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Pedagogical Natural Deduction Systems: the Propositional Case

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Abstract: This paper introduces the notion of *pedagogical natural deduction systems*, which are natural deduction systems with the following additional constraint: all hypotheses made in a proof must be motivated by some example. It is established that such systems are negationless. The expressive power of the pedagogical version of some propositional calculi are studied.
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1 Introduction

It is a common observation in teaching mathematics that the act of providing with examples of mathematical objects satisfying some definition helps to understand that definition. The mental universe of a mathematician is full of examples of objects illustrating some theoretical results and of counter-examples refuting false assertions. When a mathematics student defines a commutative field $\mathbb{K}$, a vector space $\mathcal{V}$ on $\mathbb{K}$ of dimension n, an inner product $< .,. >$ of $\mathcal{V}$ and an orthonormal basis $\mathcal{B}$ of $\mathcal{V}$, he often has in mind the Euclidean plane provided with its standard basis. In the same way, when a pupil thinks of an unspecified triangle, he has in mind the particular triangle the teacher has drawn on the blackboard. These examples associated with an abstract definition help to understand some abstract and sometimes complex notions. They motivate their introduction and answers the question "What is it used for?". This observation has some counterpart in the philosophy of mathematics, for instance Henri Poincaré suggested in [Poincaré 1913] that any definition should be immediately followed by an example. Despite this state of facts usual formal systems used in the foundations of Mathematics like Hilbert-style systems and systems based on the sequent calculus or on natural deduction systems systematically neglect this *dictum* of Poincaré.

The present article is an elementary attempt to introduce formal systems fulfilling the Poincarean point of view by including a constraint that we call the *pedagogical constraint*. We start with *natural deduction systems* as introduced by Gentzen [Gentzen 1935] and Prawitz [Prawitz 1965] manipulating *judgements* like $\Gamma \vdash F$ where Γ is a finite set $\{A_1, \cdots, A_n\}$ of formulae and F a formula, with the intuitive meaning that F holds if all formulae in Γ hold. In such systems, there is usually no built-in mechanism for manipulating definitions. Instead, in order to introduce an object x satisfying some definitions, we *assume* in the context Γ some formulae in which the variable x occurs. It is observed that the main rule in such a system creating the context Γ is the *hypothesis* rule:

$$\overline{\Gamma, F \vdash F}$$

which can be rewritten:

$$\frac{F \in \Gamma}{\Gamma \vdash F}$$

Note that in this rule the only requirement on Γ is that it is a set of formulae, possibly non-motivated or even contradictory. The pedagogical constraint will thus be the following: we require that some instance $\sigma(\Gamma)$ of Γ is provable, *i.e.* that $\vdash \sigma(A_1) \cdots \vdash \sigma(A_n)$ can be derived for some σ denoting a substitution replacing any variable of Γ by some example. Thus the *hypothesis* rule becomes the *pedagogical hypothesis* rule:

$$\frac{F \in \Gamma \qquad \vdash \sigma(\Gamma)}{\Gamma \vdash F}$$

Of course the pedagogical constraint has some important consequences on reasoning: for instance *reasoning by contradiction* is not possible either, since to prove F by assuming $\neg F$ and deriving a contradiction, one must first assume $\neg F$ which is not possible pedagogically since one has no example σ such that $\vdash \sigma(\neg F)$. In this way we see that pedagogical natural deduction systems are naturally *intuitionistic systems*. Still more radically, we see that proving a negative statement like $\neg F$ that is $F \rightarrow \perp$ is not possible since we have to assume F and derive $\perp$, but precisely we cannot assume F since we cannot have any example σ such that $\vdash \sigma(F)$. So reasoning pedagogically is not only intuitionistic, but *positive*, *i.e.* exempt of any form of negation. *Pedagogical natural deduction systems* as introduced in this paper have rather basic syntax: we simply replace the rule (Hyp) by the rule (P-Hyp), without altering the other rules. Such a change can be made in any natural deduction system, leading to an immediate and natural family of negationless formal systems based on a very simple and meaningful constraint.

In the present paper, we study the propositional calculi, since they are the simplest non-trivial natural deduction systems. The main result states the equivalence between the usual minimal propositional calculus on $\rightarrow$, $\wedge$ and $\vee$, and his pedagogical version, *i.e.* in this elementary case, the pedagogical constraint does not weaken the system. Moreover, the pedagogical constraint on classical and intuitionistic calculi makes the specific rules on negation useless, i.e. the absurdity symbol appears as a constant symbol without any specific property.

2 Related work

G.F.C. Griss gave in [Griss 1946], [Griss 1950], [Griss 1951a] and [Griss 1951b] an informal development of some part of negationless mathematics. Attempts to formalise these ideas have been made by Vredenduin [Vredenduin 1953], Gilmore [Gilmore 1953], Valpola [Valpola 1955], Nelson [Nelson 1966], [Nelson 1973] and Krivtsov [Krivtsov 2000a], [Krivtsov 2000b]. In [Mezhlumbekova 1975], Mezhlumbekova presents a translation of Heyting arithmetic into a weak system of negationless arithmetic. Quantifier-free formulas are rewritten in the form $t = 0$ where t is a term in the language of primitive recursive functionals of finite type. These attempts mostly deal with first-order logic and its extensions. They introduce quite sophisticated formal systems, with technicalities such as *quantified operators* in [Nelson 1966], or Krivtsov's *pairs of derivations* in [Krivtsov 2000a]. These authors point out the complexity of such technicalities, which creates serious difficulties in the study of these systems with usual proof-theoretical methods. Our proposal avoids all such technicalities and gives a simple account of the related ideas.

3 The usual minimal propositional calculus (MPC)

We start with a brief description of the usual intuitionistic minimal calculus on implication, conjunction and disjunction:

Definition 1. *Formulae* are defined inductively as follows:

- the constant $\top$ (the *true* formula) is a formula
- propositional variables $p, q, r, \ldots$ are formulae
- if A and B are formulae then $A \to B$ is a formula
- if A and B are formulae then $A \wedge B$ is a formula
- if A and B are formulae then $A \vee B$ is a formula

Remark. observe that we do not include negation nor absurdity in the language of formulae. However, we include the constant $\top$ in the calculus so we have a notion of *closed formulae*: a formula is closed when it does not contain any propositional variable.

