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Construction of Wavelets and Applications

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Abstract: A sequence of increasing translation invariant subspaces can be defined by the Haar-system (or generally by wavelets). The orthogonal projection to the subspaces generates a decomposition (multiresolution) of a signal. Regarding the rate of convergence and the number of operations, this kind of decomposition is much more favorable then the conventional Fourier expansion.

In this paper, starting from Haar-like systems we will introduce a new type of multiresolution. The transition to higher levels in this case, instead of dilation will be realized by a two-fold map. Starting from a convenient scaling function and two-fold map, we will introduce a large class of Haar-like systems. Besides others, the original Haar system and Haar-like systems of trigonometric polynomials, and rational functions can be constructed in this way. We will show that the restriction of Haar-like systems to an appropriate set can be identified by the original Haar-system.

Haar-like rational functions are used for the approximation of rational transfer functions which play an important role in signal processing [Bokor1 1998, Schipp01 2003, Bokor3 2003, Schipp 2002].

Key Words: Haar-like systems, multiresolution, wavelets, image and signal processing

Category: F.2.1, G.1, I.4

1 Introduction

The classical Fourier analysis where a signal is represented by its trigonometric Fourier transform, is one of the most widely spread tools in signal and image processing. Those mathematical problems, which appear at the reconstruction of continuous functions from its Fourier coefficients were already well known at the end of the $19th$ century. Du Bois-Reymond constructed a continuous function with a divergent Fourier series. Because of this problem Hilbert posed the question whether there exist any orthonormal systems for which the Fourier series with respect to this system do not posses this type of singularity.

The answer to this question was given by Alfréd Haar in 1909 [Haar 1910] when he constructed such an orthonormal system for which the Haar-Fourier series of continuous functions converge uniformly. This system, which was named after him the Haar system started to gain popularity from the 1960s. It turned out that the Haar-series possess some specific properties regarding convergence. Important function spaces can be characterized by Haar-Fourier coefficients [Schipp 2000, Schipp 1979, Gabor 1946, Walnut 2004]. The number of operations needed to compute the Haar-Fourier coefficients and reconstruct functions is much less then in the trigonometric case.

Using dilation and translation, the Haar-system can be constructed from a single function. Namely

$$h(x) = \begin{cases} 1 & (0 \le x < \frac{1}{2}) \\ -1 & (\frac{1}{2} \le x < 1) \\ 0 & (x \ge 1) \end{cases} \tag{1}$$

is the function generating the Haar-system. Using this function, the Haar-system $h_{n,k}(x)$, $(x \in \mathbb{R}_+ := [0, \infty)$), (normed by the maximum value), can be defined as follows:

$$h_{n,k}(x) = h(2^n x - k) \ (x \in \mathbb{R}_+, \ k, n \in \mathbb{N} \ := \{0, 1, 2, ...\}). \tag{2}$$

This type of construction was the basis for the wavelet constructions which started at the end of the 1980s [Daubechies 1988, Strang and Nguyen 1996, Meyer 1992, Mallat 2001]. In these new constructions, instead of h given in (1) there have been used smooth functions (the so called mother wavelets) and orthonormal or biorthogonal systems of type (2) have been constructed. These systems keep the good properties of the Haar-system, but besides that they are useful in reconstructing efficiently smooth functions as well. Because of these properties the wavelets became an efficient tool for signal and image processing [Kozaitis et al. 2005, Kozaitis and Cofer 2005, Lee and Kozaitis 2000] and [László et al. 2005].

New systems can be constructed from known orthogonal systems by argument transformation. For example, the Chebyshev system can be derived in this way from trigonometric systems. Based on this principle we constructed systems that can be used in optics and cornea topography for the mathematical description of the cornea [Schipp 2005]. In this paper a generalization of the Haar-system is given, where instead of dilation another type of argument transformation is used. These generalizations of the Haar-system contain free parameters. By an appropriate choice of these parameters we can construct Haar-like systems adapted to the specific problem. Such type of construction has been used in [Schipp at-2005], to identify transfer functions of systems which play an important role in control theory. A well known application of the classical Haar system is image compression. Based on the same principle, adaptive Haar-like systems can be constructed for functions with two variables, or for two dimensional pictures in order to analyze and compress them efficiently.

