Journal of Universal Computer Science, vol. 13, no. 11 (2007), 1779-1790
submitted: 10/9/07, accepted: 19/11/07, appeared: 28/11/07 © J.UCS

Connectivity and Reachability in Signed Networks [1]

Monica Tătărâm
(Department of Foundations of Computer Science
Faculty of Mathematics and Informatics
Str. Academiei 14, Bucharest, Sector 1, Romania RO-010014
E-mail: tataram@fmi.unibuc.ro)

Abstract: For modeling real-life situations where not only the intensity of the relation existing between elements but also its polarity is important, we have proposed (see [Marcus and Tataram 1987a]) a new type of graphs: the signed networks. In the present paper we study two of thier most important properties: connectivity and reachability, and try to use them in order to offer a strategy to improve communication in social or professional groups.

Key Words: graph theory, connectivity, geodesics
Category: G.2.2, G.2.3

1 Introduction

Graph Theory is one of the most powerful tools for modeling a large variety of phenomena and processes. One of the reasons is the number of variations to the initial definition of a graph: the digraph, the network, the signed graph, the marked graph. Most of them stemmed out of real-life situations (electric circuits, chemical formulas, shortest or complete itineraries through a finite number of locations, human relationships, genealogical trees etc.) but also from certain chapters of mathematics (like the algebraic relation theory). All of them determined a strong effort towards abstractization and formalization, so that this very versatile modeling and investigation tool can be successfully applied in a wide variety of domains and offer insights - sometimes rather counterintuitive ones - about their most hidden and difficult to approach aspects.

For instance, how can we study human relationships in a social or professional group where each person can have a strong sympathy for some person (or persons), can like other ones and - at the same time - he or she can mildly or even profoundly dislike other members of the group? Clearly, neither a signed digraph nor a network can model such situations where not only the polarity but also the intensity of the relationships are important. Thus, we proposed a new type of graph, namely a network defined over the Q-semiring $\langle Q, +, min, \infty, -\infty \rangle$ (where Q denotes the set of rational numbers), able to grasp both aspects, polarity and intensity, and we called it "a signed network" (see [Marcus and Tataram 1987a]).

[1] C. S. Calude, G. Stefanescu, and M. Zimand (Eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*

Since signed networks have proved to be very useful when studying the paradoxical and antinomic aspects of the human communication process (see [Marcus and Tataram 1987a]), the crises that define the various stages of human development, the direct and indirect influences that exist among contemporary global trends (see [Marcus and Tataram 1987b]), the evolution of economic and social indicators (see [Marcus and Tataram 1988]), we try to exploit their modeling power in order to solve some problems connected with personal status, communication, and collaboration within social or working groups. We investigate the concepts of connectivity, reachability and basis with respect to signed networks. Thus we can determine the person (or persons) that control (in a good or in a bad sense) the communication processes in the group, the ones that are best / worst suited to become leaders of the group.

2 Connectivity

Definition 1. A *signed network* is a digraph (V, X) such that to each edge $x \in X$ a rational number $q \neq 0$ is assigned, called the *edge's value*. By the *capacity* of an edge we shall mean the absolute value of the rational number assigned to that edge. We denote such a network by $SN = (V, X, Q)$, where Q is the set of rational numbers. □

Hence, a signed network can be represented by an *adjacency table* T, that is: a three-dimension massive $p \times p \times 2$, where $p = card(V)$; the cell placed at the intersection of row i with column j, $1 \leq i,\ j \leq p,\ i \neq j$, will contain a pair of rational numbers formed by the capacity and the value of the edge linking the vertex i with the vertex j, if such an edge exists in SN, and the symbol (∞, ∞), otherwise; if $i = j$, the cell will contain the pair $(0, 0)$.

Definition 2. A *signed subnetwork* of a signed network $SN = (V, X, Q)$ is a signed network $SN' = (V', X', Q)$ such that $V' \subseteq V$, $X' \subseteq X$ and the values of all edges in X' are the same as in X. □

Definition 3. A *path* P from vertex u_1 to vertex u_n, $1 \leq n \leq p = card(V)$, denoted by $P(u_1, u_n)$, is a collection of n distinct vertices $u_1, u_2, \ldots, u_n$ and of $n-1$ edges $u_1u_2, u_2u_3, \ldots, u_{n-1}u_n$ together with their corresponding values. To a path P in a signed network $SN = (V, X, Q)$ one can associate three parameters: the *length*, denoted by $n(P)$ and given by the number of its edges; the *value*, denoted by $v(P)$ and given by the algebraic sum of its edges' values; the *capacity*, denoted by $c(P)$ and given by the sum of its edges' capacities. □

If $c(P) = v(P)$, the path P is called *absolutely positive*; if $c(P) = -v(P)$, the path P is called *absolutely negative*; otherwise, P is called either *positive* or *negative* as its value $v(P)$ is a positive or a negative rational number.

