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What is Continuity, Constructively?[1]

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Abstract: The concept of continuity for mappings between metric spaces should coincide with that of uniform continuity in the case of a compact domain, and still give rise to a category. In Bishop's constructive mathematics both requests can be fulfilled simultaneously, but then the reciprocal function has to be abandoned as a continuous function unless one adopts the fan theorem. This perhaps little satisfying situation could be avoided by moving to a point–free setting, such as formal topology, in which infinite coverings are defined mainly inductively. The purpose of this paper is to discuss the earlier situation and some recent developments.
Key Words: Continuity, constructive mathematics
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1 Compactness and uniform continuity

As laid out in [11], Bishop's constructive mathematics [7, 8] simultaneously generalises three varieties of mathematics:

1. the ZFC–based, so–called classical mathematics;
2. Markov–style recursive mathematics; and
3. intuitionistic mathematics *à la* Brouwer.

In fact, the theorems of Bishop's theory are valid in all these varieties, each of which can be obtained from Bishop's own by adding appropriate principles:

1. the law of excluded middle and the axiom of choice for classical mathematics;
2. Markov's principle and Church's thesis for recursive mathematics; and
3. Brouwer's continuity principles for intuitionistic mathematics: that is, the principle of continuous choice and the fan theorem.

(Note in this context that the axiom of choice implies a fragment of the law of excluded middle [5], and that in the presence of the principle of continuous choice the fan theorem for arbitrary bars is equivalent to its restriction to detachable bars [31, Chapter 4, Section 7.4].)

1 C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

The three models of Bishop's theory which we have listed above are mutually inconsistent with particular respect to the issues of compactness and continuity. In recursive and intuitionistic mathematics, every total function on the real line that can be defined at all must be pointwise continuous (by the Kreisel–Lacombe–Shoenfield–Tsejtin theorem and by Brouwer's continuity principles, respectively), while this is obviously false in classical mathematics. The uniform continuity of a pointwise continuous function on, say, the unit interval varies with the model in a different way, which not surprisingly is the same in which the validity of the Heine–Borel covering principle depends on the model.

Before we start to discuss our main issue of uniform continuity, we recall the intimately related status of compactness. Throughout this note, let X stand for an inhabited metric space. We also adopt Bishop's definition that X is compact precisely when it is totally bounded and complete [7, Chapter 4, Section 4][8, Chapter 4, Section 4]. This choice from the various classically equivalent formulations of compactness includes, in Bishop's framework, familiar "compact" metric spaces like the unit interval. In contrast to classical mathematics, however, one cannot expect a Bishop–style constructive proof that if X is compact, then X satisfies the Heine–Borel principle

HB *every open covering of X has a finite subcovering.*

The reason is that HB fails recursively already for X being the unit interval. More specifically, a covering of the whole real line by a sequence of bounded open intervals can be constructed in recursive mathematics such that every finite collection of these intervals has total length less than any given positive number [11, Chapter 3, Theorem 4.1]. On the other hand, HB for arbitrary compact X is not only valid classically, but also in intuitionistic mathematics. To be more precise, one can prove in Bishop's theory that if X is compact, then it is a uniform quotient of the Cantor space $2^{\mathbb{N}}$ [11, Chapter 5, Theorem 1.4][31, Chapter 7, Corollary 4.4], which satisfies HB precisely when Brouwer's fan theorem for arbitrary bars is valid [31, Chapter 4, Section 7.3]; whence X satisfies HB intuitionistically whenever it is compact [11, Chapter 5][31, Chapter 6, Section 3, and Chapter 7, Section 4]. For those and other aspects of compactness in constructive mathematics, we also refer to [19].

The uniform continuity principle

UC *every pointwise continuous mapping on X is uniformly continuous*

behaves just as HB, of which it is a classical consequence [18, 4.3.31–32]. In fact, UC fails recursively for X being the unit interval, but holds classically and intuitionistically whenever X is compact. More precisely, in the presence of Church's thesis and by means of a Specker sequence one can construct a pointwise continuous function h from $[0,1]$ onto $]0,1[$ which lacks uniform continuity [11, Chapter 3, Section 3]. On the other hand, Brouwer's fan theorem for arbitrary bars guarantees the validity of UC for arbitrary compact X, through that of HB.

2 Bishop's definition

With his particular way to put continuity, Bishop managed to handle the situation of UC. To explain this, we need to recall that Bishop understood X to be locally compact if and only if every bounded subset of X is contained in a compact subset [7, Chapter 4, Section 5][8, Chapter 4, Section 6]. Local compactness is a notion depending on the metric, because so is the notion of boundedness. Every compact space is locally compact, and every locally compact space is complete and separable. The euclidean space of arbitrary dimension is locally compact, and so is every closed interval.