The Haar-system can be expressed by the characteristic functions

$$\chi_{n,k}(x) = \begin{cases} 1 \ (x <\in I_{n,k}), \\ 0 \ (x \in \mathbb{R}_+, \ x \notin I_{n,k}) \end{cases} \tag{3}$$

of the dyadic intervals $I_{n,k} := [k2^{-n}, \ (k+1)2^{-n})$ $(k, n \in \mathbb{N})$. Further on, between functions $\chi_{n,k}$ and $h_{n,k}$ the following relations are satisfied:

$$\begin{aligned} &i) \ \ h_{n,k}(x) = \chi_{n+1,2k} - \chi_{n+1,2k+1} \\ &ii) \ \chi_{n,k} = \chi_{n+1,2k} + \chi_{n+1,2k+1}(k, n \in \mathbb{N}). \end{aligned} \tag{4}$$

Equation $(4)ii)$ is called scaling equation as it connects two different scales, indexed by n and $n+1$.

The Haar-system is orthogonal with respect to the scalar product of space $L^2[0, \infty)$, that is:

$$\int_0^\infty h_{n,k}(x)h_{m,l}(x)dx = 0 \ ((n,k) \neq (m,l), n, m, k, l \in \mathbb{N}),$$

and further on, for any fixed value of $n > 0$, the system $\chi_{n,k}$ $(k \in \mathbb{N})$ is also orthogonal with respect to the same scalar product.

These systems generate a decomposition (multiresolution) of space $L^2[0, \infty)$. Specifically let us introduce the subspaces spanned by the functions $\chi_{n,k}$ $(k \in \mathbb{N})$:

$$\hat{V}_n := span\{\chi_{n,k} \ : k \in \mathbb{N}\} \ (n \in \mathbb{N}). \tag{5}$$

It follows from $(4.)ii)$. that the subspaces $\hat{V}_n$ $(n \in \mathbb{N})$ are increasing, i.e. $\hat{V}_n \subset \hat{V}_{n+1}$ $(n \in \mathbb{N})$.

The transition from subspace $\hat{V}_n$ to the next level of subspace $\hat{V}_{n+1}$, can be done by the scale transformation $A(x) = 2x$ (dilation). Namely function f belongs to space $\hat{V}_n$ if and only if for the function $(\delta_2 f)(x) := f(2x)$ $(x \geq 0)$ the relation $\delta_2 f \in \hat{V}_{n+1}$ is satisfied. It is easy to prove, that $\delta_2 \chi_{n,k} = \chi_{n+1,k}$, from which the above statement follows directly.

The support of the functions $h_{n,k}$ and $\chi_{n,k}$ is the interval $I_{n,k}$:

$$\{x : h_{n,k}(x) \neq 0\} = \{x : \chi_{n,k}(x) \neq 0\} = I_{n,k}.$$

In this paper we will examine only those types of functions which have their support in the interval $\mathbb{I} := I_{0,0} = [0, 1)$. In the Haar-Fourier expansion of these functions only those Haar-functions $h_{n,k}$, will appear which also have their support in this interval, that is for which condition $0 \leq k < 2^n$ is satisfied. In accordance with this it is convenient to index these systems continuously with natural numbers:

$$\begin{aligned} h_0(x) &:= \chi_0(x) := 1 \ (x \in [0,1)), \\ h_m &:= h_{n,k}, \ \chi_m := \chi_{n,k} \ (m = 2^n + k, k = 0, 1, ..., 2^n - 1, n \in \mathbb{N}). \end{aligned} \tag{6}$$

Instead of spaces $\hat{V}_n$ it is appropriate to introduce

$$V_n := span\{\chi_{n,k} \ : 0 \leq k < 2^n\} \ (n \in \mathbb{N}),$$

and the 2^n dimensional subspaces spanned by the Haar-functions

$$W_n := span\{h_{n,k} \ : 0 \leq k < 2^n\} \ (n \in \mathbb{N}).$$

It follows from $(4)i)$ that the orthogonal complementer of the subspace V_n with respect to V_{n+1} is W_n, that is:

$$V_n \subset V_{n+1}, \ V_{n+1} = V_n \oplus W_n \ (n \in \mathbb{N}). \tag{7}$$

