrm{Support}(x_t))$, which is the set of vertices within 1 step of the support of x_t. The algorithm outputs the set S_{t_*}, where t_* is the time step where the largest growth rate was observed.

$\texttt{FindSubgraph}(\texttt{v}, \texttt{k})$
Input: A starting vertex v and a target size k.
Output: A set of vertices S'.

1. let $T = \log(\sqrt{k})$, and let $\epsilon_t = 2^t/k$.
2. For $t = 0, \ldots, T$, compute the vector x_t iteratively using the following rule:

$$x_0 = 1_v,$$
$$x_{t+1} = \mathrm{prune}(\epsilon_{t+1}, x_t A).$$

3. For each t, compute the growth rate $g_t = \|x_t A\|/\|x_t\|$, and let t_* be value of t for which g_t is largest.
4. Output the set $S_* = N_1(\mathrm{Support}(x_{t_*}))$, which is the set of vertices within 1 step of the support of x_{t_*}.

Theorem 4. *The algorithm* $\texttt{FindSubgraph}(\texttt{v}, \texttt{k})$ *runs in time* $O(\Delta k^2)$*, and always outputs a subgraph with at most* Δk^2 *vertices, where* Δ *is the maximum degree in the graph.*

Theorem 5. *Let S be a subgraph on at most k vertices whose spectral radius is λ. Then, there exists at least one vertex $v \in S$ for which* $\texttt{FindSubgraph}(\texttt{v}, \texttt{k})$ *outputs a subgraph S' satisfying $\lambda_1(S') \geq \lambda/4$.*

Proof **(of Theorem 4).** We can bound the running time of $\texttt{FindSubgraph}(\texttt{v}, \texttt{k})$ by bounding the number of vertices in the support of x_t at each step. The pruning step ensures that x_t is at least $\epsilon_t \|x_t\|$ wherever it is nonzero, so we have

$$\|x_t\|^2 \geq |\text{Support}(x_t)|\epsilon_t^2.$$

This implies that the support of x_t satisfies $|\text{Support}(x_t)| \leq 1/\epsilon_t^2$.

At each step of the algorithm we can compute $x_t A$ and x_{t+1} from x_t, and compute the norm of $x_t A$, in time proportional to the sum of the degrees of the vertices in $\text{Support}(x_t)$, which is at most

$$O(\Delta|\text{Support}(x_t)|) = O(\Delta/\epsilon_t^2) = O(\Delta k^2 2^{-2t}).$$

The total running time is therefore

$$\sum_{t=0}^{T} O(\Delta k^2 2^{-2t}) = O(\Delta k^2).$$

The set S' output by the algorithm is equal to $N_1(\text{Support}(x_t))$ for some t, so we have $|S'| \leq \Delta/\epsilon^2 = \Delta k^2$.

Before we prove Theorem 5, we need the following lemma, which identifies at least one starting vertex within S for which we can give a good lower bound on the growth rate of the norms $\|x_t\|$.

Lemma 6. *Let S be a subset of vertices. For at least one vertex v in S, there is a nonnegative unit vector ψ such that*

1. $\text{Support}(\psi) \subseteq S$,
2. $\psi A_S \geq \lambda_1(A_S)\psi$,
3. $\psi(v) \geq \frac{1}{\sqrt{|S|}}$.

Proof. Let ψ be an eigenvector of A_S with eigenvalue $\lambda_1(A_S)$, normalized so $\|\psi\| = 1$. It is easy to see that ψ satisfies properties (1) and (2). At least one vertex v must satisfy $\psi(v) \geq 1/\sqrt{|S|}$, since otherwise we would have $\|\psi\| < 1$. Together, this v and ψ satisfy property (3).