Substitution of formulae $B_1, \ldots, B_n$ for variables $p_1, \ldots, p_n$ in a formula A can be defined as usual by induction on A. Such a substitution will be often written σ and its effect on a formula A will be written $\sigma(A)$.

The syntax of minimal calculus is given in natural deduction by the following rules. In these rules Γ is a finite set of formulae, and a judgement $\Gamma \vdash_{\mathrm{m}} A$ means that formula A holds under hypotheses Γ. Note that $\Gamma, A_1, \ldots, A_n$ is a synonym for $\Gamma \cup \{A_1, \ldots, A_n\}$:

$$\frac{}{\Gamma \vdash_{\mathrm{m}} \top}\ (\mathrm{Ax})$$

$$\frac{}{\Gamma, F \vdash_{\mathrm{m}} F}\ (\mathrm{Hyp})$$

$$\frac{\Gamma, A \vdash_{\mathrm{m}} B}{\Gamma \vdash_{\mathrm{m}} A \to B}\ (\to_{\mathrm{i}}) \qquad \frac{\Gamma \vdash_{\mathrm{m}} A \to B \quad \Gamma \vdash_{\mathrm{m}} A}{\Gamma \vdash_{\mathrm{m}} B}\ (\to_{\mathrm{e}})$$

$$\frac{\Gamma \vdash_{\mathrm{m}} A \quad \Gamma \vdash_{\mathrm{m}} B}{\Gamma \vdash_{\mathrm{m}} A \wedge B}\ (\wedge_{\mathrm{i}}) \qquad \frac{\Gamma \vdash_{\mathrm{m}} A \wedge B}{\Gamma \vdash_{\mathrm{m}} A}\ (\wedge_{\mathrm{el}}) \qquad \frac{\Gamma \vdash_{\mathrm{m}} A \wedge B}{\Gamma \vdash_{\mathrm{m}} B}\ (\wedge_{\mathrm{er}})$$

$$\frac{\Gamma \vdash_{\mathrm{m}} A}{\Gamma \vdash_{\mathrm{m}} A \vee B}\ (\vee_{\mathrm{il}}) \qquad \frac{\Gamma \vdash_{\mathrm{m}} B}{\Gamma \vdash_{\mathrm{m}} A \vee B}\ (\vee_{\mathrm{ir}})$$

$$\frac{\Gamma \vdash_{\mathrm{m}} A \vee B \quad \Gamma, A \vdash_{\mathrm{m}} C \quad \Gamma, B \vdash_{\mathrm{m}} C}{\Gamma \vdash_{\mathrm{m}} C}\ (\vee_{\mathrm{e}})$$

4 The naive pedagogical propositional calculus (N-MPC)

Now we define the simplest version of the pedagogical propositional calculus simply by replacing the (Hyp) rule of the usual system (see the previous section) by the pedagogical (P-Hyp) rule ($\Gamma \vdash_{\mathrm{n}} F$ will stand for the provability of F under the hypotheses Γ in the naive pedagogical propositional calculus):

$$\frac{F \in \Gamma \quad \vdash_{\mathrm{n}} \sigma(\Gamma)}{\Gamma \vdash_{\mathrm{n}} F}\ (\mathrm{P\text{-}Hyp})$$

In this rule, σ is any substitution, which is called a *motivation*. If Γ stands for $A_1, \ldots, A_n$ then $\vdash_{\mathrm{n}} \sigma(\Gamma)$ is $\vdash_{\mathrm{n}} \sigma(A_1), \ldots, \vdash_{\mathrm{n}} \sigma(A_n)$.

The axiom rule does not fulfil the pedagogical *dictum* because the context Γ is not motivated. As $\top$ is true even without context, we replace the (Ax) rule of MPC by the *naive pedagogical* axiom rule (N-Ax):

$$\frac{}{\vdash_{\mathrm{n}} \top}\ (\mathrm{N\text{-}Ax})$$

Example 1. We shall pedagogically prove that $(a \to b \to c) \to (a \wedge b \to c)$:

1) $\vdash_{\mathrm{n}} \top$ by (N-Ax)

2) $\top \vdash_{\mathrm{n}} \top$ by (P-Hyp) and 1)

3) $\top \vdash_{\mathrm{n}} \top \to \top$ by $(\to_{\mathrm{i}})$ and 2)

4) $\vdash_{\mathrm{n}} \top \to \top \to \top$ by $(\to_{\mathrm{i}})$ and 3)

5) $\vdash_{\mathrm{n}} \top \wedge \top$ by $(\wedge_{\mathrm{i}})$ and 1)

6) $a \to b \to c, a \wedge b \vdash_{\mathrm{n}} a \wedge b$ by (P-Hyp) and 4) 5)

7) $a \to b \to c, a \wedge b \vdash_{\mathrm{n}} a$ by $(\wedge_{\mathrm{el}})$ and 6)

8) $a \to b \to c, a \wedge b \vdash_{\mathrm{n}} a \to b \to c$ by (P-Hyp) and 4) 5)

9) $a \to b \to c, a \wedge b \vdash_{\mathrm{n}} b \to c$ by $(\to_{\mathrm{e}})$ and 7) 8)

10) $a \to b \to c, a \wedge b \vdash_{\mathrm{n}} b$ by $(\wedge_{\mathrm{er}})$ and 6)

11) $a \to b \to c, a \wedge b \vdash_{\mathrm{n}} c$ by $(\to_{\mathrm{e}})$ and 9) 10)

12) $a \to b \to c \vdash_{\mathrm{n}} a \wedge b \to c$ by $(\to_{\mathrm{i}})$ and 11)

13) $\vdash_{\mathrm{n}} (a \to b \to c) \to (a \wedge b \to c)$ by $(\to_{\mathrm{i}})$ and 12)

Considering the (P-Hyp) rule, one can require motivations to be closed. So two possibilities are now to be faced:

- *either* it is required that all formulae $\sigma(\Gamma)$ in the rule (P-Hyp) are closed
- *or* no special condition bearing on σ is required

We shall prove immediately that the first possibility does not weaken the system (*i.e.* the same judgements are provable). Indeed, if a formula is motivable by an unspecified motivation, then it is motivable with a closed motivation. Moreover, we shall prove that all formulae are motivable by a closed motivation.