The 2^N dimensional vector spaces V_N, consist of the so called dyadic step functions, constant on the intervals $I_{N,k}$ $(0 \leq k < 2^N)$. The restriction of these functions to the sets $\mathbb{I}_N := \{k2^{-N} \ : \ 0 \leq k < 2^N\}$ is a system of discrete functions, which can be conveniently used in numerical computation. Let us introduce on this space the discrete inner product

$$[f,g]_N := 2^{-N} \sum_{x \in \mathbb{I}_N} f(x)g(x). \tag{8}$$

For functions from V_N the well known inner product

$$\langle f, g \rangle := \int_0^1 f(x)g(x)dx \ (f, g \in L^2(\mathbb{I}))$$

coincides with

$$[f,g]_N = \langle f, g \rangle \ (f, g \in V_N).$$

It follows from this that the systems $\chi_{N,k}$ $(0 \leq k < 2^N)$, and h_m $(0 \leq m < 2^N)$ are orthogonal with respect to the inner product defined by (8).

We would like to emphasize that for functions defined on the interval $[0,1)$ the dilation operation has no sense. Extending these functions to the interval $[0,\infty)$ with periodicity 1, dilation can be defined through $(\delta_2 f)(x) := f(2x)$ $(x \geq 0)$.

In this paper starting out from Haar-like orthogonal systems we will introduce multiresolution. The Haar-like system $\mathcal{H}_m : X \to \mathbb{C}$ $(m \in \mathbb{N})$ defined on a set X

is generated by a basic function $\Phi : X \to \mathbb{C}$ and by map $A : X \to X$. Map Φ corresponds to the mother wavelet, and map A to dilation. Specifically when

$$X = \mathbb{I} = [0,1),\ \Phi = h, A(x) = 2x\ (mod\ 1)$$

we will get the original Haar-system. By the appropriate choice of maps Φ and A, we can construct discrete trigonometric-, and rational Haar-like functions. We will show that for any N a set $X^N := \{x_k^N : 0 \le k < 2^N\} \subset X$ of 2^N elements can be given on which system $\mathcal{H}_m^N$ $(0 \le m < 2^N)$ coincides with the discrete Haar-system:

$$h_m(k2^{-N}) = \mathcal{H}_m^N(x_k^N)\ (0 \le k,\ m < 2^N).$$

2 Construction of Haar-like Wavelets

In this section we give a procedure to construct Haar-like functions. The functions are defined on a set $X \neq \emptyset$ and are generated by a function $\Phi : X \to \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and by a twofold map $A : X \to X$. We assume that the maps Φ and A have the following properties: for every $x \in X$ there exists two $x^{'}, x^{''} \in X,\ x^{'} \neq x^{''}$ such that

$$A(x^{'}) = A(x^{''}) = x,\ \Phi(x^{'}) = -\Phi(x^{''}). \tag{9}$$

Map A is analogue of dilation δ_2, and function Φ corresponds to the mother wavelet. Define the 2^n-fold maps $A_n : X \to X$ and $\Phi_n : X \to \mathbb{T}$ by

$$\begin{aligned} &A_0(x) := x,\ A_{n+1}(x) := A(A_n(x)) = A_n(A(x)),\\ &\Phi_n(x) := \Phi(A_n(x))(n \in \mathbb{N}) \end{aligned} \tag{10}$$

and starting with $x_0^0 \in X$ introduce the sets

$$X^n := A_n^{-1}(x_0^0) := \{x \in X : A_n(x) = x_0^0\} := \{x_k^n : k = 0, 1, ..., 2^n - 1\}. \tag{11}$$

Sience $X^{n+1} = A^{-1}(A_n^{-1}(x_0^0)) = A^{-1}(X^n)$ we can choose the indices such that

$$A(x_{2k}^{n+1}) = A(x_{2k+1}^{n+1}) = x_k^n\ (k = 0, 1, ..., 2^n - 1). \tag{12}$$

If $x_0^0 = 0$ is a fixpoint of A, that is if $A(x_0^0) = x_0^0$ then $X^n \subset X^{n+1}$ $(n \in \mathbb{N})$.

