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On BCK Algebras - Part I.a: An Attempt to Treat Unitarily the Algebras of Logic. New Algebras[1]

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Abstract: Since all the algebras connected to logic have, more or less explicitely, an associated order relation, it follows that they have two presentations, dual to each other. We classify these dual presentations in "left" and "right" ones and we consider that, when dealing with several algebras in the same research, it is useful to present them unitarily, either as "left" algebras or as "right" algebras. In some circumstances, this choice is essential, for instance if we want to build the ordinal sum (product) between a BL algebra and an MV algebra. We have chosen the "left" presentation and several algebras of logic have been redefined as particular cases of BCK algebras.

We introduce several new properties of algebras of logic, besides those usually existing in the literature, which generate a more refined classification, depending on the properties satisfied. In this work (Parts I-V) we make an exhaustive study of these algebras - with two bounds and with one bound - and we present classes of finite examples, in bounded case.

In this Part I, divided in two because of its length, after surveying chronologically several algebras related to logic, as residuated lattices, Hilbert algebras, MV algebras, divisible residuated lattices, BCK algebras, Wajsberg algebras, BL algebras, MTL algebras, WNM algebras, IMTL algebras, NM algebras, we propose a methodology in two steps for the simultaneous work with them (the first part of Part I).

We then apply the methodology, redefining those algebras as particular cases of reversed left-BCK algebras. We analyse among others the properties Weak Nilpotent Minimum and Double Negation of a bounded BCK(P) lattice, we introduce new corresponding algebras and we establish hierarchies (the subsequent part of Part I).

Key Words: MV algebra, Wajsberg algebra, generalized-MV algebra, generalized-Wajsberg algebra, BCK algebra, BCK(P) lattice, residuated lattice, BL algebra, Hájek(P) algebra, generalized-BL algebra, divisible BCK(P) lattice, Hilbert algebra, Hertz algebra, Heyting algebra, weak-BL algebra, MTL algebra, IMTL algebra, WNM algebra, NM algebra, R_0 algebra, t-norm, pocrim

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1 Introduction

Surveying chronologically several algebras related to logic, we find they look rather very different:

[1] C. S. Calude, G. Stefanescu, and M. Zimand (eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*

Residuated lattices, the algebraic counterpart of logics without contraction rule, were introduced in 1924 by Krull; they have been investigated (cf. Kowalski-Ono [Kowalski and Ono 2001]) by the following researchers: Krull [Krull 1924], Dilworth in [Dilworth 1939], Ward and Dilworth in [Ward and Dilworth 1939], Ward in [Ward 1940], Balbes and Dwinger [Balbes and Dwinger 1974], Pavelka in [Pavelka 1979], Idziak in [Idziak 1984a] and others. Residuated lattices have been known under many names; cf. [Kowalski and Ono 2001], they are called *BCK lattices* in [Idziak 1984a], *full BCK-algebras* in [Okada and Terui 1999], FL_{ew}*-algebras* in [Ono and Komori 1985] and *integral, residuated, commutative l-monoids* in [Höhle 1995]; some of those definitions are free of 0. We consider in this paper their definition free of 0 (the smallest element), i.e. as posets (partially ordered sets) with greatest element 1.

Hilbert algebras were introduced in a dual form in 1950, by Leon Henk in [Henkin 1950], under the name "implicative model", as a model of positive implicative propositional calculus - an important fragment of classical propositional calculus introduced by Hilbert [Hilbert 1923], [Hilbert and Bernays 1934]. They are posets with 1. Cf. A. Diego [Diego 1966], it was Antonio Monteiro who has given the name "Hilbert algebras" to the dual algebras of Henkin's implicative models.

MV algebras were introduced in 1958, by C. C. Chang [Chang 1958], as a model of $\aleph_0$-valued Lukasiewicz logic. They are posets with 0 and 1.

An important class of residuated lattices is that of "divisible" residuated lattices (or "divisible integral, residuated, commutative l-monoids" [Höhle 1995]), introduced in 1965, in a dual, more general form, by Swamy [Swamy 1965]. The "divisible" residuated lattices are residuated lattices satisfying divisibility (div) condition, where:

(div) for all x, y, $x \wedge y = x \odot (x \rightarrow y)$.

Hence, they are posets with 1. A divisible residuated lattice is a hoop, according to [Blok and Ferreirim 2000], which is also $\vee-$semilattice. A duplicate name in the literature for bounded divisible residuated lattices is that of "bounded commutative Rl-monoid" [Rachunek and Salounová 2007] - not appropriate, in our opinion.

BCK algebras were introduced in 1966, by K. Iséki [Iséki 1966], starting from the systems of positive implicational calculus, weak positive implicational calculus by A. Church, and BCI, BCK-systems by C.A. Meredith (according to [Iséki and Tanaka 1978]). They are posets with 0. Latter on, BCK algebras with condition (S) and BCK lattices were introduced and studied by Iséki and his school. In connection with dual reversed BCK algebras, we shall need the conditions (P), (RP), (DN) and (lattice), where:

(P) for all x, y, there exists $\quad x \odot y \stackrel{notation}{=} min\{z \mid x \leq y \rightarrow z\}$,

(RP) for all x, y, z, $\quad x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$, (DN) (Double Negation) for all x, $(x^-)^- = x$,

(lattice) there exists $x \wedge y \overset{notation}{=} \inf\{x, y\}$ and $x \vee y \overset{notation}{=} \sup\{x, y\}$, for all x, y.

Wajsberg algebras were introduced in 1984, by Font, Rodriguez and Torrens [Font et al. 1984], but they were also considered earlier by Komori in the papers [Komori 1978], [Komori 1981], under the name of CN algebras; they are a model of $\aleph_0$-valued Lukasiewicz logic too, studied by Wajsberg in 1935 [Wajsberg 1935]. They are term equivalent to MV algebras [Font et al. 1984].

BL algebras were introduced in 1996, by P. Hájek in the papers [Hájek 1996], [Hájek 1998a], [Hájek 1998b], as a common generalization of MV algebras, Product algebras and Gödel algebras, in connection with continuous t-norms. They are bounded residuated lattices (i.e. residuated lattices with 0 too) satisfying (div) and the pre-linearity (prel) conditions, where:

(prel) for all x, y, $(x \to y) \vee (y \to x) = 1$.

An important class of bounded residuated lattices is that of MTL = weak-BL algebras. MTL algebras presented in [Esteva and Godo 2001] and weak-BL algebras presented in [Flondor et al. 2001] were introduced independently in 2001; they are duplicate names for the same structure. MTL algebras were introduced as algebraic model for the monoidal t-norm logic, a generalization of Hájek's Basic Logic, while weak-BL algebras were introduced as commutative weak-pseudo-BL algebras (pseudo-BL algebras [Georgescu and Iorgulescu 2000], [Di Nola et al. 2002a], [Di Nola et al. 2002b] being non-commutative generalizations of BL algebras). They are bounded residuated lattices satisfying (prel) condition.

WNM, IMTL and NM algebras, which are particular classes of MTL algebras [Esteva and Godo 2001], were introduced in 2001, too.

The first goal of this paper, Part I, is to clarify the connections between the surveyed algebras, which in many cases have been given several equivalent definitions, and to find unifying tools. It is achieved in Sections 1 and 2.

The second goal is to treat all the involved algebras unitarily, by redefining them as classes of BCK algebras, and study them gradually, from the general ones to the particular ones. It is achieved in Section 3.