Lemma 2. *For all formulae F, the judgement $\vdash_{\mathrm{n}} F_\top$ is derivable, where $F_\top$ is the formula obtained by replacing each propositional variable of F by $\top$.*

Proof. By induction on F:

- F is atomic: $F_\top = \top$ so $\vdash_{\mathrm{n}} F_\top$ is derivable by (N-Ax).
- $F = A \to B$: $F_\top = A_\top \to B_\top$ so we have to derive $\vdash_{\mathrm{n}} A_\top \to B_\top$:

  1) $\vdash_{\mathrm{n}} A_\top$ by induction hypothesis

2) $\vdash_{\rm n} B_\top$ by induction hypothesis

3) $B_\top, A_\top \vdash_{\rm n} B_\top$ by (P-Hyp) and 1) 2)

4) $B_\top \vdash_{\rm n} A_\top \to B_\top$ by $(\to_{\rm i})$ and 3)

5) $\vdash_{\rm n} B_\top \to (A_\top \to B_\top)$ by $(\to_{\rm i})$ and 4)

6) $\vdash_{\rm n} A_\top \to B_\top$ by $(\to_{\rm e})$ and 5) 2)

– $F = A \wedge B$: $F_\top = A_\top \wedge B_\top$ so we have to derive $\vdash_{\rm n} A_\top \wedge B_\top$:

1) $\vdash_{\rm n} A_\top$ by induction hypothesis

2) $\vdash_{\rm n} B_\top$ by induction hypothesis

3) $\vdash_{\rm n} A_\top \wedge B_\top$ by $(\wedge_{\rm i})$ and 1) 2)

– $F = A \vee B$: $F_\top = A_\top \vee B_\top$ so we have to derive $\vdash_{\rm n} A_\top \vee B_\top$:

1) $\vdash_{\rm n} A_\top$ by induction hypothesis

2) $\vdash_{\rm n} A_\top \vee B_\top$ by $(\vee_{\rm il})$ and 1)

□

For all formulae F, the formula $F_\top$ is closed, so all formulae F are motivable by the previous lemma, *i.e.* there is always a substitution σ such that $\vdash_{\rm n} \sigma(F)$.

5 Power and limitation of the pedagogical propositional calculus

The following lemma states that N-MPC is a subsystem of MPC.

Lemma 3. *For all derivable judgements $\Gamma \vdash_{\rm n} F$, the judgement $\Gamma \vdash_{\rm m} F$ is derivable.*

Proof. Immediate by induction on $\Gamma \vdash_{\rm n} F$. □

Theorems on formulae (*i.e.* derivable judgements without hypotheses) remains the same in MPC and N-MPC, as we see in the following result. For all sets of formulae $\Gamma = \{G_1, \cdots, G_n\}$, we shall write $\Gamma \to F$ for the formula $G_1 \to \cdots \to G_n \to F$. As usual, Γ, F denotes the set $\Gamma \cup \{F\}$.

Lemma 4. *For all derivable judgements $\Gamma \vdash_{\rm m} F$, the judgement $\vdash_{\rm n} \Gamma \to F$ is derivable.*

Proof. By induction on the derivation of $\Gamma \vdash_{\rm m} F$. We only consider the rules (Ax), (Hyp), $(\vee_{\rm il})$ and $(\vee_{\rm e})$. The others cases are rather similar:

– $\dfrac{}{\Gamma \vdash_{\mathrm{m}} \top}$ (Ax): we have to derive $\vdash_{\mathrm{n}} \Gamma \rightarrow \top$

1) $\vdash_{\mathrm{n}} \Gamma_{\top}$ by lemma 2

2) $\vdash_{\mathrm{n}} \top$ by (N-Ax)

3) $\top, \Gamma \vdash_{\mathrm{n}} \top$ by (P-Hyp) and 1) 2)

4) $\vdash_{\mathrm{n}} \top \rightarrow \Gamma \rightarrow \top$ by ($\rightarrow_{\mathrm{i}}$) and 3)

5) $\vdash_{\mathrm{n}} \Gamma \rightarrow \top$ by ($\rightarrow_{\mathrm{e}}$) and 2) 4)

– $\dfrac{}{\Gamma, F \vdash_{\mathrm{m}} F}$ (Hyp): we have to derive $\vdash_{\mathrm{n}} (\Gamma, F) \rightarrow F$

1) $\vdash_{\mathrm{n}} \Gamma_{\top}$ by lemma 2

2) $\vdash_{\mathrm{n}} F_{\top}$ by lemma 2

3) $\Gamma, F \vdash_{\mathrm{n}} F$ by (P-Hyp) and 1) 2)

4) $\vdash_{\mathrm{n}} (\Gamma, F) \rightarrow F$ by ($\rightarrow_{\mathrm{i}}$) and 3)

– $F = A \vee B$ and $\dfrac{\Gamma \vdash_{\mathrm{m}} A}{\Gamma \vdash_{\mathrm{m}} A \vee B}$ ($\vee_{\mathrm{il}}$): we have to derive $\vdash_{\mathrm{n}} \Gamma \rightarrow F$.

