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On Firmness of the State Space and Positive Elements of a Banach Algebra[1]

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Abstract: This paper is an investigation of positive elements in a Banach algebra. Under the firmness of the state space of a Banach algebra, it is shown that even powers of positive Hermitian elements are in fact positive.

Key Words: Banach algebra, constructive mathematics, state space, Hermitian elements, positive elements
Category: F.1

1 Introduction

The discussion and analysis throughout this paper will be carried out within the framework of Bishop's constructive mathematics[2].

It was during the author's postgraduate studies under the guidance and supervision of Bridges that firmness of the space of a Banach algebra was curiously looked at. Bishop and Bridges in the final chapter of [4, page 462] discuss the firmness of the spectrum of Banach algebra. It is a slight translation of that approach that motivated Bridges and the author to use firmness of the state space (which is related to the spectrum) in the investigation of positive elements [9, 13]. It should be pointed out that this short article has two main aims: first, to highlight this interesting aspect of constructive Banach algebra theory, and secondly to stand as one of the testimonies to the many areas where Bridges had been and currently working on. Furthermore, it is the intention of the author that the materials presented in this article would motivate future investigations on constructive Banach algebra theory.

The development of constructive Banach algebra theory can be traced back to Bishop's work in [2]. Bishop in the final chapter of [3] shed lights in the constructivisation process and, together with Bridges, topped it with a much smoother development in [4]. For current and recent works on constructive Banach algebra theory, see [5, 6, 8, 10, 11, 15].

There are two sections that follow immediately after this introductory one. The first contains some technical results and definitions, and the last presents the main results. Additionally, there is a brief discussion of extreme points of a state space and its connection to the character space of the Banach algebra.

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

[2] This is simply mathematics based on intuitionistic logic where 'existence' is strictly interpreted as 'computability'. Details on 'constructive mathematics' can be found in [1, 3, 4, 12].

2 Preliminary

We write B to denote a complex unital Banach algebra with identity e and B_1' the unit ball of the dual space B' of B. We define the ***state space*** of B to be set

$$V_B = \{f \in B' : f(e) = 1 = \|f\|\}.$$

For each $t > 0$ the set

$$V_B^t = \{f \in B' : \|f\| \leq 1, |1 - f(e)| \leq t\}$$

is a t–approximation to V_B.

A ***character*** of B is a bounded homomorphism of B onto $\mathbf{C}$, and the ***character space*** (or ***spectrum***) of B is the set

$$\Sigma_B = \{u \in B' : u(e) = 1, u(xy) = u(x)u(y) \text{ for all } x, y \in B\}.$$

Bishop and Bridges [4, page 452] showed that we can't hope in general to prove constructively the compactness of the spectrum. To see this, let $(a_n)_{n=0}^{\infty}$ be an increasing binary sequence and B the algebra consisting of all sequences $\mathbf{x} = (x_n)_{n=0}^{\infty}$ of complex numbers for which

$$\|\mathbf{x}\| = \sum_{n=0}^{\infty} (1 - a_n)|x_n| \tag{1}$$

exists. We define the elements $\mathbf{x}$ and $\mathbf{y} = (y_n)_{n=0}^{\infty}$ to be *equal* if $\|\mathbf{x} - \mathbf{y}\| = 0$. Then B is a Banach space equipped with norm given by (1). Moreover, if we define the product of any two elements $\mathbf{x}$ and $\mathbf{y}$ of B by

$$\mathbf{xy} = \left(\sum_{i=0}^{n} x_i y_{n-i}\right)_{n=0}^{\infty},$$

then B is a Banach algebra with identity $\mathbf{e} = (1, 0, 0, \ldots)$. Let

$$\mathbf{z} = \left(1, 2^{-1}, 2^{-2}, 2^{-3}, \ldots\right) \in B.$$

If $a_n = 1$ for some n, then the character space Σ_B of B consists of the single element $\mathbf{x} \mapsto x_0$. On the other hand, if $a_n = 0$ for all n, then to each complex number ξ with $|\xi| \leq 1$ there corresponds an element u_ξ of Σ_B defined by

$$u_\xi(\mathbf{x}) = \sum_{n=0}^{\infty} x_n \xi^n.$$

Suppose Σ_B is compact. Since the mapping $u \mapsto |u(\mathbf{z})|$ is uniformly continuous relative to the weak* topology on the unit ball of B' it maps Σ_B to a totally bounded subset of $\mathbf{R}$; whence

$$R = \sup\{|u(\mathbf{z})| : u \in \Sigma_B\}$$

exists. Either $R > 1$ or $R < 2$. In the first case, we have $a_n = 0$ for all n. In the second case, we cannot have $a_n = 0$ for all n. Thus the statement

> *The spectrum of every separable commutative unital Banach algebra is compact.*

implies WLPO.