The third goal is to study some properties of (bounded) BCK(P) algebras (BCK(P) lattices) and the new algebras obtained by adding new properties (conditions). The stress is on the conditions (DN), (WNM), but also (P1), (P2), (C), (G), (chain), where:

(WNM) (Weak Nilpotent Minimum) $(x \odot y)^- \vee [(x \wedge y) \to (x \odot y)] = 1$, for all x, y,

(P1) for all x, $x \wedge x^- = 0$,

(P2) for all x, y, z, $(z^-)^- \odot [(x \odot z) \to (y \odot z)] \leq x \to y$,

(C) (Chang) for all x, $x \vee y = (x \to y) \to y$,

(G) (Gödel) for all x, $x \odot x = x$,

(chain) for all x, y, $x \leq y$ or $y \leq x$.

This goal is achieved in Section 3.

In this paper, we shall use the universal algebra concepts of "algebra" (an *algebra* $\mathcal{A}$ is a pair (A, F), where A is a nonvoid set and F is a family of finitary operations on A) and "relational system" (a *relational system* $\mathcal{A}$ is a pair (A, R), where A is a nonvoid set and R is a family of (finitary) relations on A) from [Grätzer 1979]. More general, a *structure* (A, F, R) is a mixture of an algebra (A, F) and a relational system (A, R).

1.1 The clarification of the connections between the surveyed algebras

In order to clarify the connections between the surveyed algebras of logic, which in many cases have been given several equivalent definitions, let us make two remarks.

Remark 1.1 All the surveyed algebras have essentially an associated (attached) order relation (denoted by $\leq$), which usually does not appear explicitely in the definitions. The presence of an order relation $\leq$ implies the presence of the duality. Thus, each such algebra has a dual algebra, where the dual (inverse) order relation (denoted by $\geq$) acts (for each x, y, $x \geq y$ iff $y \leq x$); the dual of a t-norm is a t-conorm, the dual of a residuum is a co-residuum and vice-versa [Gottwald 1984], [Gottwald 1986], [Gottwald 1993].

Starting from this remark, we have obtained two criteria of classifications of the involved algebras.

• **First classification: left-algebras, right-algebras.** In order to clarify the connections between the surveyed algebras from the duality point of view, let us look at the definitions of the surveyed algebras: in BL algebras there is a t-norm $\odot$ as a principal primitive operation, while in MV algebras there is a t-conorm $\oplus$ as a principal primitive operation. In Hilbert algebras there are an implication (residuum) $\rightarrow$ and 1 as principal primitive operations, while in reversed Iséki's BCK algebras there are an implication (co-residuum) and 0 as principal primitive operations.

We shall give a name: "left-algebra" (when the principal primitive operations are the t-norms, 1 and/or the residua) **or "right-algebra"** (when the principal primitive operations are the t-conorms, 0 and/or the co-residua) **to each algebra of the pair of dual algebras**; the concepts of "left-algebra" and "right-algebra" are introduced in such a way that the two definitionally equivalent corresponding classes of algebras become a class containing only "left-algebras" and a class containing only "right-algebras". Thus, a BL algebra as defined by Hájek, is a “left-algebra”, while the MV algebra, as defined by Chang, is a “right-algebra”.

Thus, we can choose between left- and right-algebras. In some circumstances, this choice is essential, for instance if we want to build the ordinal sum (product) between a BL algebra and an MV algebra. **Our first unifying tool, when dealing with more than one algebra, consists in working only with “left-algebras”**.

• **Second classification: four possible definitions of left-algebras (right-algebras).** In order to clarify the connections between the surveyed “left-algebras” from principal primitive operations point of view, let us look at the definitions of left-algebras of logics and at the definitions of the corresponding logics; we remark that they are very different, in general: in algebras of logics, usually, the principal primitive operation is the t-norm, while in the corresponding logics, usually, the principal primitive operation is the implication = residuum. We can also remark that **basically**, “left-algebras” contain as principal primitive operations (belonging to the signature): either “a residuum $\rightarrow = \rightarrow_L$ and 1” or “a t-norm $\odot$ and 1”; a derived operation from the primitive operations (term function) can appear: either a t-norm $\odot$, associated to the residuum $\rightarrow$, or a residuum $\rightarrow$, associated to the t-norm $\odot$. We shall say that a “left-algebra” belongs either to “the general world of $\rightarrow$, 1” or to “the general world of $\odot$, 1” (see Figure 2).

Moreover, we can **refine** each of the two “general worlds” of “left-algebras” by remarking that “left-algebras” contain:

1. in “the general world of $\rightarrow$, 1”:

- 1.1: either “a residuum $\rightarrow$ and 1” as primitive operations and a derived associated t-norm $\odot$,

- 1.2: or “a residuum $\rightarrow$, an associated t-norm $\odot$ and 1” as primitive operations;

2. in “the general world of $\odot$, 1”:

- 2.1: either “a t-norm $\odot$ and 1” as primitive operations and a derived associated residuum $\rightarrow$,

- 2.2: or “a t-norm $\odot$, an associated residuum $\rightarrow$ and 1” as primitive operations.

Note that these four corresponding “worlds” of “left-algebras” exist simultaneously, they containing four corresponding definionally equivalent classes of “left-algebras”. None, one or maximum two such definitionally equivalent classes of “left-algberas” are usually appearing in the literature.

Thus, we can choose between the four definitions of “left-algebras”. **Our second unifying tool consists in working only with “left-algebras” defined by using ($\rightarrow$, 1) as principal primitive operations**.

Remark 1.2 All the surveyed algebras have at least one bound with respect to the associated order relation (i.e. there exist either the smallest element, denoted by 0, or the greatest element, denoted by 1). Among them, **many are bounded** (i.e. they have both 0 and 1) - but sometimes only one bound from the existing two appears in the signature; take for examples the MV algebras, the Wajsberg algebras, the BL algebras and the MTL algebras. **Several have only one bound**; take for examples the residuated lattices, the Hilbert algebras, the BCK algebras and the divisible residuated lattices.

Starting from this remark, we have obtained a third criterion of classification: the number of bounds.

• **Third classification: "two bounds" and "only one bound" left-algebras (right-algebras).**

In this paper Part I, we want to analyze both cases for each "left-algebra": "bounded (i.e. two bounds)" algebra and "only the bound 1" algebra, for each surveyed left-algebra; therefore, we shall define for left-algebras the "missing" algebra, either by "bounding" the algebra having "only the bound 1" (i.e. by adding the "missing" bound 0 in the signature), or by "generalizing" the bounded algebras. The definitions of the "bounding" operation and of the "generalizing" operation of left-algebras are as follows.

Definition 1.3 Let $\mathcal{A} = (A, G \cup \{1\})$ be an "X algebra", which has an order relation $\leq$ associated to $\mathcal{A}$, where $1 \in A$, 1 is the greatest element of A with respect to $\leq$ (i.e. $x \leq 1$, for all $x \in A$) and some axioms are satisfied. If there is a unique element $0 \in A$, satisfying $0 \leq x$, for all $x \in A$, then 0 is called the *zero (smallest element)* of $\mathcal{A}$. An "X algebra" with zero is called to be *bounded* and it will be denoted by: $\mathcal{A}^b = (A, G \cup \{0, 1\})$. Let $\mathbf{X}$ denote the class of "X algebras" and $\mathbf{X}^b$ denote the class of bounded "X algebras".