1) $\vdash_{\mathrm{n}} \Gamma_{\top}$ by lemma 2

2) $\vdash_{\mathrm{n}} \Gamma_{\top} \rightarrow A_{\top}$ by lemma 2

3) $\Gamma, \Gamma \rightarrow A \vdash_{\mathrm{n}} \Gamma$ by (P-Hyp) and 1) 2)

4) $\Gamma, \Gamma \rightarrow A \vdash_{\mathrm{n}} \Gamma \rightarrow A$ by (P-Hyp) and 1) 2)

5) $\Gamma, \Gamma \rightarrow A \vdash_{\mathrm{n}} A$ by ($\rightarrow_{\mathrm{e}}$) and 4) 3)

6) $\Gamma, \Gamma \rightarrow A \vdash_{\mathrm{n}} A \vee B$ by ($\vee_{\mathrm{il}}$) and 5)

7) $\vdash_{\mathrm{n}} (\Gamma \rightarrow A) \rightarrow (\Gamma \rightarrow (A \vee B))$ by ($\rightarrow_{\mathrm{i}}$) and 6)

8) $\vdash_{\mathrm{n}} \Gamma \rightarrow A$ by induction hypothesis

9) $\vdash_{\mathrm{n}} \Gamma \rightarrow (A \vee B)$ by ($\rightarrow_{\mathrm{e}}$) and 7) 8)

– $\dfrac{\Gamma \vdash_{\mathrm{m}} A \vee B \quad \Gamma, A \vdash_{\mathrm{m}} F \quad \Gamma, B \vdash_{\mathrm{m}} F}{\Gamma \vdash_{\mathrm{m}} F}$ ($\vee_{\mathrm{e}}$): we have to derive $\vdash_{\mathrm{n}} \Gamma \rightarrow F$

1) $\vdash_{\mathrm{n}} \Gamma_{\top}$ by lemma 2

2) $\vdash_{\mathrm{n}} \Gamma_{\top} \rightarrow (A \vee B)_{\top}$ by lemma 2

3) $\vdash_{\mathrm{n}} (\Gamma_{\top}, A_{\top}) \rightarrow F_{\top}$ by lemma 2

4) $\vdash_{\mathrm{n}} (\Gamma_{\top}, B_{\top}) \rightarrow F_{\top}$ by lemma 2

5) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F \vdash_{\mathrm{n}} \Gamma$ by (P-Hyp) and 1) 2) 3) 4)

6) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F \vdash_{\mathrm{n}} \Gamma \rightarrow (A \vee B)$ by (P-Hyp) and 1) 2) 3) 4)

7) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F \vdash_{\mathrm{n}} A \vee B$ by ($\rightarrow_{\mathrm{e}}$) and 6) 5)

8) $\vdash_{\mathrm{n}} A_{\top}$ by lemma 2

9) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F, A \vdash_{\mathrm{n}} (\Gamma, A)$ by (P-Hyp) and 1) 2) 3) 4) 8)

10) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F, A \vdash_{\mathrm{n}} (\Gamma, A) \rightarrow F$ by (P-Hyp) and 1) 2) 3) 4) 8)

11) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F, A \vdash_{\mathrm{n}} F$ by ($\rightarrow_{\mathrm{e}}$) and 10) 9)

12) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F, B \vdash_{\mathrm{n}} F$ with reasoning similar to 11)

13) $\Gamma, \Gamma \rightarrow (A \vee B), (\Gamma, A) \rightarrow F, (\Gamma, B) \rightarrow F \vdash_{\mathrm{n}} F$ by ($\vee_{\mathrm{e}}$) and 7) 11) 12)

14) $\vdash_{\mathrm{n}} (\Gamma \rightarrow (A \vee B)) \rightarrow ((\Gamma, A) \rightarrow F) \rightarrow ((\Gamma, B) \rightarrow F) \rightarrow (\Gamma \rightarrow F)$ by ($\rightarrow_{\mathrm{i}}$) and 13)

15) $\vdash_{\mathrm{n}} \Gamma \rightarrow (A \vee B)$ by induction hypothesis

16) $\vdash_{\mathrm{n}} (\Gamma, A) \rightarrow F$ by induction hypothesis

17) $\vdash_{\mathrm{n}} (\Gamma, B) \rightarrow F$ by induction hypothesis

18) $\vdash_{\mathrm{n}} \Gamma \rightarrow F$ by ($\rightarrow_{\mathrm{e}}$) and 14) 15) 16) 17)

□

Proposition 5. *For all formulae F, the judgement $\vdash_{\mathrm{n}} F$ is derivable if and only if the judgement $\vdash_{\mathrm{m}} F$ is derivable.*

Proof. Immediate by lemmas 4 and 3. □

However, the equivalence between MPC and N-MPC on judgements is not preserved. More precisely, we shall prove that the judgement $\top \rightarrow \top \vdash_{\mathrm{n}} \top$ is *not* derivable in N-MPC.

Definition 6. We shall define two sets of formulae $\mathcal{N}_{\mathrm{t}}$ and $\mathcal{N}_{\mathrm{f}}$. For all formulae F, the properties $F \in \mathcal{N}_{\mathrm{t}}$ and $F \in \mathcal{N}_{\mathrm{f}}$ are mutually defined by induction on F:

- $\top \notin \mathcal{N}_{\mathrm{t}}$ and $\top \in \mathcal{N}_{\mathrm{f}}$
- $p \notin \mathcal{N}_{\mathrm{t}}$ and $p \in \mathcal{N}_{\mathrm{f}}$, when p is a propositional variable
- $A \to B \in \mathcal{N}_{\mathrm{t}}$ if and only if $A \in \mathcal{N}_{\mathrm{f}}$ or $B \in \mathcal{N}_{\mathrm{t}}$, and $A \to B \in \mathcal{N}_{\mathrm{f}}$ if and only if $A \in \mathcal{N}_{\mathrm{t}}$ and $B \in \mathcal{N}_{\mathrm{f}}$
- $A \wedge B \in \mathcal{N}_{\mathrm{t}}$ if and only if $A \in \mathcal{N}_{\mathrm{t}}$ and $B \in \mathcal{N}_{\mathrm{t}}$, and $A \wedge B \in \mathcal{N}_{\mathrm{f}}$ if and only if $A \in \mathcal{N}_{\mathrm{f}}$ or $B \in \mathcal{N}_{\mathrm{f}}$
- $A \vee B \in \mathcal{N}_{\mathrm{t}}$ if and only if $A \in \mathcal{N}_{\mathrm{t}}$ or $B \in \mathcal{N}_{\mathrm{t}}$, and $A \vee B \in \mathcal{N}_{\mathrm{f}}$ if and only if $A \in \mathcal{N}_{\mathrm{f}}$ and $B \in \mathcal{N}_{\mathrm{f}}$