Recall that an element f of X', where X of a normed linear space, is ***normable*** if its ***norm***

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$$

exists. If X' is separable, and $(x_n)_{n=1}^{\infty}$ is a dense sequence in the unit ball

$$X'_1 = \{f \in X' : \forall x \in X \ (|f(x)| \leq \|x\|)\}$$

of X', then the weak* topology on X' is induced by the ***double norm***[3], defined by

$$|||f||| = \sum_{n=1}^{\infty} 2^{-n} |f(x_n)| \qquad (f \in X').$$

Proposition 1. *For all but countably many $t > 0$, V_B^t is a nonempty, weak* compact subset of B'.*

Proof. Since the mapping $f \mapsto |1 - f(e)|$ is uniformly continuous on B'_1 relative to the double norm, we see from Theorem 4.9 of [4, page 98] that for all but countably many $t > 0$, V^t is either empty or weak* compact. An application of Corollary 4.5 of [4, page 341] shows that for such t, V^t is nonempty and therefore weak* compact. Q.E.D.

We say that $t > 0$ is ***admissible*** if V_B^t is weak* compact. Note that

$$V_B = \bigcap \left\{V_B^t : t > 0 \text{ is admissible}\right\},$$

the intersection of a family of nonempty, weak* compact sets that is descending in the sense that if $0 < t' < t$, then $V_B^{t'} \subset V_B^t$. Being the intersection of a family of complete sets, V_B is complete relative to the double norm.

We say that V is ***firm*** if it is compact and $\rho_w(V^t, V) \to 0$ as $t \to 0$, where ρ_w is the Hausdorff metric on the set of weak* compact subsets of B'_1.

An element x of B is:

- ***Hermitian*** if for each $\varepsilon > 0$ there exists $t > 0$ such that $|\mathrm{Im}\, f(x)| < \varepsilon$ for all $f \in V^t$; we denote the set of all Hermitian elements of B by $\mathrm{Her}(B)$.
- ***positive*** if for each $\varepsilon > 0$ there exists $t > 0$ such that $\mathrm{Re}\, f(x) \geq -\varepsilon$ and $|\mathrm{Im}\, f(x)| < \varepsilon$ for all $f \in V^t$; we then write $x \geq 0$.

An element f of B' is a ***positive linear functional*** if $f(x) \geq 0$ for each positive element x of B; we then write $f \geq 0$.

The following lemma is stated is proved [9].

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[3] Double norms defined by different dense sequences in X are equivalent on X'_1, and X'_1 is weak* compact. Moreover, for each $x \in X$ the mapping $f \mapsto f(x)$ is uniformly continuous on X'_1 with respect to the double norm.

Lemma 2. *Suppose that the state space of B is firm. Let A be a Banach subalgebra of B, let $\{x_1, \ldots, x_N\}$ be a finitely enumerable subset of A with $x_1 = e$, and let $\varepsilon > 0$. Then there exists an admissible $t > 0$ such that for each $f \in A'$ with $\|f\| \leq 1$ and $|1 - f(e)| \leq t$, there exists $g \in V_A$ with*

$$|f(x_k) - g(x_k)| \leq \varepsilon \qquad (1 \leq k \leq N)$$

Lemma 3. *Let $(K_\lambda)_{\lambda \in L}$ be a nonempty family of totally bounded subsets of a metric space X, and let $K = \bigcap_{\lambda \in L} K_\lambda$. Suppose that for each $\varepsilon > 0$ there exists $\lambda \in L$ such that for each $x \in K_\lambda$ there exists $y \in K$ with $\|x - y\| < \varepsilon$. Then K is totally bounded. If also each K_λ is complete, then K is compact.*

Proof. Given $\varepsilon > 0$, choose $\lambda \in L$ as in the hypotheses. Let $\{x_1, \ldots, x_N\}$ be a finite ε–approximation to K_λ, and for each n choose $y_n \in K$ such that $\|x_n - y_n\| < \varepsilon$. Let $y \in K \subset K_\lambda$. Then there exists n such that $\|y - x_n\| < \varepsilon$ and therefore

$$\|y - y_n\| \leq \|y - x_n\| + \|x_n - y_n\| < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus $\{y_1, \ldots, y_n\}$ is a 2ε–approximation to K. Since $\varepsilon > 0$ is arbitrary, K is totally bounded. If also each K_λ is complete, then K is an intersection of complete sets and so is complete; whence it is compact. Q.E.D.