The generalized scaling functions are defined by

$$\begin{aligned} &\mathcal{I}_{0,0}(x)\ = 1,\ \mathcal{I}_{n,k}(x) = 2^{-n} \prod_{j=0}^{n-1} (1 + \Phi_j(x)\overline{\Phi_j}(x_k^n)),\\ &\mathcal{H}_{n,k}(x) = \Phi_n(x)\mathcal{I}_{n,k}(x)\ \ (0 \le k < 2^n, n \ge 1). \end{aligned} \tag{13}$$

The system $\mathcal{I}_{n,k}$ $(0 \le k < 2^n)$ has similar properties on the set X^n as the collection $\chi_{n,k}$ $(0 \le k < 2^n)$ on $\mathbb{I}_n$, moreover the analogue of equation (4), that is, scaling and splitting relations hold.

Theorem 1. *For any* $0 \le k < 2^n,\ n \in \mathbb{N}$ *we have*

$$\begin{aligned} &i)\ \mathcal{I}_{n,k}(x_l^n) = \chi_{n,k}(l2^{-n}) \\ &ii)\ \mathcal{I}_{n+1,2k}(x)\ +\ \mathcal{I}_{n+1,2k+1}(x) = \mathcal{I}_{n,k}(A(x)), \\ &iii)\ \mathcal{I}_{n+1,2k}(x)\ -\ \mathcal{I}_{n+1,2k+1}(x) = \Phi(x)\overline{\Phi}(x_{2k}^{n+1})\mathcal{I}_{n,k}(A(x)). \end{aligned} \tag{14}$$

To get the analogue of the discrete Haar functions for every $N \in \mathbb{N}$ we introduce the finite systems

$$\begin{aligned} &\mathcal{I}_{n,k}^N(x) := \mathcal{I}_{n,k}(A_{N-n}(x)),\ \mathcal{H}_{n,k}^N(x) := \Phi_{N-n-1}(x)\overline{\Phi}(x_{2k}^{n+1})\mathcal{I}_{n,k}^N(x) \\ &(x \in X, 0 \le n < 2^n, 0 \le n < N). \end{aligned} \tag{15}$$

On set X^N these discrete systems $\mathcal{H}_m^N := \mathcal{H}_{n,k}^N$ $(m = 2^n + k,\ 0 \le k < 2^n, 0 \le n < N)$ are the same as the discrete Haar-system. Namely we have

Theorem 2. *For any* $0 \le k < 2^n,\ 0 \le n < N$ *and any* $x \in X$ *we have*

$$\begin{aligned} &i)\ \mathcal{H}_m^N(x_{N,k}) = h_m(k2^{-N})\ (0 \le m < 2^N), \\ &ii)\ \mathcal{I}_{n+1,2k}^N(x)\ +\ \mathcal{I}_{n+1,2k+1}^N(x) = \mathcal{I}_{n,k}^N(x), \\ &iii)\ \mathcal{I}_{n+1,2k}^N(x)\ -\ \mathcal{I}_{n+1,2k+1}^N(x) = \mathcal{H}_{n,k}^N(x). \end{aligned} \tag{16}$$

2.1 Examples

The original Haar system can be obtained in this way. Namely if

$$A(x) := 2x\ (mod\ 1) = frac(2x),\ \Phi(x) := h(x)\ (x \in X := [0, 1)),$$

then (11) gives the original Haar scaling functions.

To get a Haar-like system of complex trigonometric polynomials we set

$$X := \mathbb{T}, A(z) := z^2, \Phi(z) = z\ (z \in X).$$

In this case the functions $\mathcal{I}_{n,k}$ are complex trigonometric polynomials of order 2^n.

The construction of rational Haar-like systems is based on Blaschke functions

$$B_a(z) = \frac{z-a}{1-\overline{a}z}\ (a \in \mathbb{D} := \{w \in \mathbb{C} : |w| < 1\},\ z \in \mathbb{D} \cup \mathbb{T}).$$

If the parameter a belongs to $\mathbb{D}$ then the restriction of B_a to $\mathbb{D}$ is a bijection of $\mathbb{D}$. Furthermore B_a is a one-to-one map on the unit circle $\mathbb{T}$. Starting with functions

$$A(z) := B_a(z^2),\ \Phi(z) = z\ (z \in X := \mathbb{T})$$

we get a Haar-like system of rational functions.