Definition 4. A *geodesic* from vertex t to vertex u in a signed network $SN = (V, X, Q)$ is a path from t to u of minimum length; if from t to u several such paths exist then the path with the minimum capacity is chosen; if there exist several such paths too, then the path with the minimum absolute value is chosen. The capacity of a geodesic from t to u can be taken as the *distance* from t to u (denoted by $d(t, u)$). An *absolutely positive / negative geodesic* from the vertex t to the vertex u in a signed network $SN = (V, X, Q)$ is that minimum (in the sense of the above definition) absolutely positive / negative path from t to u if such a path exists. The capacity of such a path can be taken as the *absolutely positive / negative distance* from t to u (denoted by $d_+(t, u)$, respectively $d_-(t, u)$). Obviously, there exist $t, u \in V$ and P from t to u such that P is an absolutely positive geodesic from t to u but P is not the positive geodesic from t to u. □

In [Marcus and Tataram 1987a] a method of determining the paths of length $n \geq 2$ and the geodesics between any two vertices of a signed network $SN = (V, X, Q)$, given by its adjacency table, was presented. This method was inspired from [Harary et al. 1965] and [Tomescu 1972]: first, we defined the "sum" and "product" of two tables A and B of dimension $p \times p \times 2, p \geq 2$, containing elements from $Q \bigcup \infty$ (the symbol ∞ has, by definition, the property that $\infty + a = a + \infty = \infty$ and $\infty > a, \forall a \in Q$), by the relations:

$$A + B = (a_{ij1} + b_{ij1}, a_{ij2} + b_{ij2})_{1 \leq i,j \leq p}$$
$$A \times B = (a_{ih1} + b_{hj1}, a_{ih2} + b_{hj2})_{1 \leq i,j \leq p}$$

where the index h is given by:

$$|a_{ih2} + b_{hj2}| = min\{|a_{im2} + b_{mj2}| \quad | \; a_{im1} + b_{mj1} = min\{a_{ik1} + b_{kj1} | \; 1 \leq k \leq p\}\}.$$

The power $n + 1, \; n \geq 1$, of a table A is defined by:

$$A^{n+1} = A^n \times A.$$

We also defined an order relation between two tables A and B, denoted by $\lhd$, namely: $A \lhd B$ if and only if $a_{ij1} \leq b_{ij1}, \forall \, 1 \leq i, j \leq p$. Then, we proved the following result:

Proposition 5. *([Marcus and Tataram 1987a]) Let $SN = (V, X, Q)$ be a signed network and T its adjacency table.*

1. *$\forall \, 2 \leq r \leq p - 1$ and $\forall \, 1 \leq i, j \leq p$, $p = card(V)$:* *T^r_{ij} represents the minimal capacity and value among all paths of length r existing between vertices u_i and u_j in SN.*
2. *T^{p-1} is the table of all geodesics that exist in SN.*

The proof is based on the following two lemmas:

Lemma 6. *The "product" of tables is isotonic with respect to the order relation* $\triangleleft$.

Lemma 7. *Let* T *be a table of dimension* $p \times p \times 2$, $p \geq 2$. *Then,* $\forall\, 1 \leq i, j \leq p, i \neq j$ *and* $r \geq 2$, *we have:*

$$T_{ii}^{r} = (0, 0)$$

and

$$|T_{ij2}^{r}| = min(|T_{ij2}^{r-1}|,\ min\{|t_{ih_1 2} + t_{h_1 h_2 2} + \ldots + t_{h_{r-1} j 2}| \quad | \\ t_{ih_1 1} + t_{h_1 h_2 1} + \ldots + t_{h_{r-1} j 1} = min\{t_{ik_1 1} + t_{k_1 k_2 1} + \ldots + t_{k_{r-1} j 1} \mid \\ k_1, k_2, \ldots k_{r-1} \in \{1, 2, \ldots, p\} - \{i, j\} \text{ are pairwise different } \}\}).$$