Definition 1.4 Let "Y algebra" be a bounded algebra of logic (i.e. an algebra $\mathcal{A} = (A, G \cup \{0, 1\})$, which has an order relation $\leq$ associated to $\mathcal{A}$, where $0, 1 \in A$, 0 is the smallest element of A, 1 is the greatest element of A, with respect to $\leq$, and some axioms are satisfied). Let "X algebra" denote an algebra with only the bound 1 (i.e. an algebra $\mathcal{A} = (A, F \cup \{1\})$, which has an order relation $\leq$ associated to $\mathcal{A}$, where $1 \in A$, 1 is the greatest element of A with respect to $\leq$ and some axioms are satisfied). If bounded "X algebra" is term equivalent to "Y algebra", we shall say that "X algebra" is a *generalized-"Y algebra"*. Let us denote by $\mathbf{Y}$ the class of "Y algebras" and by $\mathbf{Y}^g$ the class of generalized-"Y algebras".

Remarks 1.5

(0) The general concept of "generalized-Y algebra" and the above definition are new; for the definition, we have had in mind the following example

(see [Iorgulescu a] Remark 3.27): the implicative BCK algebra is a generalized-Boolean algebra, since a bounded implicative BCK algebra is (term equivalent to) a Boolean algebra.

(1) **There exist some bounded algebras "Y algebras", whose free of 0-reduct is a generalized-"Y algebra". In this case, we shall call the free of 0-reduct as "standard" generalization.**

Example 1.1 Take the bounded lattice (i.e. an algebra $(L, \wedge, \vee, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the idempotent, commutative, associative and absorption laws and $x \wedge 1 = x = x \vee 0$, for all $x \in L$); its free of 0-reduct is the lattice with 1 (i.e. an algebra $(L, \wedge, \vee, 1)$ of type $(2, 2, 0)$, satisfying the idempotent, commutative, associative and absorption laws and $x \wedge 1 = x$, for all $x \in L$). Since a bounded "lattice with 1" is a "bounded lattice", it follows that the lattice with 1 is the standard generalized-bounded lattice.

Example 1.2 Take the BL algebra (see Definition 2.10); its free of 0-reduct is a "residuated lattice which satisfies (div) and (prel)". Since bounded "residuated lattice which satisfies (div) and (prel)" is a BL algebra, it follows that "residuated lattice satisfying (div) and (prel)" is the standard generalized-BL algebra. Note that there exists also the "basic hoop" [Agliano et al.], which is a generalized-BL algebra too, by Definition 1.4. It remains to clarify elsewhere the connections between "basic hoops" and "residuated lattices satisfying (div) and (prel)".

(2) But, **there exist several bounded algebras "Y algebras", whose free of 0 -reduct is not a generalized-"Y algebra"**.

Example 2.1 Take the Boolean algebra (i.e. an algebra $(A, \wedge, \vee, ^-, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that $(A, \wedge, \vee, 0, 1)$ is a distributive lattice with 0 and 1, satisfying $x \wedge x^- = 0$ and $x \vee x^- = 1$, for all $x \in A$). Its free of 0-reduct is an algebra $(A, \wedge, \vee, ^-, 1)$ of type $(2, 2, 1, 0)$, such that $(A, \wedge, \vee, 1)$ is a distributive lattice with 1, satisfying $x \vee x^- = 1$, for all $x \in A$; let us call such an algebra a "Boo algebra". Since a bounded "Boo algebra" is an algebra $(A, \wedge, \vee, ^-, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that $(A, \wedge, \vee, 0, 1)$ is a distributive lattice with 0 and 1, satisfying $x \vee x^- = 1$, for all $x \in A$ - which is not a Boolean algebra, it follows that "Boo algebra" is not a generalized-Boolean algebra.

Example 2.2 Take the Wajsberg algebra (see Definition 2.9) (i.e. an algebra $(A, \rightarrow, 0, 1)$ of type $(2, 0, 0)$ such that, for all $x, y, z \in A$, axioms (W1), (W2), (W3) and (W4) hold). Its reduct free of 0 is an algebra $(A, \rightarrow, 1)$ of type $(2, 0)$ such that, for all $x, y, z \in A$, axioms (W1), (W2) and (W3) hold; let us call such an algebra a "Waj algebra". Since a bounded Waj algebra is an algebra $(A, \rightarrow, 0, 1)$ of type $(2, 0, 0)$, such that, for all $x, y, z \in A$, axioms (W1), (W2), (W3) hold and $0 \leq x$, for all x - which is not a Wajsberg algebra, it follows that Waj algebra is not a generalized-Wajsberg algebra. Later on, in [Iorgulescu a] Remarks 3.24, we shall present several generalized-Wajsberg algebras and we shall make a choice of one of them, to be studied in some respects.

(3) Note that to an "X algebra" with greatest element we can attach exactly one bounded "X algebra", while to a bounded algebra "Y algebra" we can attach more generalized-"Y algebras"; the standard generalization is taken into account in the sequel, whenever it exists. Note that $(Y^g)^b \cong Y$.

(4) The "bounding" operation is an easy one, while the "generalizing" operation is a difficult one in many cases; in this paper we have succeeded to find a generalized-Y algebra in some cases of Y algebras (bounded algebras), but several open problems still remain.

1.2 Two unifying tools

After clarifying the connections between the surveyed algebras of logic through the above three criteria, we found the unifying tools: we found that we can treat them unitarily by bringing to a "common denominator" their definitions, as in the survey paper [Iorgulescu 2003] and the preprint [Iorgulescu 2004], for bounded case. To achieve this, in **Section 2**, for the general case of algebras with only one bound, we shall present a "methodology (algorithm)" in two steps:
- the first step is to choose between the existing dual algebras: the "right-defined" or the "left-defined" algebra; our first unifying tool consists in working only with "left-algebras"; we have chosen to work with "left-defined" algebras, because most of the above mentioned algebras of logic were initially "left-defined";
- the second step is to choose between the four most usual definitions (signatures) of (bounded) "left-algebras":

$$1.1\ (\rightarrow, 1),\quad 1.2\ (\rightarrow,\ \odot, 1);\qquad\qquad 2.2\ (\odot,\ \rightarrow, 1),\quad 2.1\ (\odot, 1);$$

our second unifying tool consists in working only with the first sequence of principal primitive operations, i.e. to work only with implication and 1; we have chosen $\rightarrow$ and 1 in order to have the algebras of logic closer related to the corresponding logics and to be able to make the connections with Hilbert algebras.

Concluding, we have chosen to treat all the above mentioned algebras unitarily as special classes of reversed left-BCK algebras; in view of the above equivalences, there is no loss of generality in so doing.

Consequently, in **Section 3**, we apply the methodology and we redefine unitarily, as classes of (bounded) reversed left-BCK algebras, the surveyed algebras and study them gradually, from general ones to particular ones: the reversed left-BCK(P) algebras (which are termwise equivalent to pocrims (partialy ordered, commutative, residuated, integral monoids)), the reversed left-BCK(P) lattices (which are termwise equivalent to residuated lattices), the Hájek(P) algebras (which are termwise equivalent to BL algebras), the Wajsberg algebras (which are termwise equivalent to left-MV algebras), etc.

Starting from the remark that we can divide the properties of Hájek(P) algebras into three groups: those coming from the fact that they are bounded BCK(P) algebras, those coming from the fact that they are lattices (bounded BCK(P) lattices) and finally those coming from (div) and (prel) conditions (one will notice that very few properties of Hájek(P) algebras come from the two conditions (div) and (prel)), we shall study all the involved algebras by dividing them into three groups: (1) BCK(P) algebras which are not lattices, (2) BCK(P) algebras which are lattices (not verifying (prel) and (div) conditions) and (3) BCK(P) algebras which are lattices verifying (prel) or/and (div) conditions.