2.2 Proofs

Proof of Theorem 1.

First we prove (14)i). If $k = l$ then for $j = 0, 1, ..., n-1$ we have $1 + \Phi_j(x_l^n)\overline{\Phi_j}(x_k^n) = 2$, and by (3) then (14)$i$) is true in this case. If $k \neq l$, then it follows from (9) and (11), that there exists a number j such that

$$0 \leq j < n,\ A_j(x_k^n) \neq A_j(x_l^n),\ A_{j+1}(x_k^n) = A_{j+1}(x_l^n).$$

Set $x^{'} := A_j(x_k^n) \neq x^{''} = A_j(x_l^n)$. Then $A(x^{'}) = A(x^{''})$ and from (9) and (10) we get

$$\Phi_j(x_k^n) = \Phi(x^{'}) = -\Phi(x^{''}) = -\Phi_j(x_l^n),$$

consequently

$$1 + \Phi_j(x_l^n)\overline{\Phi_j(x_k^n)} = 1 - \Phi_j(x_l^n)\overline{\Phi_j(x_l^n)} = 1 - 1 = 0,$$

and by (3) then (14)i) is proved.

To prove (14)ii) and (14)iii) observe that from the definition of Φ_j and from (10) and (12) for $t = 0, 1$ we get

$$\begin{aligned}\mathcal{I}_{n+1,2k+t}(x) &= 2^{-(n+1)} \prod_{j=0}^{n} (1 + \Phi_j(x)\overline{\Phi_j}(x_{2k+t}^{n+1})) \\ &= \frac{1 + \Phi(x)\overline{\Phi}(x_{2k+t}^{n+1})}{2} 2^{-n} \prod_{j=0}^{n-1} (1 + \Phi_j(A(x))\overline{\Phi_j}(A(x_{2k+t}^{n})) \\ &= \frac{1 + \Phi(x)\overline{\Phi}(x_{2k+t}^{n+1})}{2} \mathcal{I}_{n,k}(A(x)). \end{aligned} \qquad (17)$$

Since $\Phi(x_{2k+1}^{n+1}) = -\Phi(x_{2k}^{n+1})$, taking the sum and difference of functions $\mathcal{I}_{n+1,2k+t}$ $(t = 0, 1)$ we get (14)ii) and (14)iii).

Proof of Theorem 2.

Applying (14)ii) and (14)iii) for $A_{N-n-1}(x)$ instead of x in equation (15) we get

$$\mathcal{I}_{n+1,2k}^N(x) + \mathcal{I}_{n+1,2k+1}^N(x) = \mathcal{I}_{n+1,2k}(A_{N-n-1}(x)) + \mathcal{I}_{n+1,2k+1}^N(A_{N-n-1}(x)) =$$

$$= \mathcal{I}_{n,k}(A(A_{N-n-1}(x))) = \mathcal{I}_{n,k}(A_{N-n}(x)) = \mathcal{I}_{n,k}^N(x),$$

$$\mathcal{I}_{n+1,2k}^N(x) - \mathcal{I}_{n+1,2k+1}^N(x) = \mathcal{I}_{n+1,2k}(A_{N-n-1}(x)) - \mathcal{I}_{n+1,2k+1}^N(A_{N-n-1}(x)) =$$

$$= \Phi(A_{N-n-1}(x))\Phi(x_{2k}^{n+1})\mathcal{I}_{n,k}(A_{N-n}(x)) = \mathcal{H}_{n,k}^N(x).$$

and $(16)ii)$, and $(16)iii)$ is proved.