Definition 8. A signed network $SN = (V, X, Q)$ is *(absolutely) positive unilateral* if between any two vertices $t, u \in V$ there exists at least one (absolutely) positive path either from t to u or from u to t. A signed network $SN = (V, X, Q)$ is *(absolutely) positive strong* if between any two vertices $t, u \in V$ there exists at least one (absolutely) positive path from t to u and at least one (absolutely) positive path from u to t. Similarly, we define an (absolutely) negative unilateral signed network and an (absolutely) negative strong signed network. □

Definition 9. An *(absolutely) positive unilateral component* of a signed network $SN = (V, X, Q)$ is a signed subnetwork $\overline{SN} = (\overline{V}, \overline{X}, Q)$ of SN which is (absolutely) positive unilateral. An *(absolutely) positive strong component* of a signed network $SN = (V, X, Q)$ is a signed subnetwork $\overline{SN} = (\overline{V}, \overline{X}, Q)$ of SN which is (absolutely) positive strong. Similarly, we define an (absolutely) negative unilateral component, respectively an (absolutely) negative strong component of a signed network. □

3 Reachability

Definition 10. Let $SN = (V, X, Q)$ be a signed network and u and t two vertices in V. We say that t is *reachable in an (absolutely) positive sense* from u iff there exists an (absolutely) positive path from u to t. Similarly we define a vertex that is reachable in an (absolutely) negative sense from another one. □

Definition 11. Let $SN = (V, X, Q)$ be a signed network and $u \in V$ one of its vertices. Then we define the following sets of vertices in V:

$$R_s^{+}(u) = \{v \in V \mid \exists\, P(u, v) : c(P) = v(P)\},$$

$$R_s^{-}(u) = \{v \in V \mid \exists\, P(u, v) : c(P) = -v(P)\}.$$

□

Definition 12. Let $SN = (V, X, Q)$ be a signed network and $u \in V$ one of its vertices. Then we define the following sets of vertices in V:

$$R_w^+(u) = \{v \in V | \, \exists P(u,v) : v(P) \geq 0\},$$

$$R_w^-(u) = \{v \in V | \, \exists P(u,v) : v(P) < 0\}.$$

□

Definition 13. Let $SN = (V, X, Q)$ be a signed network and $U \subseteq V$. We denote by

$R_s^+(U) = \bigcup_{u \in U} R_s^+(u)$, $R_s^-(U) = \bigcup_{u \in U} R_s^-(u)$, and

$R_w^+(U) = \bigcup_{u \in U} R_w^+(u)$, $R_w^-(U) = \bigcup_{u \in U} R_w^-(u)$.

□

Definition 14. Let $SN = (V, X, Q)$ be a signed network and $B \subset V$ a set with the following properties:

1. $\forall \, t \in V : \, \exists \, u \in B : t \in R_s^+(u)$;
2. $\nexists \, B' \subset B : \forall \, t \in V : \, \exists \, z \in B' : t \in R_s^+(z)$.

Then, the set B is called *an absolutely positive basis* for SN. Similarly, we define an absolutely negative basis for SN. We denote them B_s^+ and B_s^-, respectively. □

Definition 15. Let $SN = (V, X, Q)$ be a signed network and $B \subset V$ a set with the following properties:

1. $\forall \, t \in V : \, \exists \, u \in B : t \in R_w^+(u)$;
2. $\nexists \, B' \subset B : \forall \, t \in V : \, \exists \, z \in B' : t \in R_w^+(z)$.

Then, the set B is called *a positive basis* for SN. Similarly, we define a negative basis for SN. We denote them by B_w^+ and B_w^-, respectively. □

Definition 16. Let $SN = (V, X, Q)$ be a signed network and $B \subset V$ a set with the following properties:

1. $\forall \, t \in V : \, \exists \, u \in B : t \in R_w^+(u) \cup R_w^-(u)$;
2. $\nexists \, B' \subset B : \forall \, t \in V : \, \exists \, z \in B' : t \in R_w^+(z) \cup R_w^-(z)$.