Proof **(of Theorem 5).** Let $S_t = N_1(\text{Support}(x_t))$ be the set of vertices within one step of the support of x_t. The spectral radius $\lambda_1(S_t)$ of the induced subgraph on S_t is at least as large as the growth rate $g_t = \|x_t A\|/\|x_t\|$, by the following equation.

$$\|x_t A\| = \|x_t A_{S_t}\| \leq \|x_t\| \lambda_1(S_t). \tag{3}$$

If at any step the growth rate satisfies $g_t \geq \lambda/4$, then the subgraph output by the algorithm will have spectral radius at least $\lambda/4$, as desired. For the remainder of the proof, we will assume the growth rate is less than $\lambda/4$ at each step, which implies that for every $t \leq T$,

$$\|x_t\| \leq \|x_{t-1} A\| < \|x_0\|(\lambda/4)^t = (\lambda/4)^t. \tag{4}$$

From this, we will derive a contradiction.

We now make an assumption on the starting vertex v. We assume there exists a nonnegative vector ψ with the following properties: $\psi A \geq \lambda \psi$, $\text{Support}(\psi) \subseteq S$, and $\psi(v) \geq 1/\sqrt{|S|}$. We know from Lemma 6 that such a vector exists for at least one vertex in S. Under this assumption, we will prove the following lower bound on the inner product of x_t with ψ.

$$\langle x_t, \psi \rangle \geq \frac{1}{\sqrt{|S|}} (\lambda/2)^t \quad \text{for every } t \leq T. \tag{5}$$

When we prove equation (5), it will contradict equation (4) when $t = T = \log(\sqrt{|S|})$, and we will be done.

We will prove that equation (5) holds by induction. We know it holds for $t = 0$. The only difficulty in the induction step is to bound the effect of the pruning step on the projection of x_t onto ψ. We define r_t to be the vector that is removed during the pruning step.

$$r_t = x_{t-1} A - x_t.$$

Let r'_t be the vector that is equal to r_t on the support of S, and 0 elsewhere. Since the support of ψ is contained in S, we have

$$\langle r_t, \psi \rangle = \langle r'_t, \psi \rangle \leq \|r'_t\| \|\psi\| = \|r'_t\|.$$

Since the value of r_t at any given vertex is at most $\epsilon_t \|x_{t-1} A\|$, we have

$$\langle r_t, \psi \rangle \leq \|r'_t\| \leq \epsilon_t \|x_{t-1} A\| \sqrt{|S|}.$$

We will now complete the proof by induction by assuming that $\langle x_{t-1}, \psi \rangle \geq (1/\sqrt{|S|})(\lambda/2)^{t-1}$ and proving that $\langle x_t, \psi \rangle \geq (1/\sqrt{|S|})(\lambda/2)^t$.

$$\begin{aligned}
\langle x_t, \psi \rangle &= \langle x_{t-1} A, \psi \rangle - \langle r_t, \psi \rangle \\
&\geq \lambda \langle x_{t-1}, \psi \rangle - \epsilon_t \|x_{t-1} A\| \sqrt{|S|} \\
&\geq \lambda (1/\sqrt{|S|})(\lambda/2)^{t-1} - \epsilon_t \sqrt{S} (\lambda/4)^t \\
&= \left((2/\sqrt{|S|}) - \epsilon_t 2^{-t} \sqrt{S} \right) (\lambda/2)^t \\
&\geq (1/\sqrt{|S|})(\lambda/2)^t.
\end{aligned}$$

The last step follows because we have set ϵ_t so that

$$\epsilon_t = 2^t/k \leq 2^t/|S|.$$

We can now prove the main theorem of this section by describing the algorithm `SpectralRadius(k)`.

Proof **(of Theorem 2).** The algorithm `SpectralRadius(k)` runs the subroutine `FindSubgraph(v, k)` for each vertex v in the graph, and outputs the subgraph with largest spectral radius found from any vertex. The running time and approximation guarantee are immediate from Theorem 4 and Theorem 5.