Thus, Section 3 will involve many old, but redefined, algebras and also new algebras, which are all particular cases of (bounded) BCK algebras, in an attempt to unify their treatment. Some hierarchies concerning bounded BCK(P) lattices verifying (DN), (WNM) conditions are finally presented.

In **Section 4**, the last of this Part I, we present conclusions, the resuming list of open problems presented in the previous sections and final remarks.

We overview now the contents of next Parts II-V [Iorgulescu b] - [Iorgulescu e]:

In Part II [Iorgulescu b], we find equivalent conditions with divisibility (div) and prelinearity (prel) and we decompose (div) and (prel) in the general case of BCK(P) lattices. We introduce then the new algebras: (bounded) α, β, γ, δ, ε, π algebras and $\alpha\beta$, $\alpha\gamma$, ..., $\alpha\beta\gamma\delta\varepsilon\pi$ algebras and we establish connections with the old algebras and hierarchies. Finally, we introduce and study the ordinal sum (product) of two bounded BCK algebras.

In Part III [Iorgulescu c], we give classes of examples of finite proper Wajsberg algebras (MV algebras) and of Wajsberg algebras satisfying (WNM) condition, of Hájek(P) algebras (BL algebras) and of Hájek(P) algebras satisfying (WNM) condition and, finally, of divisible bounded BCK(P) lattices and of divisible bounded BCK(P) lattices satisfying (WNM) condition.

In Part IV [Iorgulescu d], we give classes of examples of finite proper IMTL algebras and of NM algebras, of MTL algebras and of WNM algebras, of bounded $\alpha\gamma$ algebras and of bounded $\alpha\gamma$ algebras satisfying the condition (WNM).

In Part V [Iorgulescu e], we give classes of examples of finite proper bounded α, β, γ, $\beta\gamma$ algebras and BCK(P) lattices (residuated lattices), satisfying or not satisfying the conditions (WNM) and (DN). We give finally examples of some finite proper bounded BCK algebras.

The groundwork of this big research in five parts was made in 2003 and was announced on January 10^{th}, 2004, in the five preprints [Iorgulescu 2004] - [Iorgulescu 2004].

This Part I is a paper of "macromathematics" and not of "micromathematics", since it connects almost all algebras of logic in a comprehensive unifying study, we hope. We use essentially the results from the paper [Iorgulescu 2003] and from the preprint [Iorgulescu 2004], but the paper is self-contained as much as possible. Note that in [Iorgulescu 2003] and [Iorgulescu 2004] only the bounded case is considered. The old results are presented without proof.

Hence, this Part I is organized as follows:

1. Introduction - this section.

2. A methodology: How to bring to a “common denominator” the definitions of algebras of logic.

3. A unitary treatment of algebras of logic as particular cases of (bounded) reversed left-BCK algebras. New algebras.

4. Conclusions.

Sections 1 and 2 are treated in this first part of Part I, while Sections 3 and 4 are treated in [Iorgulescu a], the subsequent part of Part I.

2 A methodology: How to bring to a “common denominator” the definitions of algebras of logic

After surveying, in chronological order, in Section 1, some algebras connected with logic, we propose in this section a **methodology in two steps** to bring their definitions to a “common denominator”, in order to better see the connections between these algebras and to be able to treat them unitarily.

Note that in this paper, we deal simultaneously with the two cases:
- the more general case of algebras with one bound,
- the case of bounded algebras.

The methodology is defined for both cases.

Between algebras, or classes of algebras, or categories of algebras, let $\cong$ mean “are term equivalent”, or “are termwise equivalent” (“are definitionally equivalent”) , or “are categorically isomorphic”, respectively, and $=$ mean “is a duplicate name” through this paper. When $\mathbf{X} \cong \mathbf{Y}$, we shall also write $\mathbf{X}$ ($\mathbf{Y}$).

2.1 The first step: choose between “right” or “left” algebras

We know that there are two possible dual definitions of algebras of logic and we have given them names [Flondor et al. 2001], [Iorgulescu 2004], [Iorgulescu 2003]: “right-definition” and “left-definition”. When an algebra is “right-defined” (“left-defined”), we shall say that it is a *right-algebra (left-algebra*, respectively).

Hence, the notions of “right” and “left” algebras are dual; they are connected with the right-continuity of a t-conorm and with the left-continuity of a t-norm, respectively. We can also say that they are connected with the “positive (right)” cone and with the “negative (left)” cone, respectively, of a commutative l-group (i.e. commutative lattice-ordered group).

Roughly speaking, we could say that an algebra endowed with a partial order is a "right" algebra if it contains a t-conorm (and/or a coresiduum) and 0 as principal operations, and is a "left" algebra if it contains a t-norm (and/or a residuum) and 1 as principal operations (see [Iorgulescu a] Definition 3.6 of a residuum).

Recall that t-norms (triangular norms) and t-conorms were defined initially on the real interval $[0, 1]$, then on a poset with one bound [Iorgulescu 2003], as follows:

Definition 2.1 A binary operation $\bigcirc$ on the poset $(A, \leq)$ with smallest element 0 (with greatest element 1) is a *t-conorm* (*t-norm*) iff it is commutative, associative, non-decreasing (isotone) in the first argument and hence in the second argument too, and it has 0 (1) as neutral element.

Usually, a t-norm will be denoted by "$\odot$" and a t-conorm by "$\oplus$".

Recall also the following definitions:

(j) A *partially ordered, abelian (i.e. commutative), integral right-monoid* or a *right-pocim* for short is a structure $(A, \leq, \oplus, 0)$ such that: $(A, \leq)$ is a poset with smallest element 0, $(A, \oplus, 0)$ is a commutative right-monoid (i.e. $\oplus$ is commutative, associative and has 0 as neutral element) and $\oplus$ is non-decreasing in the first argument and hence in the second argument; *integral* means that the smallest element of the poset $(A, \leq)$ coincides with the neutral element of the commutative right-monoid.

(j') A *partially ordered, commutative, integral left-monoid* or a *left-pocim* for short is a structure $(A, \leq, \odot, 1)$ such that: $(A, \leq)$ is a poset with greatest element 1, $(A, \odot, 1)$ is an abelian left-monoid and $\odot$ is non-decreasing in the first argument and hence in the second argument too.

Recal also [Iorgulescu 2003], [Iorgulescu 2004] that the statement: "$\oplus$ is a t-conorm on the poset $(A, \leq)$ with smallest element 0" is equivalent to the statement: "the structure $(A, \leq, \oplus, 0)$ is a right-pocim" and that dually, the statement: "$\odot$ is a t-norm on the poset $(A, \leq)$ with greatest element 1" is equivalent to the statement: "the structure $(A, \leq, \odot, 1)$ is a left-pocim".

Examples.

1) Let $(B, \wedge, \vee, ^-, 0, 1)$ be a Boolean algebra. Then, $\wedge$ is a t-norm, $\vee$ is a t-conorm and their associated residua are defined as follows:
$x \to_\wedge y = (x \wedge y^-)^- = x^- \vee y$, $x \to_\vee y = (x \vee y^-)^- = x^- \wedge y = (x^- \to_\wedge y^-)^-$.
Hence, $(B, \wedge, \vee, ^-, 0, 1)$ is the left-Boolean algebra, while $(B, \vee, \wedge, ^-, 1, 0)$ is the right-Boolean algebra.

1') Let us consider the termwise equivalent Boolean algebras $(B, \wedge, ^-, 1)$ and $(B, \vee, ^-, 0)$. Then, $(B, \wedge, ^-, 1)$ is the left-Boolean algebra, while $(B, \vee, ^-, 0)$ is the right-Boolean algebra.

2) Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. Since $\wedge$ is a t-norm and $\vee$ is a t-conorm, it follows that $(L, \wedge, \vee, 0, 1)$ is the left-lattice, while the dual lattice, $(L, \vee, \wedge, 1, 0)$, is the right-lattice.

2') Generalization. Let $(L, \wedge, \vee)$ be a ("Dedekind") lattice. Then, $(L, \wedge, \vee)$ is the left-lattice, while the dual, $(L, \vee, \wedge)$, is the right-lattice. If $(L, \leq)$ is an "Ore" lattice (i.e. for all $x, y \in L$, there exist $\inf\{x, y\}$ and $\sup\{x, y\}$), then the dual "Ore" lattice is $(L, \geq)$, where $x \geq y$ iff $y \leq x$, for all $x, y \in L$. Then, the well-known bijection between "Ore" and "Dedekind" lattices can be formulated as follows: "$(L, \leq)$ iff $(L, \vee, \wedge)$" (right-lattices) and then the dual bijection is "$(L, \geq)$ iff $(L, \wedge, \vee)$" (left-lattices).

3) Let $(L, \wedge, \vee, 1)$ be a lattice with greatest element, 1 (one bound), as for example: if $\mathbf{Z}^- = \{0, -1, -2, -3, \ldots\}$ (the set of negative integers), then $(\mathbf{Z}^-, \wedge = \min, \vee = \max, 0)$ is a lattice with greatest element. Hence, $(L, \wedge, \vee, 1)$ is a left-lattice, which has no (self) dual.

4) Dually, let $(L, \wedge, \vee, 0)$ be a lattice with smallest element, 0 (one bound), as for example: if $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$ (the set of natural numbers=positive integers), then $(\mathbf{N}, \wedge = \min, \vee = \max, 0)$ is a lattice with smallest element. Hence, $(L, \vee, \wedge, 0)$ is a right-lattice, which has no (self) dual.

5) Let $(G, \wedge, \vee, +, -, 0)$ be a commutative l-group (lattice-ordered group). Let $G^+ = \{x \in G \mid x \geq 0\}$ be the positive cone and $G^- = \{x \in G \mid x \leq 0\}$ be the negative cone. Then, $+$ is a t-conorm pe $(G^+, \leq, 0)$ and is a t-norm on $(G^-, \leq, 0)$. Note that $x \leq y$ in G^+ iff $-x \geq -y$ in G^-. Hence, $(G^+, \leq, +, 0)$ is a right-pocim, while $(G^-, \leq, +, 0)$ is a left-pocim.

6) MV algebras and BCK algebras were initially defined as "right" algebras, while residuated lattices, Hilbert algebras, Wajsberg algebras and BL algebras were initially defined as "left" algebras.

Note that "right" and "left" algebras are not "symmetric" algebras, while a commutative l-group, for example, is symmetric, in the following sense: a commutative l-group is a kind of pair of a "left" algebra (its negative cone) and a "right" algebra (its positive cone).

The passage from the (definition of) "right" algebra to its dual, the (definition of) "left" algebra, is made by replacing: the t-conorm $\oplus$ by the t-norm $\odot$, the co-residuum $\rightarrow_R$ by the residuum $\rightarrow \overset{notation}{=} \rightarrow_L$ ("R" comes from "right", "L" comes from "left"), the negation $^{-_R}$ (where $x^{-_R} = x \rightarrow_R 1$) by the negation $^- \overset{notation}{=} {}^{-_L}$ (where $x^- = x \rightarrow 0$), 0 by 1 (and 1 by 0), the binary relation $\leq$ by its inverse relation, $\geq$.

The passage from the "left" algebra to its dual, the "right" algebra, is made by replacing: the t-norm $\odot$ by the t-conorm $\oplus$, the residuum $\rightarrow \overset{notation}{=} \rightarrow_L$ by the co-residuum $\rightarrow_R$, the negation $^- \overset{notation}{=} {}^{-_L}$ by the negation $^{-_R}$, 1 by 0 (and 0 by 1), the binary relation $\geq$ by its inverse relation, $\leq$.

We claim that it is useful, when a research concerns several related algebras, to define all the involved algebras in the same way, i.e. either as "right" algebras or as "left" algebras; in some circumstances, this choice is essential, for instance when we want to build the ordinal sum (product) between a BL algebra and an MV algebra. This is the first step of our methodology: to choose between "right" or "left" algebras.

We have chosen to work with "left" algebras, because most of the algebras of logic were initially left-defined. Note that we shall freely write $x \geq y$ or $y \leq x$ in that case.

Recall now the definitions of the surveyed algebras in Section 1, in chronological order too, by adding the definitions of the missing "bounded" or "generalized-" algebras.

- Residuated lattices have been introduced as "left" algebras. We shall use the following definition, free of 0:

Definition 2.2
1) A *residuated lattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 1)$ of type $(2, 2, 2, 2, 1)$, verifying:

($L^{\cdot,1}$) $(A, \wedge, \vee, 1)$ is a lattice with greatest element 1 (under $\geq$),

(X2) $(A, \odot, 1)$ is an abelian (i.e. commutative) left-monoid,

(RP) for all $x, y, z \in A$, $y \rightarrow z \geq x \iff z \geq x \odot y$.

1') A *bounded residuated lattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ verifying ($L^{0,1}$), (X2) and (RP), where:

($L^{0,1}$) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.

Let **R-L** and **R-L**^{b} denote the class (or the category) of residuated lattices and of bounded residuated lattices, respectively.

In a bounded residuated lattice we can define a negation, $^-$, by: $x^- = x \rightarrow 0$, for all x. A bounded residuated lattice verifying (DN) (Double Negation) condition is also called a "Girard monoid".

- Hilbert algebras were introduced as "left" algebras, as the dual of Henkin's "right" implicative models.

Definition 2.3 (see [Diego 1966])

A *Hilbert algebra* is an algebra $(A, \rightarrow, 1)$ of type $(2, 0)$, satisfying, for all $x, y, z \in A$:

(h1) $x \rightarrow (y \rightarrow x) = 1$,

(h2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(h3) if $x \rightarrow y = y \rightarrow x = 1$, then $x = y$,

or, equivalently [Diego 1966]

Definition 2.4 A *Hilbert algebra* is an algebra $(A, \rightarrow)$ of type (2), if A is a non-void set, satisfying, for all $x, y, z \in A$:
(H1) $(x \rightarrow x) \rightarrow x = x$,
(H2) $x \rightarrow x = y \rightarrow y$,
(H3) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,
(H4) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$.

- MV algebras were introduced as "right" algebras. We shall give here only the definition of left-MV algebras, which duals the right-definition presented in [Cignoli et al. 2000].

Definition 2.5 [Flondor et al. 2001], [Iorgulescu 2003] An *MV algebra* is an algebra $(A, \odot, ^-, 1)$ of type (2,1,0), satisfying, for all $x, y, z \in A$:
(MV1-L) $x \odot (y \odot z) = (x \odot y) \odot z$,
(MV2-L) $x \odot y = y \odot x$,
(MV3-L) $x \odot 1 = x$,
(MV4-L) $(x^-)^- = x$,
(MV5-L) $x \odot 1^- = 1^-$,
(MV6-L) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.