Then, the set B is called *a basis* for SN and we denote it by B. □

Definition 17. Let $SN = (V, X, Q)$ be a signed network and $u \in V$ one of its vertices. We denote by

$$in^{+}(u) = \{t \in V \,|\, \exists\,(t, u) \; and \; v(t, u) \in Q_{+}^{*}\},$$

and respectively

$$in^{-}(u) = \{t \in V \,|\, \exists\,(t, u) \; and \; v(t, u) \in Q_{-}^{*}\},$$

the set of all its *positive*, respectively *negative direct ascendants*. We denote by $in(u)$ the set of all the direct ascendants of the vertex u. □

Theorem 18. *Let $SN = (V, X, Q)$ be a signed network. A set $B \subset V$ is an absolutely positive basis for SN if and only if it satisfies the following conditions:*

(i) $R_s^{+}(B) = V$;

(ii) $\nexists\, u \in B$ *such that* $\exists\, t \in B$ *with the property that* $\exists\, P(t, u) : c(P) = v(P)$.

Proof. (i) Is immediate, considering the Definition 14.

(ii) Let us suppose, for the sake of contradiction, that $\exists\, B' \subset B$ such that B' is an absolutely positive basis for SN. Let $u \in B - B'$; according to the hypothesis (ii), $u \notin R_s^{+}(B)$ which implies that $u \notin R_s^{+}(B')$. Thus, we obtained a contradiction.

Reciprocally, let B be the smallest subset of V such that $R_s^{+}(B) = V$. Let us suppose, for the sake of contradiction, that $\exists\, t, u \in B$ such that $u \in R_s^{+}(t)$. This means that $R_s^{+}(u) \subset R_s^{+}(t)$. Let us denote $B' = B - \{u\}$; it follows that $R_s^{+}(B') = V$ which contradicts the hypothesis that B is minimal.

Let us observe that this result holds for every type of basis in a signed network.

Corollary 19. *Every signed network admits an (absolutely) positive basis, an (absolutely) negative basis, respectively a basis.*

Proof. We consider the case of the absolutely positive basis; the other cases follow similarly.

We apply Theorem 18. If $B = V$, the condition (i) in Theorem 18 is trivially satisfied but, in most cases, the condition (ii) is not satisfied. Let B_s^{+} be a set of vertices from V which is minimal with respect to condition (i). Then, according to the proof of Theorem 18, B_s^{+} must satisfy the condition (ii). Consequently, following the characterization of the absolutely positive bases given in Theorem 18, B_s^{+} is an absolutely positive basis for SN.

Corollary 20.

(i) Let U, U' be two absolutely positive strong components of a signed network SN and let $t, u \in B_s^{+}$; if $t \in U$ and $u \in U'$ then $U = U'$.

(ii) Let U, U' be two positive strong components of a signed network SN and let $t, u \in B_w^+$; if $t \in U$ and $u \in U'$ then $U = U'$.

(iii) Let U, U' be two absolutely negative strong components of a signed network SN and let $t, u \in B_s^-$; if $t \in U$ and $u \in U'$ then $U = U'$.

(iv) Let U, U' be two absolutely positive strong components of a signed network SN and let $t, u \in B_w^-$; if $t \in U$ and $u \in U'$ then $U = U'$.

Proof. All four assertions follow immediately from condition (ii) of Theorem 18.

Proposition 21. *Let $SN = (V, X, Q)$ be a signed network.*

(i) If $in^+(u) = \emptyset$ then $u \in B_s^+$, $\forall\, B_s^+$ of SN;

(ii) If $in^-(u) = \emptyset$ then $u \in B_s^-$, $\forall\, B_s^-$ of SN;

(iii) If $in^+(u) = in^-(u) = \emptyset$, then the vertex u belongs to every basis of SN.

Proof. (i) and (ii) are immediate, considering the hypothesis, the Definition 4 and, respectively, Definitions 14, 15, 16 given above.

(iii) We give a proof by contradiction. Let $u \in V$ be such that $in^+(u) = in^-(u) = \emptyset$ and suppose that, for instance, $u \notin B_w^+, \forall B_w^+$ of SN. Then, according to Definition 15: $\exists\, t \in B_w^+ : u \in R_w^+(t)$, that is: $\exists\, P = t, t_1, t_2, \ldots t_n, u$ such that $v(P) \geq 0$. If $v(t_n, u) > 0$ then $in^-(u) = \emptyset$ but $in^+(u) \ni t$; similarly: if $v(t_n, u) < 0$ then $in^+(u) = \emptyset$ but $in^-(u) \ni t$. Hence, in both cases, we contradict the hypothesis that both sets, $in^+(u)$ and $in^-(u)$, are empty.