4 Large induced subgraphs with small spectral radius

Let G be an undirected and connected graph. A subset of vertices S of G is called independent if no two vertices in S are adjacent. The independence number $\alpha(G)$ of a graph G is the largest size of an independent set. The vertex cover number $\beta(G)$ of G is the smallest cardinality of a subset of vertices that is incident to all the edges of G. It is easy to see that $\alpha(G) + \beta(G) = n$ for any graph G on n vertices.

Given $\lambda > 0$, a subset $S \subset V(G)$ is called λ-independent if the subgraph induced by S has spectral radius at most λ. Let $\alpha_\lambda(G)$ denote the largest order of a λ-independent subset of G. Obviously, $\alpha_0(G)$ equals the independence number $\alpha(G)$ of G. Actually, since $\lambda_1(H) \geq 1$ for any nonempty graph H, it follows that $\alpha_0(G) = \alpha_\lambda(G)$ for any $\lambda \in [0, 1)$. If $\lambda < \mu$, then $\alpha_\lambda(G) \leq \alpha_\mu(G)$. Also, if λ equals the spectral radius of G, then $\alpha_\lambda(G) = n$.

Recall that a partition $(A_1, \ldots, A_l)$ of the vertex set of a graph G is called equitable the following happens for any $1 \leq i, j \leq l$: for any $x \in A_i$, the number of neighbours of x which are contained in A_j depends only on i and j.

Theorem 7. *Let G be a connected d-regular graph on n vertices. Then*

$$\alpha_\lambda(G) \leq n\frac{\lambda - \lambda_n}{d - \lambda_n} \tag{6}$$

Equality holds iff G has an equitable partition (S, S^c) with S inducing a λ-regular subgraph.

Proof. Let S be a subset of $V(G)$ such that the subgraph induced by S has spectral radius at most λ. Denote by d_S the average degree of the subgraph induced by S. Consider the partition of $V(G) = S \cup S^c$ which gives the following quotient matrix (see [Haemers 1995])

$$\begin{bmatrix} d_S & d - d_S \\ d - d_{S^c} & d_{S^c} \end{bmatrix}$$

It is known (see [Haemers 1995]) that the eigenvalues of the quotient matrix interlace the eigenvalues of G. Since the eigenvalues of the quotient matrix are d and $\frac{nd_S - s}{n-s}$, it follows that

$$\lambda_n(G) \leq \frac{nd_S - s}{n - s}$$

which implies

$$s \leq n\frac{d_S - \lambda_n}{d - \lambda_n}$$

Since $d_S \leq \lambda$, this proves inequality (6). Equality happens in the previous inequality if and only if the subgraph induced by S is λ-regular.

Equality happens if and only the partition (S, S^c) is equitable and S induces a λ-regular subgraph of G.

This result implies Corollary 4.4 from [Cardoso et al.].

The λ-clustering number $\chi_\lambda(G)$ of a connected graph G is the minimum k such that G has a partition into k subsets such that each subset induces a subgraph whose spectral radius is at most λ (see [Bilu 2006]). Bilu (see Theorem 3 in [Bilu 2006]) showed that the λ-clustering number of a graph is at least $\frac{\lambda_1 - \lambda_n}{\lambda - \lambda_n}$.

It follows easily that $\chi_\lambda(G) \cdot \alpha_\lambda(G) \geq n$ for any connected graph G on n vertices. The previous results imply the following corollary which is Theorem 3 from [Bilu 2006] for regular graphs.

Corollary 1. *If G is a connected d-regular graph on n vertices, then*

$$\chi_\lambda(G) \geq \frac{d - \lambda_n}{\lambda - \lambda_n} \tag{7}$$

Another generalization of graph coloring was considered in [Linial et al.]. Given an integer t, let $mcc_t(G)$ be the smallest integer m such that the vertices of G can be t-colored so that no monochromatic component has cardinality exceeding m. It is easy to see that $mcc_t(G) = 1$ if and only if $\chi(G) \leq t$. Also, $mcc_t(G) = 2$ if and only if $\chi_1(G) \leq t$. In general, we have that $\chi_{t-1}(G) \leq mcc_t(G) \leq \chi_{\sqrt{t-1}}(G)$.