Lemma 7. *For all formulae F, we have $F \in \mathcal{N}_{\mathrm{f}}$ if and only if $F \notin \mathcal{N}_{\mathrm{t}}$.*

Proof. By induction on F:

- $F = \top$: we have $\top \in \mathcal{N}_{\mathrm{f}}$ and $\top \notin \mathcal{N}_{\mathrm{t}}$
- $F = p$: we have $p \in \mathcal{N}_{\mathrm{f}}$ and $p \notin \mathcal{N}_{\mathrm{t}}$
- $F = A \to B$: we prove the two direction of the equivalence separately.
  - if $A \to B \in \mathcal{N}_{\mathrm{f}}$: we have $A \in \mathcal{N}_{\mathrm{t}}$ and $B \in \mathcal{N}_{\mathrm{f}}$, so by induction hypothesis we have $A \notin \mathcal{N}_{\mathrm{f}}$ and $B \notin \mathcal{N}_{\mathrm{t}}$, thus $A \to B \notin \mathcal{N}_{\mathrm{t}}$
  - if $A \to B \notin \mathcal{N}_{\mathrm{t}}$: we have $A \notin \mathcal{N}_{\mathrm{f}}$ and $B \notin \mathcal{N}_{\mathrm{t}}$, so by induction hypothesis we have $A \in \mathcal{N}_{\mathrm{t}}$ and $B \in \mathcal{N}_{\mathrm{f}}$, thus $A \to B \in \mathcal{N}_{\mathrm{f}}$
- $F = A \wedge B$: we prove the two direction of the equivalence separately.
  - if $A \wedge B \in \mathcal{N}_{\mathrm{f}}$: we have $A \in \mathcal{N}_{\mathrm{f}}$ or $B \in \mathcal{N}_{\mathrm{f}}$, so by induction hypothesis we have $A \notin \mathcal{N}_{\mathrm{t}}$ or $B \notin \mathcal{N}_{\mathrm{t}}$, thus $A \wedge B \notin \mathcal{N}_{\mathrm{t}}$ in both cases
  - if $A \wedge B \notin \mathcal{N}_{\mathrm{t}}$: we have $A \notin \mathcal{N}_{\mathrm{t}}$ or $B \notin \mathcal{N}_{\mathrm{t}}$, so by induction hypothesis we have $A \in \mathcal{N}_{\mathrm{f}}$ or $B \in \mathcal{N}_{\mathrm{f}}$, thus $A \wedge B \in \mathcal{N}_{\mathrm{f}}$
- $F = A \vee B$: we prove the two direction of the equivalence separately.
  - if $A \vee B \in \mathcal{N}_{\mathrm{f}}$: we have $A \in \mathcal{N}_{\mathrm{f}}$ and $B \in \mathcal{N}_{\mathrm{f}}$, so by induction hypothesis we have $A \notin \mathcal{N}_{\mathrm{t}}$ and $B \notin \mathcal{N}_{\mathrm{t}}$, thus $A \vee B \notin \mathcal{N}_{\mathrm{t}}$
  - if $A \vee B \notin \mathcal{N}_{\mathrm{t}}$: we have $A \notin \mathcal{N}_{\mathrm{t}}$ and $B \notin \mathcal{N}_{\mathrm{t}}$, so by induction hypothesis we have $A \in \mathcal{N}_{\mathrm{f}}$ and $B \in \mathcal{N}_{\mathrm{f}}$, thus $A \vee B \in \mathcal{N}_{\mathrm{f}}$

□

Lemma 8. *For all derivable judgements $\Gamma \vdash_{\mathrm{n}} F$, if $F \in \mathcal{N}_{\mathrm{f}}$ and $\Gamma \neq \emptyset$ then there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$.*

Proof. By induction on the derivation of $\Gamma \vdash_{\mathrm{n}} F$:

- $\dfrac{}{\vdash_{\mathrm{n}} \top}$ (N-Ax): $\Gamma = \emptyset$, contradicting the hypothesis

- $\dfrac{F \in \Gamma \quad \vdash_{\mathrm{n}} \sigma(\Gamma)}{\Gamma \vdash_{\mathrm{n}} F}$ (P-Hyp): $F \in \Gamma$ and $F \in \mathcal{N}_{\mathrm{f}}$ by hypothesis

- $\dfrac{\Gamma, A \vdash_{\mathrm{n}} B}{\Gamma \vdash_{\mathrm{n}} A \rightarrow B}$ $(\rightarrow_{\mathrm{i}})$: by definition of $\mathcal{N}_{\mathrm{f}}$ we have $A \in \mathcal{N}_{\mathrm{t}}$ and $B \in \mathcal{N}_{\mathrm{f}}$. By induction hypothesis there are two cases: $A \in \mathcal{N}_{\mathrm{f}}$ or there is a formula $B' \in \Gamma \cap \mathcal{N}_{\mathrm{f}}$.
  - if $A \in \mathcal{N}_{\mathrm{f}}$: by the lemma 7 we have $A \notin \mathcal{N}_{\mathrm{t}}$, but $A \in \mathcal{N}_{\mathrm{t}}$, which contradict the hypothesis
  - if $B' \in \Gamma \cap \mathcal{S}$: B' is appropriate

- $\dfrac{\Gamma \vdash_{\mathrm{n}} A \rightarrow B \quad \Gamma \vdash_{\mathrm{n}} A}{\Gamma \vdash_{\mathrm{n}} B}$ $(\rightarrow_{\mathrm{e}})$: there are two cases: $A \in \mathcal{N}_{\mathrm{f}}$ or $A \notin \mathcal{N}_{\mathrm{f}}$.
  - if $A \in \mathcal{N}_{\mathrm{f}}$: by induction hypothesis on $\Gamma \vdash_{\mathrm{n}} A$ there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$
  - if $A \notin \mathcal{N}_{\mathrm{f}}$: $B \in \mathcal{N}_{\mathrm{f}}$ and $A \in \mathcal{N}_{\mathrm{t}}$ by the lemma 7, so $A \rightarrow B \in \mathcal{N}_{\mathrm{f}}$. By induction hypothesis on $\Gamma \vdash_{\mathrm{n}} A \rightarrow B$, there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$