Theorem 22.

(i) Let $SN = (V, X, Q)$ be an absolutely positive unilateral signed network. Then, $\exists B_s^+$ for SN if and only if $\exists u \in V : in^+(u) = \emptyset$ and u is the only vertex in V having this property;

(ii) Let $SN = (V, X, Q)$ be an absolutely negative unilateral signed network. Then, $\exists\, B_s^-$ for SN if and only if $\exists\, u \in V : in^-(u) = \emptyset$ and u is the only vertex in V having this property;

(iii) Let $SN = (V, X, Q)$ be a positive unilateral signed network. Then, $\exists\, B_w^+$ for SN if and only if $\exists\, u \in V : in^+(u) = in^-(u) = \emptyset$ and u is the only vertex in V having this property;

(iv) Let $SN = (V, X, Q)$ be a negative unilateral signed network. Then, $\exists\, B_w^-$ for SN if and only if $\exists\, u \in V : in^+(u) = in^-(u) = \emptyset$ and u is the only vertex in V having this property;

(v) Let $SN = (V, X, Q)$ be a unilateral signed network. Then, $\exists\, B$ for SN if and only if $\exists\, u \in V : in^+(u) = in^-(u) = \emptyset$ and u is the only vertex in V having this property;

Proof. (i) "$\Longrightarrow$" Let $SN = (V, X, Q)$ be an absolutely positive unilateral signed network and u its unique vertex having the property that $in^+(u) = \emptyset$. We want to prove that its unique absolutely positive basis is $B_s^+ = \{u\}$. The existence of an absolutely positive basis, in the sense of Definition 14, is granted by the hypothesis that SN is absolutely positive unilateral, while the fact that the vertex u belongs to any such basis is granted by Proposition 21(i).

Let us suppose, for the sake of contradiction, that $\exists\, t \in V$ such that $B_s^+ = \{u, t\}$. According to Theorem 18 (ii), $\nexists\, P(u, t) : c(P) = v(P)$ and $\nexists\, P'(t, u) : c(P') = v(P')$, which contradicts the hypothesis that SN is absolutely positive unilateral.

Let us now suppose, for the sake of contradiction, that $\exists\, t \in V$ such that $D_s^+ = \{t\}$ is another absolutely positive basis for SN, besides B_s^+ (we supposed that $card(B_s^+) = 1$ in order to ensure the minimality condition). In this case, $R_s^+(t) = V$, which means that the vertex u too can be reached from t by means of an absolutely positive path. This conclusion contradicts the hypothesis that $in^+(u) = \emptyset$.

"$\Longleftarrow$" Let SN be an absolutely positive signed network and B_s^+ its unique absolutely positive basis. We want to prove that SN admits only one vertex with no positive ascendants, namely the vertex that constitutes the basis B_s^+. Let us suppose, for the sake of contradiction, that $B_s^+ = \{u\}$ and that $in^+(u) \neq \emptyset$. Hence, there exists at least one vertex $t \in SN$ such that: $\exists (t, u) \in X : v(t, u) > 0$. Consequently, $R_s^+(u) \subset R_s^+(t)$. But: $R_s^+(u) = V$ because $\{u\} = B_s^+$, which gives us the contradiction we looked for.

Let us now suppose, for the sake of contradiction, that $\exists\, u, t \in V$ such that $in^+(u) = in^+(t) = \emptyset$. According to Proposition 21(i), the vertex t - as well as the vertex u - belong to every absolutely positive basis of SN. If $\{t\} \neq B_s^+$ we contradict the hypothesis that B_s^+ is unique.

If $t \in B_s^+$, then, according to Proposition 21(ii), $\nexists\, P(t, u) : c(t, u) = v(t, u)$, which contradicts the hypothesis that SN is absolutely positive unilateral.

(ii)–(iv): The assertions (ii), (iii), (iv), and (v) can be proved in a similar way.

Theorem 23. *Let $SN = (V, X, Q)$ be an acyclic signed network. Then:*

(i) $\exists\, B_s^+$ for SN and $B_s^+ = \{u \mid in^+(u) = \emptyset\}$;

(ii) $\exists\, B_s^-$ for SN and $B_s^- = \{u \mid in^-(u) = \emptyset\}$;

(iii) $\exists\, B$ for SN and $B = \{u \mid in^+(u) = in^-(u) = \emptyset\}$;

Proof. (i) Let $SN = (V, X, Q)$ be an acyclic signed network. First, we will use Theorem 18 to prove that B_s^+ exists, then we will prove that it is unique.