Inequality (6) can be extended to irregular graphs as follows. Our approach is inspired from [Godsil and Newman]. For $\lambda > 0$, let $S \subset V(G)$ be a λ-independent set on s vertices. Let λ_1 denote the spectral radius of G and let x be the positive eigenvector corresponding to λ_1 such that $\sum_{i \in S} x_i^2 = s$.

Let z be the restriction of x to the set S, that is $z_i = x_i$ if $i \in S$ and $z_i = 0$ otherwise. The matrix $B = A - \lambda_n I$ is positive semidefinite and thus, $y^t B y \geq 0$ for any $y \in \mathbb{R}^n$. Let $y = z - \frac{s}{x^t x}x$. Then the previous inequality implies

$$\left(z - \frac{s}{x^t x}x\right)(A - \lambda_n I)\left(z - \frac{s}{x^t x}x\right) \geq 0$$

Expanding the left hand-side, we obtain

$$z^tAz - 2s\frac{\lambda_1 x^t z}{x^t x} + s^2\frac{\lambda_1}{x^t x} - \lambda_n\left(z^t z - 2s\frac{x^t z}{x^t x} + \frac{s^2}{x^t x}\right) \geq 0$$

Since the subgraph induced by S has spectral radius less than λ, it follows that $z^tAz \leq \lambda z^t z = \lambda s$. Using this fact in the previous inequality and simplifying, we get

$$s \leq \frac{\lambda - \lambda_n}{\lambda_1 - \lambda_n} x^t x \tag{8}$$

Note that for regular graphs we can take x to be the all one vector in which case we retrieve Theorem 6.

We conclude this section with some remarks regarding the connections between the spectral radius of a graph and its structure. In [Nikiforov a], Nikiforov proved the following result.

Theorem 8. *If a connected graph G is K_{r+1}-free, then $\lambda_1(G) \leq \lambda_1(T_r(n))$, where $T_r(n)$ is the r-partite Turán graph on n vertices. Equality happens iff $G \sim T_r(n)$.*

We give now a new and short proof of this theorem.

Proof. If G is K_{r+1}-free, then, combining the results from [Erdös 1970] and [Biyikoglu and J. Leydold] (see also [Bollobás], Theorem 1.4, page 295), we deduce that there exists an r-partite graph H on n vertices whose degree sequence is larger than the degree sequence of G in the majorization order such that $\lambda_1(G) \leq \lambda_1(H)$. The result follows now easily using the results from [Feng et al. 2007].

In [Bollobás and Nikiforov 2007], Bollobás and Nikiforov proved the following theorem which again connects the spectral radius of a graph G and the number of complete subgraphs in G. For $i \geq 2$, let $k_i(G)$ denote the number of complete subgraphs on i vertices contained in G.

Theorem 9. *If G is a connected graph, then for $r \geq 2$,*

$$\lambda_1^{r+1}(G) - \sum_{s=2}^{r}(s-1)k_s(G)\lambda_1^{r+1-s}(G) \leq (r+1)k_{r+1}(G)$$

This result was used in [Bollobás and Nikiforov 2007] to prove the following theorem.

Theorem 10. *If G is a graph on n vertices and $r \geq 2$, then*

$$k_{r+1}(G) \geq \left(\frac{\lambda_1(G)}{n} - 1 + \frac{1}{r}\right)\frac{r(r-1)}{r+1}\left(\frac{n}{r}\right)^{r+1}$$

We now improve Theorem 9.