3 Firmness and positivity

Proposition 4. *If the state space of B is firm, then so is the state space of every separable Banach subalgebra of B.*

Proof. Let A be a separable Banach subalgebra of B, $(x_n)_{n=1}^{\infty}$ a dense sequence in the unit ball of A, and $\|\!|\cdot|\!\|$ the corresponding double norm on A'. Given $\varepsilon > 0$, choose N such that $\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon$. Using Lemma 2, choose $t > 0$ such that

- V_B^t and V_A^t are weak* compact,
- $\rho_w(V_B^t, V_B) < \varepsilon$, and
- for each $f \in V_A^t$ there exists $g \in V_A$ such that

$$|f(x_k) - g(x_k)| \leq \varepsilon \qquad (1 \leq k \leq N). \tag{2}$$

Let $f \in V_A^t$, and choose $g \in V_A$ such that (2) holds. We have, in A_1',

$$\begin{aligned}
\|\!|f - g|\!\| &= \sum_{n=1}^{\infty} 2^{-n} |(f-g)(x_n)| \\
&= \sum_{n=1}^{N} 2^{-n} |f(x_n) - g(x_n)| + \sum_{n=N+1}^{\infty} 2^{-n} |f(x_n) - g(x_n)| \\
&\leq \sum_{n=1}^{N} 2^{-n}\varepsilon + 2 \sum_{n=N+1}^{\infty} 2^{-n} \\
&< 3\varepsilon.
\end{aligned}$$

It follows from Lemma 3 that V_A is weak* compact. It is then clear from the foregoing that $\rho_w(V_A^t, V_A) \to 0$ as $t \to 0$. Q.E.D.

Let K be a convex subset of a normed space, and let $x_0 \in K$. We say that x_0 is

– a ***classical extreme point*** of K if

$$\forall x, y \in K \left(x_0 = \frac{1}{2}(x+y) \Rightarrow x = y = x_0 \right);$$

– an ***extreme point*** of K if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \in K \left(\left\| x_0 - \frac{1}{2}(x+y) \right\| < \delta \Rightarrow \|x - y\| < \varepsilon \right).$$

An extreme point is a classical extreme point, and the converse holds classically.

The proof of the next result is similar to that given in [14, page 38] for the special case where B is a Banach algebra of functions. This is proved in [9, 13].

Proposition 5. *Let A be a commutative, unital Banach algebra generated by Hermitian elements, and*

$$K^0 = \{ f \in A' : f \geq 0, \, f(e) \leq 1 \}.$$

Then every classical extreme point of K^0 is an element of Σ_B.

Lemma 6. *For each $t \in (0,1)$, if $0 < \alpha, \beta \leq 1$ and $1 - \frac{1}{2}(\alpha + \beta) < t/2$, then $\alpha > 1 - t$ and $\beta > 1 - t$.*

Proof. If $1 - \frac{1}{2}(\alpha + \beta) < t/2$, then

$$0 \leq \frac{1}{2}(1-\alpha) + \frac{1}{2}(1-\beta) < \frac{t}{2},$$

so both $\frac{1}{2}(1-\alpha) < t/2$ and $\frac{1}{2}(1-\beta) < t/2$. Hence $\alpha > 1 - t$ and $\beta > 1 - t$. Q.E.D.

Proposition 7. *If the state space V of B is firm, then every extreme point of V is a character of B.*

Proof. Let $|||\cdot|||$ be the double norm corresponding to a dense sequence $(x_n)_{n=1}^{\infty}$ in the unit ball of B with $x_1 = e$. Noting that $V \subset K^0$, we show that every extreme point of V is also one of K^0. Accordingly, let f_0 be an extreme point of V, and let $\varepsilon > 0$. Choose $\delta_1 \in (0, \varepsilon)$ such that if $f, g \in V$ and $\left|\left|\left| \frac{1}{2}(f+g) - f_0 \right|\right|\right| < \delta_1$, then $|||f - g||| < \varepsilon$. Then choose an admissible $t > 0$ such that $\rho_w(V^t, V) < \delta_1/2$. Finally, choose $\delta_2 > 0$ such that if $f, g \in B'$ and $|||f - g||| < \delta_2$, then $|f(e) - g(e)| < t/2$. Now let

$$\delta = \min \left\{ \frac{1}{2}\delta_1, \delta_2 \right\},$$

and consider $f, g \in K^0$ with $\left|\left|\left| \frac{1}{2}(f+g) - f_0 \right|\right|\right| < \delta$. Since

$$\left| \frac{1}{2}(f+g)(e) - 1 \right| = \left| \frac{1}{2}(f+g)(e) - f_0(e) \right| < \frac{t}{2},$$

we have $|1 - f(e)| < t$ and $|1 - g(e)| < t$, by Lemma 6; whence $f, g \in V^t$, and therefore there exist $f', g' \in V$ such that $\|f - f'\| < \delta_1/2$ and $\|g - g'\| < \delta_1/2$. We now have

$$\left\|\frac{1}{2}(f' + g') - f_0\right\| \leq \left\|\frac{1}{2}(f + g) - f_0\right\| + \frac{1}{2}\|f - f'\| + \frac{1}{2}\|g - g'\|$$
$$< \frac{1}{2}\delta_1 + \frac{1}{4}\delta_1 + \frac{1}{4}\delta_1 = \delta_1.$$