As with compactness, Bishop again made an appropriate choice when he defined in [7, Chapter 4, Section 5][8, Chapter 4, Section 6] that

> *a mapping on a locally compact metric space is continuous precisely when it is uniformly continuous on every compact subset of its domain.*

Besides excluding the recursive counterexample h mentioned above, this definition of continuity works quite well for mappings on locally compact metric spaces—to which Bishop explicitly restricted his attention—at least as far as composition is concerned.

Contrary to common belief, namely, the composition $\psi \circ \varphi : X \to Z$ of two continuous mappings $\varphi : X \to Y$ and $\psi : Y \to Z$ between metric spaces is again continuous provided that continuity is understood in Bishop's sense. This is because in Bishop's definition of continuity the domains of all the mappings under consideration, which in our case are X and Y, must be locally compact. In fact, we only need the intermediate metric space Y to be locally compact.

To see this, consider a compact subset X_0 of X. By definition, φ is uniformly continuous on X_0; whence $\varphi(X_0)$ is totally bounded: the image of a totally bounded subset under a uniformly continuous mapping is totally bounded [11, Chapter 2, Proposition 4.3]. In particular, $\varphi(X_0)$ is a bounded subset of Y. Since, in addition, Y is locally compact, there is a compact subset Y_0 of Y that contains $\varphi(X_0)$. As ψ is uniformly continuous on Y_0, also $\varphi \circ \psi$ is uniformly continuous on X_0: the composition of two uniformly continous mappings is uniformly continous.

A drawback of Bishop's concept of continuity is that it does not include the reciprocal function, for the lack of a locally compact domain. Although $x \mapsto 1/x$ as defined on $]0,1]$ is uniformly continuous on every compact subset of $]0,1]$ (by [11, Chapter 2, Lemma 3.3], any such subset is bounded away from 0), it falls short from being continuous in Bishop's sense: as a whole, its domain is not locally compact, because it is not complete.

Without requiring the intermediate metric space to be locally compact, forming compositions fails drastically to preserve uniform continuity on compact subsets. Let g be the uniformly continuous function on $[0,1]$, constructed by means of Church's thesis [11, Chapter 6, Corollary 2.9], that maps $[0,1]$ onto $]0,1]$. Although $1/x$ is uniformly continuous on every compact subset of $]0,1]$, the composition $1/g$ of g with $1/x$, which is defined on $[0,1]$, cannot be uniformly continous. In fact, $1/g$ is unbounded, because g has infimum 0.

The example of this function g shows furthermore that one cannot prove in Bishop's framework that uniformly continuous images of compact spaces are again compact, as is the case in classical mathematics. (The problem is the preservation of completeness.) This observation necessitates once more Bishop's request that a continuous function have a locally compact domain, and somehow hints at the variant of continuity we remember next.

3 Bridges's modification

Following an idea of Bishop, Bridges healed the defect that the former's concept of continuity is no longer closed under composition as soon as the precondition is dropped that the domains be locally compact. The solution is to require instead more from the functions. Bridges defined in [9, Chapter 2, Section 7] that

> *a mapping on an arbitrary metric space is continuous if and only if it is uniformly continuous near every subset of the domain which is a compact image.*

This definition requires two explanations: first, a compact image is the image of a compact space under a uniformly continuous mapping; secondly, a mapping f on X is called uniformly continuous near a subset X_0 of X if for every $\varepsilon > 0$ there is $\delta > 0$ such that $f(y)$ is within ε of $f(x)$ for each $x \in X_0$ and for all the $y \in X$ which not necessarily belong to X_0 but still are within δ of x.

Bridges thus arrived at a concept of continuity for mappings between arbitrary metric spaces which is closed under composition, and coincides with Bishop's for every mapping whose domain is locally compact [9, Chapter 2, Section 7]. Every compact space is a compact image, and if a mapping is uniformly continuous near a subset, then it is uniformly continous on this subset. Hence Bridges's variant of continuity entails uniform continuity on compact subsets (in fact, on compact images) for mappings between arbitrary metric spaces.

Unlike uniform continuity on compact subsets, Bridges's modified concept still excludes the function $1/x$—at least unless one accepts the fan theorem (see below). However, Bridges's notion of continuity differs from Bishop's original one inasmuch as while "only" the lack of a locally compact domain as required in the definition excludes $1/x$ from the latter, one can give a more concrete reason in the case of the former. More specifically, $]0,1]$ is a compact image recursively, thanks to the presence of the function g recalled above. Hence if one exclusively works in the recursive setting, and assumes that $1/x$ be uniformly continuous near compact images, then it would be uniformly continuous on the whole of $]0,1]$, which is absurd.