The MV algebras are bounded structures, where $0=1^-$, verifying (DN) condition. Note that an equivalent definition is the following, where the two bounds are visible:

Definition 2.6 An *MV algebra* is an algebra $(A, \odot, \rightarrow, 0, 1)$ of type (2,2,0,0), satisfying, for all $x, y, z \in A$:
(MV1-L) $x \odot (y \odot z) = (x \odot y) \odot z$,
(MV2-L) $x \odot y = y \odot x$,
(MV3-L) $x \odot 1 = x$,
(MV4-L) $(x \rightarrow 0) \rightarrow 0 = x$,
(MV5-L) $x \odot 0 = 0$,
(MV6-L) $(((x \rightarrow 0) \odot y) \rightarrow 0) \odot y = (((y \rightarrow 0) \odot x) \rightarrow 0) \odot x$.

The MV algebras come from commutative l-groups. Mundici [Mundici 1986b] proved that MV algebras are intervals in commuative l-groups.

We shall present later ([Iorgulescu a] Remarks 3.24) a generalized-MV algebra.

- An important class of residuated lattices is that of "divisible" residuated lattices.

Definition 2.7
(1) A *divisible* residuated lattice (or "divisible integral, residuated, commutative l-monoids", in [Höhle 1995], Lemma 2.5) is a residuated lattice satisfying the condition (div).

(1') A *bounded divisible* residuated lattice or a *divisible bounded* residuated lattice is a bounded residuated lattice satisfying the condition (div). A duplicate name in the literature is "bounded commutative Rl-monoid" - not appropriate, in our opinion.

Let **divR-L** and **divR-L**^{b} denote the class (or the category) of divisible residuated lattices and of bounded divisible residuated lattices, respectively.

A bounded divisible residuated lattice satisfies condition (DN) if and only if it is an MV algebra.

- The definitions of BCK algebras and of related algebras will be given in Section 3.

- Wajsberg algebras were introduced as "left" algebras [Font et al. 1984], as follows:

Definition 2.8 A *Wajsberg algebra* is an algebra $(A, \to, ^-, 1)$ of type $(2, 1, 0)$ such that, for all $x, y, z \in A$:
(W1) $1 \to x = x$,
(W2) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(W3) $(x \to y) \to y = (y \to x) \to x$,
(W4) $(x^- \to y^-) \to (y \to x) = 1$.

Note that an equivalent definition is the following, where the two bounds are visible:

Definition 2.9 A *Wajsberg algebra* is an algebra $(A, \to, 0, 1)$ of type $(2, 0, 0)$ such that, for all $x, y, z \in A$:
(W1) $1 \to x = x$,
(W2) $(y \to z) \to [(z \to x) \to (y \to x)] = 1$,
(W3) $(x \to y) \to y = (y \to x) \to x$,
(W4) $((x \to 0) \to (y \to 0)) \to (y \to x) = 1$.

We shall present later ([Iorgulescu a] Remarks 3.24) a generalized-Wajsberg algebra.

Wajsberg algebras are bounded algebras, with $0 = 1^-$, verifying condition (DN). Let **W** denote the class (or the category) of Wajsberg algebras.

Wajsberg algebras are termwise equivalent to MV algebras (see [Font et al. 1984], Theorems 4 and 5): **W** $\cong$ **MV**.

- BL algebras were introduced as "left" algebras. The starting point in defining and studying Basic Logic and BL algebras were the algebras of the form $([0, 1], \min, \max, \odot, \to, 0, 1)$, where $\odot$ is a continuous t-norm on $[0, 1]$ and $\to$ is the associated residuum; these algebras are called *standard* BL algebras. The most important continuous t-norms on $[0, 1]$ are the following three: Lukasiewicz t-norm, Product t-norm, Gödel t-norm. These three t-norms and their associated residua are the following:

(1) Lukasiewicz:

$$x \odot_L y = \max(0, x+y-1), \quad x \to_L y = \begin{cases} 1, \text{ if } x \leq y \\ 1-x+y, \text{ if } x > y \end{cases} = \min(1, 1-x+y);$$

(2) Product (Gaines):

$$x \odot_P y = xy, \quad x \to_P y = \begin{cases} 1, \text{ if } x \leq y \\ y/x, \text{ if } x > y, \end{cases} \quad \text{(Goguen implication)}$$

(3) Gödel (Brouwer):

$$x \odot_G y = \min(x, y), \quad x \to_G y = \begin{cases} 1, \text{ if } x \leq y \\ y, \text{ if } x > y, \end{cases} \quad \text{(Gödel implication).}$$

They correspond to the most significant fuzzy logics: Lukasiewicz logic, Product logic and Gödel logic, respectively. The MV algebras, the Product algebras and the Gödel algebras constitute the algebraic models for these three types of logics. The class of BL algebras contains the MV algebras [Chang 1958], [Cignoli et al. 2000], the Product algebras [Hájek et al. 1996], [Mangani 1973], [Hájek 1998a] and the Gödel algebras [Hájek 1998a] (or linear Heyting algebras (A. Monteiro; cf. L. Monteiro [Monteiro 1970]) or L-algebras (Horn [Horn 1969]); cf. [Boicescu et al. 1991]).

Definition 2.10
(1) A *BL algebra* [Hájek 1998a] is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$, such that:
($R^{0,1}$) $\mathcal{A}$ is a bounded residuated lattice,
(div) for all $x, y \in A$, $x \wedge y = x \odot (x \to y)$ (divisibility),
(prel) for all $x, y \in A$, $(x \to y) \vee (y \to x) = 1$ (pre-linearity).
(1') A *generalized-BL algebra* (standard generalization) is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, 1)$, such that ($R^{\cdot,1}$), (div), (prel) hold, where:
($R^{\cdot,1}$) $\mathcal{A}$ is a residuated lattice.

A BL algebra is an MV algebra iff it satisfies condition (DN) (Double Negation).

The BL algebra $([0, 1], \min, \max, \odot_L, \to_L, 0, 1)$, determined by the above Lukasiewicz t-norm, is the standard (canonical) (left-) MV algebra.

Definition 2.11 A *Product algebra* [Hájek 1998a] is a BL algebra $\mathcal{A}$ which fulfills the following two conditions:
(P1) for every $x \in A$, $x \wedge x^- = 0$,
(P2) for every $x, y, z \in A$, $(z^-)^- \odot [(x \odot z) \to (y \odot z)] \leq x \to y$.

The standard Product algebra is the BL algebra $([0,1], \min, \max, \odot_P, \rightarrow_P, 0, 1)$, determined by the above Product t-norm.

A BL algebra which fulfills the condition (P1) is usually called a *SBL algebra*. Let us name *SSBL algebras* the BL algebras fulfilling the condition (P2).

Let **Product**, **SBL**, **SSBL** denote the classes of Product algebras, SBL and SSBL algebras, respectively.

Definition 2.12 A *Gödel algebra* [Hájek 1998a] is a BL algebra $\mathcal{A}$ which fulfills the condition (G) (idempotent multiplication):

(G) for each $x \in A$, $x \odot x = x$.

The standard Gödel algebra is the BL algebra $([0,1], \min, \max, \odot_G, \rightarrow_G, 0, 1)$, determined by the above Gödel t-norm. Recall that Gödel algebras are exactly the Heyting algebras verifying (prel) condition (i.e. the linear Heyting algebras). Hence, a generalized-Gödel algebra is the relatively pseudocomplemented lattice verifying (prel) condition. Let **Gödel** denote the class of Gödel algebras.

Open problem 2.13 Define generalized-Product algebras.

- Another important class of bounded residuated lattices is that of weak-BL = MTL algebras.

Definition 2.14

(1) A *weak-BL algebra* [Flondor et al. 2001] or a *MTL (Monoidal t-norm Based) algebra* [Esteva and Godo 2001] is a bounded residuated lattice satisfying the condition (prel).