Notice that

$B = \{u \in V \mid in^+(u) = \emptyset\}$
$\Rightarrow \nexists t \in V$ such that $P(t, u) : c(P) = v(P)$
$\Rightarrow \nexists z \in B$ such that $P(z, u) : c(P) = v(P)$.

So, we have to prove that $R_s^+(B) = V$.

Notice that

$t_0 \in V$
$\Rightarrow in^+(t_0) \neq \emptyset$ (otherwise, $t_0 \in B$)
$\Rightarrow \exists t_1 \in V : v(t_1, t_0) > 0$.

We distinguish two cases:

(a) $in^+(t_1) = \emptyset$
This implies that $t_1 \in B \Rightarrow R_s^+(B) = V$ and the existence is proved.

(b) $in^+(t_1) \neq \emptyset$
This implies that $\exists t_2 \in V : v(t_2, t_1) > 0$.
By repeating the above argument we obtain a sequence of vertices, denoted $t_0, t_1, \ldots, t_k \in V$ such that either

(1) $in^+(t_k) = \emptyset$, hence $t_k \in B$ or

(2) $in^+(t_k) \neq \emptyset$, and then $\exists t_{k+1} \in V : v(t_{k+1}, t_k) > 0$.

Let us denote by P the path $t_k, (t_k, t_{k-1}), t_{k-1}, \ldots, t_1, (t_1, t_0), t_0$ thus obtained. All the vertices in P are distinct since we supposed that SN is acyclic. Also, P is finite since SN has a finite number of vertices. Consequently, $t_k \in B$ and $c(P) = v(P)$. By Definition 11, it follows that $t_0 \in R_s^+(B)$, hence: $R_s^+(B) = V$.

The unicity of B_s^+ follows from Proposition 21(i), since B contains only those vertices in V that have no positive direct ascendants.

(ii) The proof is analogue.

(iii) By replacing the hypothesis $in^+(u) = \emptyset$ with the hypothesis $in^+(u) = in^-(u) = \emptyset$ and the condition $c(P) = v(P)$ with the condition $v(P) > 0$, respectively $v(P) < 0$, we obtain similar proofs for the last three situations.

4 An application: Communication in social or professional groups

Let us suppose that we study a group of seven persons, engaged in some professional collaboration, in order to improve communication between them and thus their work performance.

First, by means of interviews, questionnaires, direct observation etc. we establish pairs of people that like/dislike working together to a higher or lower degree.

Then, we represent these working relationships by means of a signed network with seven vertices, $\mathbf{SN = (V,X,Q)}$, connected by edges whenever a person i likes/dislikes to work with a person j, $1 \leq i \neq j \leq 7$. To each vertex (i, j) we assign the value 2 or 1, depending on the fact that the worker i strongly likes to collaborate with worker j or only prefers to work with him; on the contrary, we assign the value -2 or -1 to vertex (i, j) if the worker i strongly dislikes to collaborate with worker j or only avoids working with him. The working relationships in this professional group can be described by the following adjacency table (see Definition 1):

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{pmatrix}
(0,0) & \infty & \infty & \infty & \infty & \infty & \infty \\
(1,1) & (0,0) & \infty & (2,-2) & (1,-1) & \infty & \infty \\
(1,1) & (2,-2) & (0,0) & (1,-1) & (1,-1) & \infty & \infty \\
(1,1) & (2,-2) & \infty & (0,0) & (1,1) & \infty & \infty \\
(2,2) & (2,2) & (1,1) & \infty & (0,0) & \infty & \infty \\
(2,2) & \infty & \infty & \infty & \infty & (0,0) & \infty \\
(2,2) & (1,1) & (1,1) & (1,1) & (2,2) & (2,2) & (0,0)
\end{pmatrix}
\end{array}
$$

By means of this adjacency table we now can point out:

- the shortest communication channels that exist in the group (by using Proposition 5 that computes all the geodesics that exist in this signed network);
- the working subgroups that should obtain best, good, poor, respectively worst results by working together (by determining the (absolutely) positive or negative strong components).