- $\dfrac{\Gamma \vdash_{\mathrm{n}} A \quad \Gamma \vdash_{\mathrm{n}} B}{\Gamma \vdash_{\mathrm{n}} A \wedge B}$ $(\wedge_{\mathrm{i}})$: $A \wedge B \in \mathcal{N}_{\mathrm{f}}$, so $A \in \mathcal{N}_{\mathrm{f}}$ or $B \in \mathcal{N}_{\mathrm{f}}$:
  - if $A \in \mathcal{N}_{\mathrm{f}}$: by induction hypothesis on $\Gamma \vdash_{\mathrm{n}} A$ there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$
  - if $B \in \mathcal{N}_{\mathrm{f}}$: similar to the precedent case

- $\dfrac{\Gamma \vdash_{\mathrm{n}} A \wedge B}{\Gamma \vdash_{\mathrm{n}} A}$ $(\wedge_{\mathrm{el}})$: $A \in \mathcal{N}_{\mathrm{f}}$ so $A \wedge B \in \mathcal{N}_{\mathrm{f}}$ and by induction hypothesis there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$

- $\dfrac{\Gamma \vdash_{\mathrm{n}} A}{\Gamma \vdash_{\mathrm{n}} A \vee B}$ $(\vee_{\mathrm{il}})$: $A \vee B \in \mathcal{N}_{\mathrm{f}}$ so $A \in \mathcal{N}_{\mathrm{f}}$ and by induction hypothesis there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$

- $\dfrac{\Gamma \vdash_{\mathrm{n}} A \vee B \quad \Gamma, A \vdash_{\mathrm{n}} C \quad \Gamma, B \vdash_{\mathrm{n}} C}{\Gamma \vdash_{\mathrm{n}} C}$ $(\vee_{\mathrm{e}})$: by induction hypothesis on $\Gamma, A \vdash_{\mathrm{n}} C$, either $A \in \mathcal{N}_{\mathrm{f}}$ or there is a formula $B' \in \Gamma \cap \mathcal{N}_{\mathrm{f}}$:
  - if $A \in \mathcal{N}_{\mathrm{f}}$: by induction hypothesis on $\Gamma, B \vdash_{\mathrm{n}} C$, either $B \in \mathcal{N}_{\mathrm{f}}$ or there is a formula $B'' \in \Gamma \cap \mathcal{N}_{\mathrm{f}}$:

  - ∗ if $B \in \mathcal{N}_{\mathrm{f}}$: $A \vee B \in \mathcal{N}_{\mathrm{f}}$ and by induction hypothesis on $\Gamma \vdash_{\mathrm{n}} A \vee B$ there is a formula in $\Gamma \cap \mathcal{N}_{\mathrm{f}}$
  - ∗ if $B'' \in \Gamma \cap \mathcal{N}_{\mathrm{f}}$: B'' is appropriate
- if $B' \in \Gamma \cap \mathcal{N}_{\mathrm{f}}$: B' is appropriate

□

Proposition 9. *The judgement* $\top \to \top \vdash_{\mathrm{n}} \top$ *is not derivable.*

Proof. We have $\top \in \mathcal{N}_{\mathrm{f}}$, so by the lemma 8 we have $\top \to \top \in \mathcal{N}_{\mathrm{f}}$. Since $\top \in \mathcal{N}_{\mathrm{f}}$ we have $\top \to \top \in \mathcal{N}_{\mathrm{t}}$, so $\top \to \top \notin \mathcal{N}_{\mathrm{f}}$ by the lemma 7, which is absurd. □

6 Beyond the limitation of N-MPC

Derivable judgements without hypotheses (*i.e.* theorems) are the same in N-MPC and MPC. But when the context is not empty, we loose the equivalence since $\top \to \top \vdash_{\mathrm{n}} \top$ is not derivable. Hence we are led to replace the rule (N-Ax) by the following rule we call the *pedagogical* axiom rule (P-Ax):

$$\frac{\vdash_{\mathrm{p}} \sigma(\Gamma)}{\Gamma \vdash_{\mathrm{p}} \top}\ \text{(P-Ax)}$$

We call P-MPC (*i.e.* Pedagogical Minimal Propositional Calculus) the new pedagogical system we obtain and we write $\Gamma \vdash_{\mathrm{p}} F$ for the provability of F under the hypotheses Γ in this system. The rule (P-Ax) is identical to the rule (N-Ax) when the context Γ is empty. One may ask why we do not choose an unconstrained rule, as a true formula remains intuitionistically true independently of the context. But if we want to *pedagogically* access to a true formula through a proof, the manipulated context must be motivable. Indeed, a context is pedagogically acceptable only if it is motivable.

As in lemma 2, we immediately see that all formulae are motivable in P-MPC by a closed motivation. Moreover, MPC and P-MPC are equivalent on judgements:

Proposition 10. *For all sets of formulae* $\Gamma \cup \{F\}$*, the judgement* $\Gamma \vdash_{\mathrm{p}} F$ *is derivable if and only if the judgement* $\Gamma \vdash_{\mathrm{m}} F$ *is derivable.*

Proof. We prove the two directions of the equivalence separately:

⇐) by induction on $\Gamma \vdash_{\mathrm{m}} F$; the rules (Ax) and (Hyp) are the only non-immediate cases:

- $\dfrac{}{\Gamma \vdash_{\mathrm{m}} \top}$ (Ax): we have $\vdash_{\mathrm{p}} \Gamma_{\top}$ by lemma 2, so we can derive $\Gamma \vdash_{\mathrm{p}} \top$ by the (P-Ax) rule

- $\overline{\Gamma, F \vdash_{\mathrm{m}} F}$ (Hyp): we have $\vdash_{\mathrm{p}} \Gamma_{\top}$ and $\vdash_{\mathrm{p}} F_{\top}$ by lemma 2, so we can derive $\Gamma, F \vdash_{\mathrm{p}} F$ by the (P-Hyp) rule

$\Rightarrow$) immediate by induction on $\Gamma \vdash_{\mathrm{p}} F$

□

7 What about negation?

Definition 11. $\bot$-*formulae* are formulae with the additional constant $\bot$ (the *absurd* formula).