Theorem 11. *If G is a connected graph with clique number $\omega \geq 3$, then for $2 \leq r \leq \omega$, we have*

$$\lambda_1^{r+1}(G) - \textstyle\sum_{s=2}^{r}(s-1)k_s(G)\lambda_1^{r+1-s}(G) + \lambda_1^r - \sum_{s=2}^{r}(s-1)k_s(G)\lambda_1^{r-s}(G) \\ \leq (r+1)k_{r+1}(G)$$

Proof. Given $u \in V(G)$ and $l \geq 0$, let $w_l(u)$ denote the number of l-walks $u = u_1, u_2, \ldots, u_l$ starting at u. Denote by $w_l(G) = \sum_{u \in V(G)} w_l$ the number of l-walks in G. As shown in [Bollobás and Nikiforov 2007], for each $2 \leq s \leq \omega(G)$ and $l \geq 2$, we have that

$$w_{l+r}(G) - \sum_{s=2}^{r}(s-1)k_s(G)w_{l+r-s}(G) \leq \sum_{u \in V(G)} k_{r+1}(u)w_l(u) \tag{9}$$

In [Bollobás and Nikiforov 2007], the authors use now the inequality $w_l(u) \leq w_{l-1}(G)$ and proceed to deduce Theorem 9. Note that the stronger inequality $w_l(u) \leq w_{l-1}(G) - w_{l-1}(u)$ is true because

$$w_{l-1}(G) = \sum_{v \in V(G)} w_{l-1}(v) \geq w_{l-1}(u) + \sum_{v \sim u} w_{l-1}(v) = w_{l-1}(u) + w_l(u)$$

Using the inequality $w_l(u) \leq w_{l-1}(G) - w_{l-1}(u)$ in (9), we obtain

$$w_{l+r}(G) - \textstyle\sum_{s=2}^{r}(s-1)k_s(G)w_{l+r-s}(G) \\ \leq (r+1)k_{r+1}(G)W_{l-1}(G) - \sum_{u \in V(G)} k_{r+1}(u)w_{l-1}(u)$$

Using (9) again with $l-1$ instead of l, we obtain that

$$w_{l+r-1}(G) - \sum_{s=2}^{r}(s-1)k_s(G)w_{l+r-1-s}(G) \leq \sum_{u \in V(G)} k_{r+1}(u)w_{l-1}(u)$$

Combining the last two inequalities, we obtain that

$$w_{l+r}(G) - \textstyle\sum_{s=2}^{r}(s-1)k_s(G)w_{l+r-s}(G) + w_{l+r-1}(G) \\ - \sum_{s=2}^{r}(s-1)k_s(G)w_{l+r-1-s}(G) \leq (r+1)k_{r+1}(G)w_{l-1}(G) \tag{10}$$

The proof now proceeds similarly to [Bollobás and Nikiforov 2007]. It is known (see [Cvetkovic et al. 1980], page 44) that, there exists non-negative constants $c_1 > 0, c_2, \ldots, c_n$ such that $w_l(G) = \sum_{i=1}^{n} c_i \lambda_i^{l-1}(G)$ for each $l \geq 1$. Since G is connected and $\omega(G) \geq 3$, it follows that $\lambda_1(G) > |\lambda_i(G)|$ for each $2 \leq i \leq n$. This implies that for each fixed q, we have that

$$\lim_{l \to +\infty} \frac{w_{l+q}(G)}{w_{l-1}(G)} = \lambda_1^{q+1}(G)$$

Dividing (10) by $w_{l-1}(G)$ and using the above result, we deduce that

$$\lambda_1^{r+1}(G) - \sum_{s=2}^{r}(s-1)k_s(G)\lambda_1^{r+1-s}(G) + \lambda_1^r - \sum_{s=2}^{r}(s-1)k_s(G)\lambda_1^{r-s}(G) \\ \leq (r+1)k_{r+1}(G)$$

which finishes the proof.

It would be interesting to calculate α_λ for special families of graphs such as hypercubes or Kneser graphs for example. Computing $\alpha_\lambda(G)$ is NP-complete for $\lambda = O(1)$. At the same time, $\alpha_\lambda(G) = n$ if $\lambda = \lambda_1(G)$. Another interesting open problem is to determine in what range of λ, is computing α_λ NP-complete.

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