Since the number of vertices in this signed network is very small, one does not need to apply in this aim graph-theoretic algorithms: a simple, direct computation shows that, for instance: there are no absolutely positive or negative strong components but there are absolutely positive, respectively negative unilateral components: the set of vertices $\{1, 6, 7\}$ forms an absolutely positive unilateral component (showing that workers 1, 6, and 7 can have a very good working relationship), while the set of vertices $\{2, 3, 4\}$ forms an absolutely negative unilateral component (showing that workers 2, 3, and 4 should not be asked to work together!)

If we need to coordinate communication among the members of the group so that the messages reach their addressees without any distortions, we should apply Proposition 5 and Definitions 11 and 12 in order to compute - for each member of the group - the reachability sets; then, we can be sure that each member t in $R_s^+(u)$, respectively $R_w^+(u)$, will receive the messages sent by member u completely, respectively partially correct, while the members z in $R_w^-(u)$ or $R_s^-(u)$ will receive the messages sent by member u with flaws or - even worst - with errors. For instance, the messages sent by member number 5 will be received in best conditions by members number 1, 2, 3 and only in quite good conditions by member number 4, since $R_s^+(5) = \{1, 2, 3\}$ and $R_w^+(5) = \{1, 2, 3, 4\}$. The messages sent by member number 2 will reach member number 1 quite late or with distortions, while the same messages may arrive at members number 4 and 5 with significant errors, since $R_w^+(2) = \{4, 5\}$ and $R_w^-(2) = \{1, 4, 5\}$.

Finally, let us suppose that we have to choose a leader that can best coordinate the work of each member of the group. In this case, we should apply Theorem 23 and determine the absolutely positive basis B_s^+ of the signed network that describes the group. In our example, member number 7 best qualifies for group leader, since $B_s^+ = \{7\}$.

5 Conclusions

Signed networks seem to be a powerful modeling tool and, at the same time, an intriguing topic of theoretical investigation. For instance, one can study the concept of condensation in signed networks and use it to investigate the intensity and the polarity of indirect relationships existing in very large groups of individuals. Or, one can study the concept of equilibrium and use it to appreciate the dynamics of direct and indirect relationships with respect to both their intensity and polarity.

The last topic of study has been proposed by Professor Gheorghe Stefanescu. The author of this paper is in debt to him, as well as to the other two editors and to the referees, for their valuable remarks and suggestions.

References

[Chaty 1988] Chaty, G.: "On signed digraphs with all cycles negative"; Discrete Applied Mathematics 20 (1988), 83-85.

[Gross and Yellen 2004] Gross, J.L., Yellen, J. (Eds.): "Handbook of Graph Theory"; CRC Press LLC, Boca Raton, Fl (2004).

[Harary et al. 1965] Harary, F., Norman, R.Z., Cartwright, D: "Structural Models: An Introduction to the Theory of Directed Graphs"; John Wiley, New York (1965).

[Harary et al. 1985] Harary, F., Lundgren, R.J., Maybee, J.S.: "On signed digraphs with all cycles negative"; Discrete Applied Mathematics 12 (1985), 155-164.

[Marcus and Tataram 1987a] Marcus, S., Tataram, M.: "Human Communication - A Graph-theoretic Approach"; Rev. Roum. Linguistique, CLTA 24, 2 (1987), 129-137.

[Marcus and Tataram 1987b] Marcus, S., Tataram, M.: "Semiotics of Individual and Global Crises; Their Indicators and Interactions"; Rev. Roum. Linguistique, CLTA 24, 1 (1987), 57 - 76.

[Marcus and Tataram 1988] Marcus, S., Tataram, M.: "Social Indicators in a Semiotic and Communicational Perspective"; Rev. Roum. Linguistique, CLTA 25, 2 (1988), 145 - 165.

[Popescu and Tomescu 1996] Popescu, D., Tomescu, I.: "Negative cycles in complete signed graphs"; Discrete Applied Mathematics 68 (1996), 145-152.

[Tomescu 1972] Tomescu, I.: "Introducere in Combinatorica"; Ed. Tehnica, Bucuresti (1972).

[Vedam and Venkatasubramanian 1997] Vedam, H., Venkatasubramanian, V.: "Signed digraph based multiple fault diagnosis"; Computers and Chemical Engineering 21, 1 (1997), 655-660.

[von Reinhard 2000] von Reinhard, D.: "Graph Theory", Second Edition; Springer Verlag (2000).