We call P-IPC the pedagogical version of the usual intuitionistic propositional calculus, *i.e.* the system P-MPC extended to $\bot$-formulae and including the intuitionistic absurdity rule ($\bot_{\mathrm{i}}$) ($\Gamma \vdash_{\mathrm{i}} F$ will stand for the provability of F under hypotheses Γ in P-IPC):

$$\frac{\Gamma \vdash_{\mathrm{i}} \bot}{\Gamma \vdash_{\mathrm{i}} F}\,(\bot_{\mathrm{i}})$$

Similarly, we call P-CPC the pedagogical version of the usual classical propositional calculus, *i.e.* the system P-MPC extended to $\bot$-formulae and including the classical absurdity rule ($\bot_{\mathrm{c}}$) ($\Gamma \vdash_{\mathrm{c}} F$ will stand for the provability of F under hypotheses Γ in P-CPC):

$$\frac{\Gamma, F \to \bot \vdash_{\mathrm{c}} \bot}{\Gamma \vdash_{\mathrm{c}} F}\,(\bot_{\mathrm{c}})$$

We shall prove that the rules ($\bot_{\mathrm{i}}$) and ($\bot_{\mathrm{i}}$) are useless in such pedagogical systems: they do not appear in any derivations.

Definition 12. The set $\mathcal{B}$ of $\bot$-formulae is defined by induction on the $\bot$-formulae:

- $\bot \in \mathcal{B}$
- $\top \notin \mathcal{B}$
- $p \notin \mathcal{B}$ when p is a propositional variable
- $A \to B \in \mathcal{B}$ if and only if $B \in \mathcal{B}$
- $A \wedge B \in \mathcal{B}$ if and only if $A \in \mathcal{B}$ or $B \in \mathcal{B}$
- $A \vee B \in \mathcal{B}$ if and only if $A \in \mathcal{B}$ and $B \in \mathcal{B}$

Lemma 13. *For all* $\bot$*-formulae* $B \in \mathcal{B}$ *and for all substitutions* σ*, we have* $\sigma(B) \in \mathcal{B}$*.*

Proof. By induction on B:

- $B = \top$: impossible since $\top \notin \mathcal{B}$
- $B = \bot$: $\sigma(B) = \bot$, so $\sigma(B) \in \mathcal{B}$
- $B = p$ with p a propositional variable: impossible since $p \notin \mathcal{B}$
- $B = F \to B'$: we have $\sigma(B) = \sigma(F) \to \sigma(B')$. By induction hypothesis on B' we have $\sigma(B') \in \mathcal{B}$. So $\sigma(B) \in \mathcal{B}$
- $B = B_1 \wedge B_2$: we have $\sigma(B) = \sigma(B_1) \wedge \sigma(B_2)$. By definition of $\mathcal{B}$ we have $B_1 \in \mathcal{B}$ or $B_2 \in \mathcal{B}$:
  - if $B_1 \in \mathcal{B}$: $\sigma(B_1) \in \mathcal{B}$ by induction hypothesis on B_1, so $\sigma(B) \in \mathcal{B}$
  - if $B_2 \in \mathcal{B}$: similar to the precedent case
- $B = B_1 \vee B_2$: we have $\sigma(B) = \sigma(B_1) \vee \sigma(B_2)$. By induction hypothesis on B_1 we have $\sigma(B_1) \in \mathcal{B}$ and by induction hypothesis on B_2 we have $\sigma(B_2) \in \mathcal{B}$, so $\sigma(B) \in \mathcal{B}$

$\square$

The following lemma holds for P-IPC, but it also holds for P-CPC:

Lemma 14. *For all derivable judgements* $\Gamma \vdash_{\mathrm{i}} F$ *on* $\bot$*-formulae we have* $F \notin \mathcal{B}$*.*

Proof. By induction on $\Gamma \vdash_{\mathrm{i}} F$:

- $\dfrac{\vdash_{\mathrm{i}} \sigma(\Gamma)}{\Gamma \vdash_{\mathrm{i}} \top}$ (P-Ax): $\top \notin \mathcal{B}$ by definition of $\mathcal{B}$
- $\dfrac{F \in \Gamma \quad \vdash_{\mathrm{i}} \sigma(\Gamma)}{\Gamma \vdash_{\mathrm{i}} F}$ (P-Hyp): $\sigma(F) \notin \mathcal{B}$ by induction hypothesis, so $F \notin \mathcal{B}$ according to the contrapositive of the lemma 13
- $\dfrac{\Gamma, A \vdash_{\mathrm{i}} B}{\Gamma \vdash_{\mathrm{i}} A \to B}$ ($\to_{\mathrm{i}}$): $B \notin \mathcal{B}$ by induction hypothesis, so $A \to B \notin \mathcal{B}$
- $\dfrac{\Gamma \vdash_{\mathrm{i}} A \to B \quad \Gamma \vdash_{\mathrm{i}} A}{\Gamma \vdash_{\mathrm{i}} B}$ ($\to_{\mathrm{e}}$): $A \to B \notin \mathcal{B}$ by induction hypothesis, so $B \notin \mathcal{B}$
- $\dfrac{\Gamma \vdash_{\mathrm{i}} A \quad \Gamma \vdash_{\mathrm{i}} B}{\Gamma \vdash_{\mathrm{i}} A \wedge B}$ ($\wedge_{\mathrm{i}}$): $A \notin \mathcal{B}$ and $B \notin \mathcal{B}$ by induction hypothesis, so $A \wedge B \notin \mathcal{B}$