It follows from (12) that

$$A_j(x_k^n) = x_{[k2^{-j}]}^{n-j} \ (0 \le k < 2^n, 0 \le j \le n, n \in \mathbb{N}),$$

where $[t]$ denotes the integer part of the number $t \in \mathbb{R}$. This and (15) imply

$$\mathcal{I}_{n+1,2k}^N(x_l^N) = \mathcal{I}_{n,k}(A_{N-n}(x_l^N)) = \mathcal{I}_{n,k}(x_{[l2^{n-N}]}^n) = \chi_{n,k}(l2^{-N}),$$

$$\Phi_{N-n-1}(x_l^N)\overline{\Phi}(x_{2k}^{n+1}) = \Phi(x_{[l2^{n+1-N}]}^{n+1})\overline{\Phi}(x_{2k}^{n+1}) = \Phi(x_{2s+r}^{n+1})\overline{\Phi}(x_{2k}^{n+1}),$$

where $2s + r := [l2^{n+1-N}]$ and $r = 0, 1$. Thus from (15) we get

$$\mathcal{H}_{n,k}^N(x_l^N) = \Phi(x_{2s+r}^{n+1})\overline{\Phi}(x_{2k}^{n+1})\chi_{n,k}(l2^{-N})$$

and consequently

$$\mathcal{H}_{n,k}^N(x_l^N) = \begin{cases} 0 & (l2^{-N} \notin [k2^{-n}, (k+1)2^{-n})), \\ 1 & (l2^{-N} \in [k2^{-n}, (k+1/2)2^{-n})) \\ -1 & (l2^{-N} \in [(k+1/2)2^{-n}, (k+1)2^{-n})). \end{cases}$$

Thus $(16)i)$ is proved.

3 Applications

In this section we will show trigonometric and rational Haar-like functions constructed as desctibed in the previous section for both, the continuous and discrete case as well.

3.1 Haar-, and Haar scaling functions

The continuous and discrete Haar scaling functions $\mathcal{I}_{n,k}$, $\mathcal{I}_{n,k}^N$ and Haar functions $\mathcal{H}_{n,k}$, $\mathcal{H}_{n,k}^N$ are represented on the next figures for trigonometric polynomials i.e. $a = 0$ and for the general case, $a \neq 0$, i.e. rational functions, by using the corresponding values of the parameter a.

We would like to emphasize that Haar-like trigonometric polynomials $\mathcal{H}_{n,k}$ corresponding to the parameter $a = 0$, form an orthogonal system with respect to the ususal scalar product of the interval $I_{0,0}$.

Figure 1. represents the real and imaginary parts of the scaling function $\mathcal{I}_{n,k}$ generated by the two-fold map $A(z) = z^2$, that is for the the case $a = 0$ which corresponds to the continuous case of Haar-like scaling systems of complex trigonometric polynomials.

Figure 2. represents the real and imaginary parts of the continuous Haar function $\mathcal{H}_{n,k}$ generated by the two-fold map $A(z) = z^2$.