(1') A *generalized-weak-BL algebra* or a *weak-generalized-BL algebra* or a *generalized-MTL algebra* is a residuated lattices satisfying (prel).

- Let us recall also the following particular cases of MTL algebras presented in [Esteva and Godo 2001]:

Definition 2.15

(1) An *IMTL algebra (Involutive Monoidal t-norm based Logic)* is an MTL algebra satisfying the condition (DN).

(2) A *WNM algebra* is an MTL algebra satisfying the additional condition (WNM) (Weak Nilpotent Minimum):

(WNM) for all x, y, $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 1$.

(3) An *NM (Nilpotent Minimum) algebra* is an IMTL algebra satisfying the condition (WNM), or a WNM algebra satisfying the condition (DN).

Hence we have: **NM** = **IMTL** + (WNM) = **WNM** + (DN).

Open problem 2.16 Define generalized-WNM algebras, generalized-IMTL algebras and generalized-NM algebras.

Remarks 2.17 (1) We shall stress in Parts III [Iorgulescu c], IV [Iorgulescu d] of this paper, the importance of NM algebras versus MV algebras, that the class of MV algebras and the class of NM algebras are incomparable (with respect to set inclusion).

(2) We shall obtain new algebras by adding the condition (WNM) to bounded residuated lattices.

- Recall [Pei 2003] that the IMTL algebras, introduced in 2001 by Esteva and Godo [Esteva and Godo 2001], are termwise equivalent to weak-R_0 algebras, introduced in 1997 by G.J. Wang [Wang 1997] and that NM algebras are termwise equivalent to R_0 algebras, introduced also in 1997 by G.J. Wang [Wang 1997], [Pei 2003], as left algebras, as follows:

Definition 2.18 [Pei 2003]
(1) A *weak-R_0 algebra* is an algebra $\mathcal{M} = (M, \wedge, \vee, \rightarrow, ^-, 0, 1)$ of the type $(2, 2, 2, 1, 0, 0)$, such that:
• $(M, \wedge, \vee, 0, 1)$ is a bounded lattice, $\leq$ being the order relation,
• "$^-$" is an order reversing involution with respect to $\leq$,
• the following conditions hold: for all $x, y, z \in M$,
(R_1) $x^- \rightarrow y^- = y \rightarrow x$,
(R_2) $1 \rightarrow x = x$,
(R_3) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
(R_4) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(R_5) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$.
(2) An R_0 *algebra* is a weak-R_0 algebra verifying the additional condition (R_6):
(R_6) $(x \rightarrow y) \vee ((x \rightarrow y) \rightarrow (x^- \vee y)) = 1$.

Hence, we have: weak-R$_0$ $\cong$ **IMTL**, $\mathbf{R}_0$ $\cong$ **NM**.

2.2 The second step: choose between four possible definitions of left-algebras (right-algebras)

In left-algebras connected with commutative logic (residuated lattices, Hilbert algebras, MV algebras, Wajsberg algebras, BL algebras etc.) we have two (adjoint) operations: the implication (residuum) ($\rightarrow$) and the product (t-norm) ($\odot$). As it was largely developed in the survey-paper [Iorgulescu 2003], there are two main ways of studying these algebras as posets $(A, \geq, 1)$ with greatest element:

(1) either

1.1 to start only with the residuum $\rightarrow$ as a primitive operation (i.e. to start with the BCK algebra), and then its associated (derived) t-norm $\odot$ is defined, whenever it exists, by the condition:

(P) for all x, y, there exists $x \odot y \overset{notation}{=} min\{z \mid x \leq y \rightarrow z\}$, (see the definitions of BCK algebras, BCK(P) algebras, Wajsberg algebras etc.)

or, alternatively,

1.2 to start with both $\rightarrow$ and $\odot$ (in this order), verifying then the condition:

(RP) for all x, y, z, $\quad x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z$,

as very seldom is the case,

(2) or

2.1 to start only with the t-norm $\odot$ as a primitive operation (i.e. to start with the commutative partially ordered monoid) and then its asociated (derived) residuum $\rightarrow$ is defined, whenever it exists, by the condition:

(R) for all y, z, there exists $\quad y \rightarrow z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}$,

(see the definitions of commutative left-monoids, left-pocrims, Heyting algebras, MV algebras etc.)

or, alternatively,

2.2 to start with both $\odot$ and $\rightarrow$ (in this order), verifying then the condition:

(PR) for all x, y, z, $\quad x \leq y \rightarrow z \Longleftrightarrow x \odot y \leq z$,

as very often is the case (see the definitions of residuated lattices, BL algebras etc.).

Note [Iorgulescu 2003] that (RP)=(PR).

Thus, there exist four (two plus two) different basic definitions for a "left" algebra of logic having both $\odot$ and $\rightarrow$ and usually one, maximum two, among the four different types are used in the literature, for each algebra.

The four different basic definitions of algebras determine a **left-"Mendeleev-type" table (matrix) (Sergiu Rudeanu's remark) with** 4 **columns** and as many rows as distinct (i.e. not termwise equivalent) algebras are involved in the study. The given algebras will fill some cells of the table and the empty cells of each row will be filled with the "missing" algebras; the only problem is **the problem of names** for the "missing" algebras.

Since it was proved explicitely in [Iorgulescu 2003] (Theorems 2.13, 2.50, 2.55) the termwise equivalence existing between the algebras in Figure 1, the "four parents algebras", it follows that all the particular algebras, the "four descendents", sitting on the same row, bellow the row of the "parents", are termwise equivalent at their turns. Hence, between the algebras of the same row we should use the sign $\cong$.

We shall use the sign "$\equiv$" between the algebras of the same column which are termwise equivalent and we shall read it: "is an equivalent definition of".

$$\mathbf{r\text{-}BCK(P)} \underset{\pi^*}{\overset{\pi}{\rightleftarrows}} \mathbf{r\text{-}BCK(RP)} \underset{\gamma'}{\overset{\delta'}{\rightleftarrows}} \mathbf{X\text{-}BCK(RP)} \underset{\rho}{\overset{\rho^*}{\rightleftarrows}} \mathbf{X\text{-}BCK(R)} \equiv \text{pocrims}$$

Figure 1: The "four parents algebras", where r- means "reversed-".

Note that there exists a dual **(right-) "Mendeleev-type" table (matrix) with** 4 **columns**, determined by the four different possible definitions of right-algebras which are (particular cases of) BCK(S) algebras or dual pocrims.

We give now two examples of left-"Mendeleev-type" tables, one in the bounded case and the other in the one bound case.

Example 1. The following four bounded left-algebras: the reversed bounded BCK(P) algebras **(r-BCK(P)**^{b}**)**, the bounded pocrims, the bounded residuated lattices **(R-L**^{b}**)** and the BL algebras (**BL**) determine a "Mendeleev-type" table (matrix) with 4 columns and 3 rows, where only four cells are filled. We have then introduced the "missing" algebras [Iorgulescu 2003] in the empty cells. The complete table of all $12 = 4 \times 3$ algebras is presented in [Iorgulescu 2004], where the initial four algebras are marked by a bullet. Since the 12 algebras are (direct or indirect) generalizations (or ascendents) of Wajsberg algebras and of MV algebras, we have added a fourth row to the table, the row of Wajsberg and MV algebras, where we have filled only two columns.