- $\dfrac{\Gamma \vdash_{\rm i} A \wedge B}{\Gamma \vdash_{\rm i} A}$ $(\wedge_{\rm el})$: $A \wedge B \notin \mathcal{B}$ by induction hypothesis, so $A \notin \mathcal{B}$
- $\dfrac{\Gamma \vdash_{\rm i} A}{\Gamma \vdash_{\rm i} A \vee B}$ $(\vee_{\rm il})$: $A \notin \mathcal{B}$ by induction hypothesis, so $A \vee B \notin \mathcal{B}$
- $\dfrac{\Gamma \vdash_{\rm i} A \vee B \quad \Gamma, A \vdash_{\rm i} C \quad \Gamma, B \vdash_{\rm i} C}{\Gamma \vdash_{\rm i} C}$ $(\vee_{\rm e})$: by induction hypothesis on $\Gamma, A \vdash C$ we have $C \notin \mathcal{B}$
- $\dfrac{\Gamma \vdash_{\rm i} \bot}{\Gamma \vdash_{\rm i} F}$ $(\bot_{\rm i})$: by induction hypothesis we have $\bot \notin \mathcal{B}$ which is absurd, so $F \notin \mathcal{B}$

□

Lemma 15. *For all derivable judgements $\Gamma \vdash_{\rm c} F$ on $\bot$-formulae we have $F \notin \mathcal{B}$.*

Proof. By induction on $\Gamma \vdash_{\rm c} F$. The proof is similar to that of the previous lemma, so we only treat the case of the rule $(\bot_{\rm c})$:

- $\dfrac{\Gamma, F \to \bot \vdash_{\rm c} \bot}{\Gamma \vdash_{\rm c} F}$ $(\bot_{\rm c})$: by induction hypothesis we have $\bot \notin \mathcal{B}$ which is absurd, so $F \notin \mathcal{B}$

□

Proposition 16. *In all derivations of judgements P-IPC and P-CPC, there is no occurrence of the rule $(\bot_{\rm i})$ neither of the rule $(\bot_{\rm c})$.*

Proof. In this proof, $\Gamma \vdash F$ will stand indifferently for $\Gamma \vdash_{\rm i} F$ and for $\Gamma \vdash_{\rm c} F$. The rule $(\bot_{\rm i})$ or the rule $(\bot_{\rm c})$ appears in the derivation of $\Gamma \vdash F$ if some judgements of the form $\Gamma \vdash \bot$ appear in the derivation. This never happens according to the propositions 14 and 15 because $\bot \in \mathcal{B}$. □

Observe that the symbol $\bot$ can appear in some derivable judgement like $\vdash \top \vee \bot$. But the symbol $\bot$ do not have the same significance as in intuitionistic or classical systems: without the absurdity rules, $\bot$ is an harmless formal constant.

8 Conclusion

We have established that the minimal propositional logic is implicitly pedagogical. Of course, the same question can be asked for stronger systems such as first-order logic, second-order propositional calculus and real-size systems like Peano arithmetics. Since pedagogical systems are intrinsically positive (*i.e.* exempt of negation), one may expect important changes in the pedagogical versions of systems in which absurdity is definable (such as $\forall \alpha . \alpha$ in the second-order propositional calculus, or $0 = 1$ in arithmetics).

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References

[Colson 1986] Colson L.: "Quelques remarques sur l'environnement dans la Théorie Intuitionniste des Types". D.E.A. report, University of Paris Sud - Orsay, (September 1986).

[Gentzen 1935] Gentzen G.: "Investigations into Logical Deduction" Mathematische Zeitschrift 39, (1935), pp. 176-210, 405-431.
Reprinted in "The Collected Papers of Gerhard Gentzen" Studies in Logic and the Foundations of Mathematics, North-Holland (1969).

[Gilmore 1953] Gilmore P.C.G.: "The effect of Griss' criticism of the intuitionistic logic on deductive theories formalized within the intuitionistic logic" Indagationes Mathematicæ 15, (1953), pp. 162-174, 175-186.

[Griss 1946] Griss G.F.C.: "Negationless intuitionistic mathematics" Indagationes Mathematicæ 8, (1946), pp. 675-681.

[Griss 1950] Griss G.F.C.: "Negationless intuitionistic mathematics II" Indagationes Mathematicæ 12, (1950), pp. 108-115.

[Griss 1951a] Griss G.F.C.: "Negationless intuitionistic mathematics III" Indagationes Mathematicæ 13, (1951), pp. 193-199.

[Griss 1951b] Griss G.F.C.: "Negationless intuitionistic mathematics IVa, IVb" Indagationes Mathematicæ 13, (1951), pp. 452-462, 463-471.

[Heyting 1956] Heyting A.: "Intuitionism: An Introduction." Studies in Logic and the Foundations of Mathematics, North-Holland (1956).

[Krivtsov 2000a] Krivtsov V.N.: "A negationless interpretation of intuitionistic theories I" Studia Logica 64(3), (2000), pp. 323-344.

[Krivtsov 2000b] Krivtsov V.N.:, "A negationless interpretation of intuitionistic theories II" Studia Logica 65(2), (2000), pp. 155-179.

[Mezhlumbekova 1975] Mezhlumbekova V.: "Deductive Capabilities of Negationless Intuitionistic Arithmetic" Moscow University Mathematical Bulletin, Vol. 30(2), (1975)

[Mints 2006] Mints G.: "Notes on Constructive Negation" Synthese, Vol. 148(3), (2006), pp. 701-717.

[Nelson 1966] Nelson D.: "Non-null implication" Journal of Symbolic Logic 31, (1966), pp. 562-572.

[Nelson 1973] Nelson D.: "A complete negationless system" Studia Logica 32, (1973), pp. 41-49.

[Poincaré 1913] Poincaré H.: "Dernières Pensées" Flammarion, Paris (1913).
English translation in "Last thoughts", Dover Publications, N. Y., (1963).

[Prawitz 1965] Prawitz D.: "Natural Deduction A Proof-Theoretical Study." Almquist and Wiksell, Stockholm (1965).

[Troelstra and Van Dalen 1988] Troelstra A.S., Van Dalen D.: "Constructivism in Mathematics: An Introduction" Studies in Logic and the Foundations of Mathematics, North-Holland (1988), volume 2.

[Valpola 1955] Valpola V.: "Ein System der negationslosen Logik mit ausschliesslich realisierbaren Prädicaten" Acta Philosophica Fennica 9, (1955), pp. 1-247.

[Vredenduin 1953] Vredenduin P.G.J.: "The logic of negationless mathematics" Compositio Mathematica 11, (1953), pp. 204-277.