Hence $\|f' - g'\| < \varepsilon$, and therefore

$$\|f - g\| \leq \|f - f'\| + \|f' - g'\| + \|g - g'\| < \varepsilon + \delta_1 < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof that f_0 is an extreme point, and therefore a classical extreme point, of K^0. By Proposition 5, f_0 is a character of B. Q.E.D.

Proposition 8. *If V is weak* compact, then every element of V is a convex combination of characters of B.*

Proof. It is easily shown that V is convex. An application of the Krein–Milman Theorem [4, page 363, (7.5)] shows that V is the closed convex hull of its extreme points; so we can apply Proposition 7. Q.E.D.

Corollary 9. *If the state space of B is firm, then the character space of every separable commutative Banach subalgebra of B is nonempty.*

Proof. Let A be a separable commutative Banach subalgebra of B. Proposition 4 shows that V_A is firm; in particular, it is compact and so has extreme points. By Proposition 7, those extreme points are characters of A. Q.E.D.

Proposition 10. *Let V be firm. Then $a \in \mathrm{Her}(B)$ is positive if and only if $f(a) \geq 0$ for each $f \in V$.*

Proof. If a is positive, then for each $\varepsilon > 0$ there exists an admissible $t > 0$ such that $\mathrm{Re}\, g(a) \geq -\varepsilon$ and $|\mathrm{Im}\, g(a)| < \varepsilon$ for all $g \in V^t$. If $f \in V$, then $f \in V^t$ and so $\mathrm{Re}\, f(a) \geq -\varepsilon$ and $|\mathrm{Im}\, f(a)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $f(a) = \mathrm{Re}\, f(a) \geq 0$.

Conversely, suppose that $f(a) \geq 0$ for each $f \in V$. Since there exist admissible numbers $t > 0$ such that $\rho_w(V, V^t)$ is arbitrarily small, we can choose an admissible t such that for each $g \in V^t$ there exists $f \in V$ with $|g(a) - f(a)| < \varepsilon$. It now follows that for each $g \in V^t$,

$$|\mathrm{Im}\, g(a)| \leq |\mathrm{Im}\, f(a)| + |g(a) - f(a)| \leq 0 + \varepsilon = \varepsilon$$

and

$$\mathrm{Re}\, g(a) > \mathrm{Re}\, f(a) - \varepsilon \geq 0 - \varepsilon = -\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $a \geq 0$. Q.E.D.

Theorem 11. *Let a be a Hermitian element of a complex unital Banach algebra B that has firm state space. Then a^n is positive for each even positive integer n.*

Proof. Let A be a (separable) closed subalgebra of B generated by Hermitian elements a and identity e. By Proposition 4, the state space V_A of A is firm. It follows from Proposition 8 that for each $f \in V_A$ there exist characters $u_1, \ldots, u_m$ of A, and nonnegative numbers $\lambda_1, \ldots, \lambda_m$, such that $\sum_{i=1}^m \lambda_i = 1$ and $\|f - \sum_{i=1}^m \lambda_i u_i\|$ is arbitrary small. In particular, given any n and $\varepsilon > 0$, choose u_i and λ_i such that $|(f - \sum_{i=1}^m \lambda_i u_i)(a^n)| < \varepsilon$. If $a \geq 0$ and n is even, then

$$\begin{aligned} \operatorname{Re} f(a^n) &\geq \operatorname{Re} \sum_{i=1}^m \lambda_i u_i(a^n) - \left| f(a^n) - \sum_{i=1}^m \lambda_i u_i(a^n) \right| \\ &\geq \operatorname{Re} \sum_{i=1}^m \lambda_i u_i(a)^n - \varepsilon \\ &\geq -\varepsilon, \end{aligned}$$

the last step following from Proposition 10. Since $\varepsilon > 0$ is arbitrary, we have $\operatorname{Re} f(a^n) \geq 0$ for each $f \in V$; whence, again by Proposition 10, $a^n \geq 0$. Q.E.D.

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