Bridges's modified concept also shows a possible defect from the predicative perspective. While the compact subsets of a metric space form the completion of the finite subsets with respect to the Hausdorff metric, it is questionable whether the subsets that are compact images form a set. In all, one might still miss the definitive concept of continuity for constructive mathematics *à la* Bishop. We should anyway appreciate the elegant way in which Bishop circumvented problematic cases, such as the function $1/x$, due to which way he enabled his theory to accommodate recursive mathematics as well.

4 The fan theorem

An equivalent of the fan theorem for detachable bars [11, Chapter 6, Corollary 2.8][6], the positivity principle

PP *every uniformly continuous real–valued function on a compact metric space that attains only positive values has positive infimum,*

rules out recursive phenomena like the aforementioned function g. Hence PP is likely to enable $1/x$ to lie within the realm of continous functions.

Waaldijk [30] has indeed shown that over Bishop's theory the fan theorem for detachable bars is necessary and sufficient for bringing $1/x$ into the modified concept of continuity worked out by Bridges. We now summarise Waaldijk's proof, which is based on the observation that PP can be put equivalently as

every compact image contained in $]0,+\infty[$ *is bounded away from* 0.

In this form, PP implies that $1/x$ is continuous in Bridges's sense, for it is uniformly continuous on every set of the form $]r,+\infty[$ with $r>0$. Conversely, PP follows from the assumption that $1/x$ is uniformly continuous near every compact image contained in $]0,+\infty[$. In fact, if $f: X \to]0,+\infty[$ is uniformly continuous and X is compact, then $f(X)$ is a compact image contained in $]0,+\infty[$. In particular, $1/x$ is uniformly continous on $f(X)$, so that $1/f$ is uniformly continuous on X, and thus bounded—that is, f has positive infimum.

Referring to this observation, Waaldijk [30] proposes that the fan theorem for detachable bars be added to Bishop–style constructive mathematics. However, if this would be accepted, then the recursive model had to be abandoned. This is not only because PP is incompatible with the presence of recursive functions like

g, but also since this version of the fan theorem conflicts directly with Church's thesis [11, Chapter 5, Proposition 3.1].

As the classical contrapositive of weak König's lemma, the fan theorem for detachable bars is classically valid, which the principle of continuous choice, in the sense of [11, p. 107], is not. This other cornerstone of intuitionistic mathematics is not only inconsistent with Church's thesis, but also with classical logic [11, Chapter 5, Proposition 2.1, Theorem 2.2].

So adding only the fan theorem, without continuous choice, would cause fewer doubts than moving to full–fledged intuitionism, at least from the classical point of view. Can different frameworks for constructive mathematics still cope in a satisfying way with the phenomenon of continuity, but without invoking the fan theorem and thus without renouncing recursive mathematics?

5 The point–free perspective

In the point–free setting of formal topology *à la* Martin–Löf and Sambin (see [28] for a comprehensive survey), HB for compact intervals as proved by Cederquist and Negri [12] is a consequence of the restriction to inductively generated coverings which is characteristic of Martin–Löf type theory [21, 22, 23], the background of formal topology. Building upon this formal version of HB, Palmgren has recently answered our question in the affirmative [26].

To this end, Palmgren has shown that every function on the real line which is continuous in Bishop's sense can be represented by a continuous mapping—that is, a morphism of formal topologies—on the formal topology of the real numbers. Before we can explain this part of Palmgren's work, we need to recall how the real numbers can be put as a formal topology, following Negri and Soravia [24].

The initial data is the set S of all pairs (p, q) of rational numbers with $p < q$, which are called basic opens of this formal topology. Each $(p, q) \in S$ is thought to represent the open interval $]p, q[\subseteq \mathbb{R}$. In this interpretation, a formal real is a subset ξ of S that behaves as if the $]p, q[$ with $(p, q) \in \xi$ were a neighbourhood filter of some $x \in \mathbb{R}$.

This construction meets the intended meaning of a real number in Bishop's theory, which is cast into axioms in [10]. Moreover, every constructively reasonable model of $\mathbb{R}$, such as the located Dedekind cuts, is isomorphic to the formal reals, by assigning the formal real $\overline{x} = \{(p, q) \in S : p < x < q\}$ to each $x \in \mathbb{R}$. Note that the basic opens are the primitive objects, and the formal reals the derived ones.

Now let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function in Bishop's sense. Palmgren sets $(p, q)\, A_f\, (r, s)$ precisely when $f\,(]p, q[) \subseteq\,]r, s[$, and shows that one thus gets an continuous mapping $A_f \subseteq S \times S$ on the formal topology of the real numbers: that is, a relation whose required properties reflect how a continuous function f acts on the open intervals represented by the basic opens.