Example 2. The following five generalized left-algebras: the reversed BCK(P) algebras **(r-BCK(P))**, the pocrims, the reversed BCK(P) lattices **(r-BCK(P)-L)**, the residuated lattices **(R-L)** and generalized-BL algebras (**BL**^{g}) determine a "Mendeleev-type" table (matrix) with 4 columns and 3 rows too, where only five cells are filled. One can introduce the "missing" algebras, and put them in the empty cells. The complete table of all $12 = 4 \times 3$ generalized algebras is presented in Figure 2, where the initial five algebras are marked by a bullet. Since the 12 algebras are (direct or indirect) ascendents (i.e. generalizations) of generalized-Wajsberg algebras and of generalized-MV algebras, we have added a fourth row to the table, the row of generalized-Wajsberg and generalized-MV algebras, not completely filled.

Recall that the axioms appearing in the tables from [Iorgulescu 2004] (the bounded case) and Figure 2 (the generalized case) are the following (see the papers [Iorgulescu 2003], [Iorgulescu 2004]):

$(A^{\cdot,1}) = (\mathrm{X}^{\cdot,1})$=(A1) $(A, \geq, 1)$ is a poset with greatest element 1,

$(A^{0,1}) = (\mathrm{X}^{0,1})$ $(A, \geq, 0, 1)$ is a poset with greatest element, 1, and smallest element, 0,

(A2) $(A, \rightarrow, 1)$ verifies: for all x, y, z, (R1) $1 \rightarrow x = x$, (R2) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,

(A3) $x \rightarrow y = 1 \Longleftrightarrow x \leq y$, for all x, y,

The world of generalized left-algebras			
The general world of $\rightarrow, 1$		The general world of $\odot, 1$	
The world of $\rightarrow, 1$	The world of $\rightarrow, \odot, 1$	The world of $\odot, \rightarrow, 1$	The world of $\odot, 1$
• **r-BCK(P)** $(A, \geq, \rightarrow, 1)$ $(A^{\cdot,1})$,(A2),(A3),(A4), (P)	**r-BCK(RP)** $(A, \geq, \rightarrow, \odot, 1)$ $(A^{\cdot,1})$, (A2), (A3), (RP)	**X-BCK(RP)** $(A, \geq, \odot, \rightarrow, 1)$ $(A^{\cdot,1})$, (X2), (RP)	• **X-BCK(R)** = pocrim $(A, \geq, \odot, 1)$ $(A^{\cdot,1})$, (X2), (X3), (R)
• **r-BCK(P)-L** $(A, \wedge, \vee, \rightarrow, 1)$ $(L^{\cdot,1})$,(A2),(A3),(A4), (P)	**r-BCK(RP)-L** $(A, \wedge, \vee, \rightarrow, \odot, 1)$ $(L^{\cdot,1})$, (A2), (A3), (RP)	• **X-BCK(RP)-L** **= R-L** $(A, \wedge, \vee, \odot, \rightarrow, 1)$ $(L^{\cdot,1})$, (X2), (RP)	**X-BCK(R)-L** **= X-R-L** $(A, \wedge, \vee, \odot, 1)$ $(L^{\cdot,1})$, (X2), (X3), (R)
r-Ha(P)^g $(A, \wedge, \vee, \rightarrow, 1)$ $(L^{\cdot,1})$,(A2),(A3),(A4), (P) (div),(prel)	**r-Ha(RP)**^{g} $(A, \wedge, \vee, \rightarrow, \odot, 1)$ $(L^{\cdot,1})$, (A2), (A3), (RP), (div), (prel)	• **X-Ha(RP)**^{g}= **BL**^{g} $(A, \wedge, \vee, \odot, \rightarrow, 1)$ $(L^{\cdot,1})$, (X2), (RP), (div),(prel)	**X-Ha(R)**^{g}= **X-BL**^{g} $(A, \wedge, \vee, \odot, 1)$ $(L^{\cdot,1})$, (X2), (X3), (R), (div), (prel)
W^g $(A, \rightarrow, 1)$		**MV**^{g} $(A, \odot, \rightarrow, 1)$	

Figure 2: The table with four columns corresponding to the four different definitions of generalized-algebras, where ^{g} means "generalized-"

(A4) $x \leq y \Longrightarrow z \to x \leq z \to y$, for all x, y, z,
(A5) $(x \to y) \to y = (y \to x) \to x$, for all x, y,
(A6) $(x^- \to y^-) \to (y \to x) = 1$,
(X2) $(A, \odot, 1)$ is an abelian (i.e. commutative) left-monoid,
(X3) $x \leq y \Rightarrow x \odot z \leq y \odot z$, for every x, y, z,
(X4) $x \odot 1^- = 1^-$, for all x,
(X5) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$, for all x, y,
(DN) $(x^-)^- = x$, for all x,
(P) there exists $x \odot y \overset{notation}{=} min\{z \mid x \leq y \to z\}$, for all x, y,
(R) there exists $y \to z \overset{notation}{=} \max\{x \mid x \odot y \leq z\}$, for all y, z,
(RP)= (PR) $x \odot y \leq z \Longleftrightarrow x \leq y \to z$, for all x, y, z,
$(L^{\cdot,1})$ $(A, \wedge, \vee, 1)$ is a lattice with last element, 1,
$(L^{0,1})$ $(A, \wedge, \vee, 0, 1)$ is a bounded lattice,
(div) $x \wedge y = x \odot (x \to y)$, for all x, y,
(prel) $(x \to y) \vee (y \to x) = 1$, for all x, y.

Remark 2.19 The above left-"Mendeleev-type" tables (matrices) with 4 columns can be completed with other rows, corresponding to other algebras of logic. Thus, a comprehensive left-"Mendeleev-type" table of all algebras related to logic which are particular cases of BCK(P) algebras (pocrims) (or a set of many small "Mendeleev-type" tables, built on groups of such algebras), together with a comprehensive "map" of the hierarchies of all these algebras (or a set of many small "maps" of hierarchies corresponding to those groups of algebras) will give - in our oppinion - a more clear and precise view of the domain (mathematical logic and algebraic logic) (as well as an atlas clarifies, for the reader, by its maps, the geographical position of one state versus other states, or a city map clarifies the geographical position of one building versus other buildings).

We claim that it is useful, when a research concerns several related left-algebras, to define all the involved algebras in the same manner, i.e. in only one of the four different ways (the same for right-algebras); this is the second step of our methodology: to choose between the four definitions.

We have chosen the first definition, (1.1), i.e. we have chosen to start only with implication (= residuum, see [Iorgulescu a] Definition 3.6) and 1, because in this way we are much closer to logic and we can make the connections with Hilbert algebras (while choosing (2.1), (2.2) we are closer to algebra, analysis). Hence, we have chosen to treat all the above mentioned algebras unitarily as special classes of reversed left-BCK algebras - instead of treating them unitarily as special classes of partially ordered (left-) monoids.

Consequently, in the next section, we shall start with $\to$ and 1, i.e. we shall work with (bounded) BCK(P) algebras, (bounded) BCK(P) lattices, (generalized) Hájek(P) algebras and (generalized) Wajsberg algebras (i.e. algebras from the first column of the table in Figure 2), etc.

Following these comments, weak R_0-algebras and R_0-algebras "go with" Hájek(P) algebras and with Wajsberg algebras in column 1 of the table in [Iorgulescu 2004], while IMTL algebras and NM algebras "go with" BL algebras, in the 3^{rd} column of that table. Consequently, we should normally refer to "weak-R_0 algebras, R_0 algebras, axiom (R_6), Hájek(P) algebras and Wajsberg algebras", but sometimes we shall refer to "IMTL algebras, NM algebras, axiom (WNM), BL algebras and even MV algebras (from the 4^{th} column)" too.

The subsequent Sections, 3 and 4, of Part I are presented in [Iorgulescu a].

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