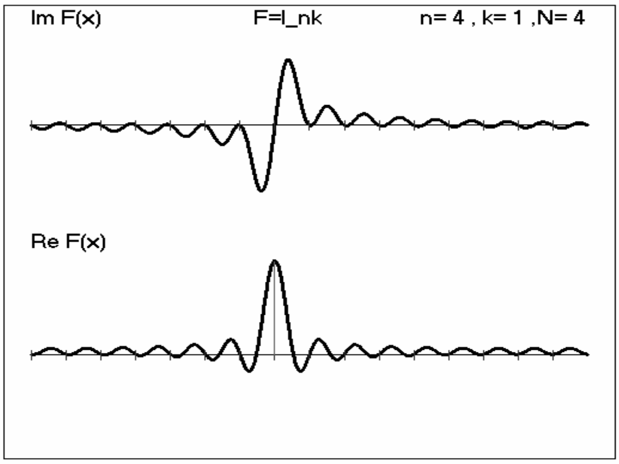

Figure 1: A Scaling function; a=0, continuous

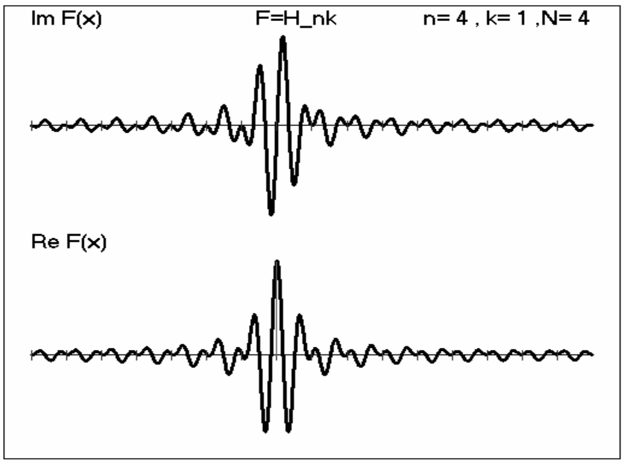

Figure 2: A Haar function; a=0, continuous

Figure 3. represents the real and imaginary parts of the scaling function $\mathcal{I}_{n,k}^N$ generated by the two-fold map $A(z) = z^2$, and the corresponding scaling function $\chi_{n,k}^N$. This case corresponds to a discrete Haar-like scaling system of complex trigonometric polynomials.

Figure 4. represents the real and imaginary parts of the Haar-function $\mathcal{H}_{n,k}^N$

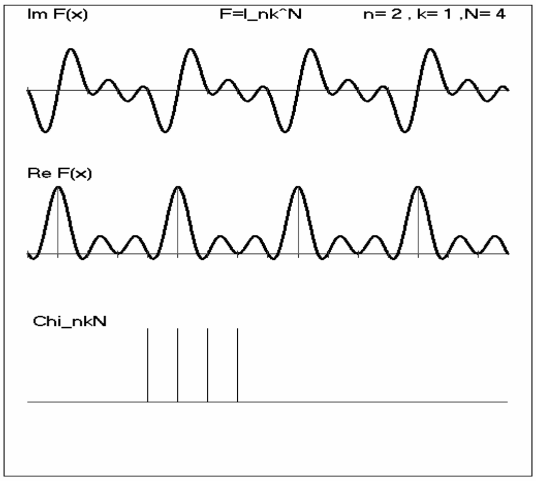

Figure 3: A Scaling function; a=0, discrete

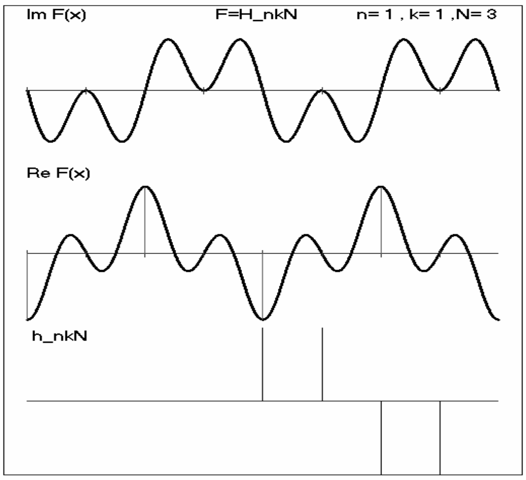

Figure 4: A Haar function; a=0, discrete

generated by the two-fold map $A(z) = z^2$, together with the corresponding Haar-function $h_{n,k}^N$.

Figures 5., 6., 7., and 8. represent the real and imaginary parts of the continuous and discrete rational scaling functions and Haar functions respectively, generated by the twofold map $A(z) = B_a(z^2)$, with $Re(a) = 0.6$, $Im(a) = 0$.