As any continuous mapping of this kind, A_f induces a function a_f on the formal reals. This is given by assigning to every formal real ξ the formal real $a_f(\xi)$ with $(r,s) \in a_f(\xi)$ if and only if $(p,q)\, A_f\, (r,s)$ for some $(p,q) \in \xi$. By the definition of A_f, we have $a_f(\overline{x}) = \overline{f(x)}$ for every $x \in \mathbb{R}$, in which sense f is represented by A_f.

After all this, Palmgren proves that, by virtue of the aforementioned formal variant of HB, a function on a compact interval of formal reals is uniformly continuous whenever it is induced by a continuous mapping from the formal topology of this compact interval to that of the real line. In addition, the continuous mappings between formal topologies are automatically closed under composition, which essentially is the composition of relations, and include $1/x$ in a natural way. Indeed, the continuous mapping $A_{1/x}$ corresponding to $1/x$ is given by setting $(p,q)\, A_{1/x}\, (r,s)$ precisely when either $q < 0$ or $0 < p$ and both $r \leq 1/q$ and $1/p \leq s$: that is, in intuitive terms, $0 \notin\,]p,q[$ and $]1/q, 1/p[\, \subseteq\,]r,s[$.

Curi [16] has lifted Palmgren's achievements from the context of real intervals to arbitrary formal metric spaces. Apart from any talk of uniform continuity, Palmgren and Curi have thus established, via formal topology, a link between Martin–Löf type theory and Bishop–style constructive mathematics. To perform this task was overdue inasmuch as the former had originally been intended as a formal system for the latter: as "a full scale system for formalizing intuitionistic mathematics as developed, for example, in the book by Bishop" [21, 23].[2]

The feasibility of an undertaking like this was already clear from Bishop's book [7], which is generally free of truly impredicative moves that would have made impossible to formalise it in a predicative system like Martin–Löf's. Also, the constructive Zermelo–Fraenkel set theory (CZF) founded by Aczel [1, 2, 3, 5] both was designed as a predicative framework for Bishop's constructive mathematics and could still be interpreted within type theory.

6 Sets of points

One issue, however, was sometimes felt to be problematic until recently: already the reals, let alone the points of any more involved space, might not form a set in the sense of type theory, and thus might lie beyond its range. Now it has been done away, as follows, with the objection that the point spaces right at the basis of analysis might not be sets.

First, Curi arrived at the conclusion that the points of a locally compact regular formal topology form a set [14, 15, 17]. This class of formal topologies is the point–free counterpart of that of locally compact Hausdorff spaces; it

[2] Martin–Löf's first book [20], however, was written independently of Bishop's [7], which "the author [Martin–Löf] did not get access to until after this manuscript [Martin–Löf's] was finished" [20, p. 13].

includes all formal metric spaces, and thus the formal real line. More generally, all points of a regular formal topology are maximal (Sambin [28, p. 363]), so that they form a set, as Palmgren showed in [25]. Curi and Palmgren have then generalised their results to sets of continuous mappings [16, 27], of which the sets of points they considered before are particular cases.

Aczel [4] gave a variant of Palmgren's type–theoretic construction in CZF, by which he could weaken Curi's hypothesis of local compactness to the one of set presentability. A fragment of CZF suffices to show that every complete separable metric space is a set [13], which result confirms in an unexpected way Bishop's intuition to concentrate on separable metric spaces in constructive analysis. In [7, Appendix A], Bishop even argued for his view of the concept of an nonseparable metric space as an instance of "pseudo-generality" from the constructive perspective.

7 Inductive definitions

It is in order to remember that the predominant role of inductive definitions within Martin–Löf type theory originated in nothing but the intention to eliminate the fan theorem [20], if not Brouwer's choice sequences in general. On the other hand, it is the peculiar character of choice sequences, the seminal concept of intuitionistic analysis, on which the intuitionistic argument for the fan theorem is based [11, Chapter 5, Section 3].

As we have indicated in [29], the fan theorem is another instance of the somewhat heuristic argument that uniformisation principles of a certain kind be constructively acceptable (our prime example was countable choice). This argument reads as follows: "if the only conceivable way to constructively verify a $\forall\exists$–statement is to prove the corresponding $\exists\forall$–statement, then both statements are to be identified with each other."

By the exclusive use of inductive definitions, one can avoid any such reference to heuristics, because to construct an object following the prescribed rules is then by definition the only way to build it, along which the object can thus legitimately be decomposed into its parts. We have seen above how inductive definitions, standing behind the validity of HB in the point–free setting of formal topology, help to circumvent the unwanted phenomena of continuity that seem to be unavoidable in any point–set setting like Bishop's.

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