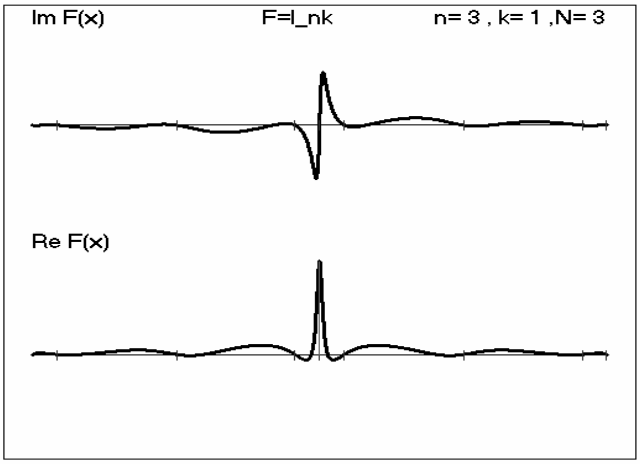

Figure 5: A Scaling function; a=0.6, continuous

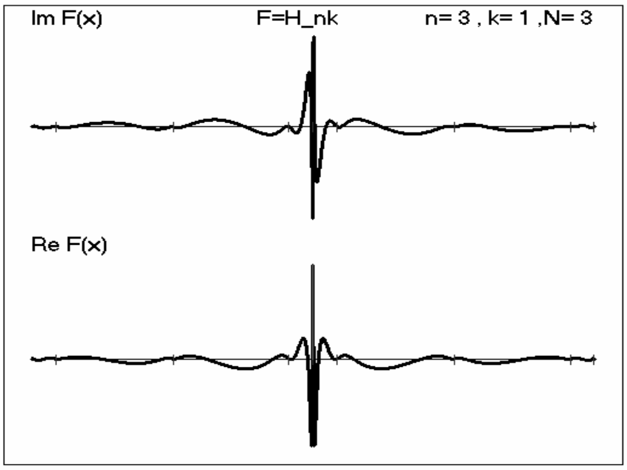

Figure 6: A Haar function; a=0.6, continuous

4 Conclusions

Wavelets are localized functions. Because of their favorable properties, they have important applications in many areas such as signal and image processing, edge detection, denoising and image compression. Due to this advantage of the wavelet

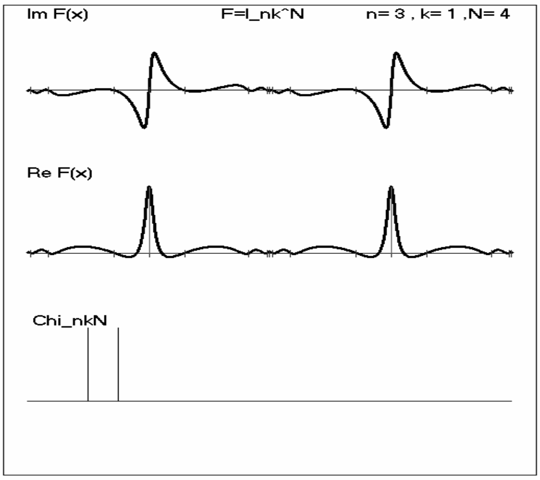

Figure 7: A Scaling function; a=0.6, discrete

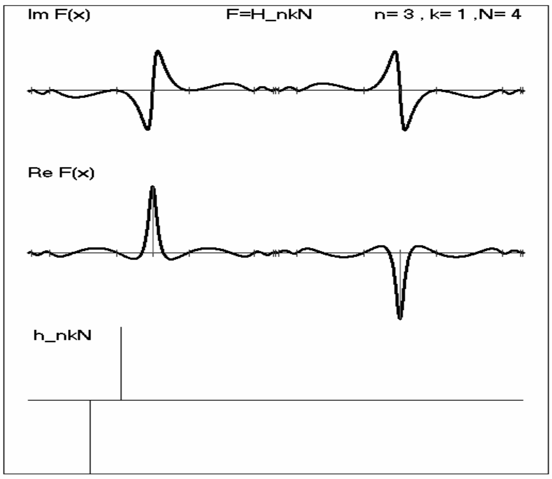

Figure 8: A Haar function; a=0.6, discrete

transform when compared to the Fourier transform it became an important and growing area to study.

In this paper we introduced a new type of multiresolution constructed by a large class of Haar-like systems. The systems are generated by the mother wavelet Φ and by a two-fold map A. We have shown that by an appropriate choice of Φ and A Haar-like trigonometric functions and Haar-like rational functions can be obtained.

These type of Haar-like systems can be used in optics and cornea topography for the mathematical description of the cornea [Schipp 2005]. By an appropriate choice of the parameters, Haar-like systems adapted to specific problems can be constructed. Such type of systems are used in [Schipp at-2005], to identify transfer functions of systems which play an important role in control theory. A well known application of the classical Haar system is image compression. Based on the same principle, adaptive Haar-like systems can be constructed for functions with two variables, or for two dimensional pictures in order to analyze and compress them efficiently, without significantly affecting the quality of the image.

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