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On Complements of Sets and the Efremovič Condition in Pre–apartness Spaces[1]

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Abstract: In this paper we study various properties of complements of sets and the Efremovič separation property in a symmetric pre–apartness space.
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The constructive theory of apartness[2] (point–set and set–set) has been developed within the framework of Bishop's constructive mathematics BISH [1, 2, 3, 13] in a series of papers over the past five years [17, 5, 12, 14, 7]. In this paper we derive some basic properties of complements of sets in pre–apartness spaces and discuss a strong separation property.

Our starting point is a set X equipped with an inequality relation applicable to points of X, and a symmetric relation $\bowtie$ applicable to subsets of X. The inequality satisfies two simple properties

$$\begin{aligned} &x \neq y \Rightarrow y \neq x \\ &x \neq y \Rightarrow \neg(x = y). \end{aligned}$$

For a point x of X we write $x \bowtie S$ as shorthand for $\{x\} \bowtie S$. There are three notions of complement applicable to a subset S of X :

– the **logical complement**

$$\neg S = \{x \in X : x \notin S\},$$

– the **complement**

$$\sim S = \{x \in X : \forall s \in S\, (x \neq s)\},$$

– and the **apartness complement**

$$-S = \{x \in X : x \bowtie S\}.$$

The pair $(X, \bowtie)$ is called a **symmetric pre–apartness space** if the following axioms are satisfied.

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*
[2] The motivation for this theory lay in the classical theory of nearness and proximity; see [8, 9, 11].

B1 $X \bowtie \emptyset$.
B2 $S \bowtie T \Rightarrow S \subset\sim T$.
B3 $R \bowtie (S \cup T) \Leftrightarrow R \bowtie S \wedge R \bowtie T$.
B4 $-S \subset \sim T \Rightarrow -S \subset -T$.

Throughout this paper, unless otherwise specified, X will stand for a symmetric pre–apartness space.

Using the above system of axioms, one can easily show that

$$-S \subset\sim S \subset \neg S.$$

The canonical example of a symmetric pre–apartness space is a uniform space $(X,\mathcal{U})$ where two sets S and T are apart if there exists an entourage $U \in \mathcal{U}$ such that we have $S \times T \subset\sim U$. In addition to the classical definition of a uniform space, in BISH we assume that the underlying set X comes equipped with an inequality relation, and the collection $\mathcal{U}$ satisfies the following condition

$$\forall U \in \mathcal{U}\ \exists V \in \mathcal{U}(V^2 \subset U \wedge X \times X = U \cup \sim V).$$

For more details on uniform spaces see, for instance, [7].

Two very useful properties that will be implicitly used in this paper are presented in the following lemma. The proof is straightforward and we will omit it.

Lemma 1. *In any symmetric pre–apartness space we have*

- $(S \bowtie T \wedge A \subset S \wedge B \subset T) \Rightarrow A \bowtie B$.
- $S \bowtie T \Rightarrow S \subset -T$.

An interesting feature of the complements in a pre–apartness space is the following.

Lemma 2. *For any subset S of X we have*

$$-S = - \sim\sim S = - \sim -S.$$

Proof. For the first equality, since

$$S \subset\sim\sim S, \tag{1}$$

the inclusion from right to left is clear; the reverse inclusion follows from $-S \subset\sim (\sim\sim S)$ and B4. For the second equality, first note that since we have $-S \subset\sim S$ we immediately get $- \sim -S \subset - \sim\sim S$; for the reverse inclusion, using (1) for $-S$ and the first equality, we get $- \sim\sim S \subset\sim\sim -S$, and the desired conclusion now follows from B4 .

The apartness complements in a pre–apartness space X form a base for a topology, the **apartness topology**, on X. The open sets in this topology are called **nearly open sets**. In other words, a set is open in this topology if it can be written as a union of apartness complements. The **closure** of a set S is defined by

$$\overline{S} = \{x \in X : \forall U(x \in -U \Rightarrow S - U \neq \emptyset)\},$$

and the **interior** of S, is

$$\mathrm{Int}(S) = \{x \in S : \exists U(x \in -U \subset S)\}.$$

Lemma 3. *If X satisfies the decision condition*[3]

A5 $x \in -S \Rightarrow \forall y \in X\, (x \neq y \vee y \in -S)\,,$

then for any subset S of X we have

$$-S = -\overline{S}.$$

Proof. As $S \subset \overline{S}$, one inclusion is clear. Let now $x \in -S$; by A5, for any z in $\overline{S}$, either $z \neq x$ or $z \in -S$. If $z \in -S$, by definition of $\overline{S}$ it follows that $S \cap -S \neq \emptyset$—a contradiction, so the second alternative is ruled out; hence $-S \subset\sim \overline{S}$. B4 now shows that $-S \subset -\overline{S}$.

Before displaying more properties of complements, we introduce the **Efremovič condition**:

$$S \bowtie T \Rightarrow \exists E\, (S \bowtie E \wedge T \bowtie\sim E)\,.$$

This is the strongest of all the separation properties normally considered for a pre–apartness space X. In the classical theory of proximity spaces this property is part of the axioms system, and the topology induced by the proximity relation turns out to be $\mathrm{T}_{3.5}$—that is completely regular.

Proposition 4. *Every uniform space satisfies the Efremovič condition.*

Proof. Let $S \bowtie T$ in X, and construct a 3–chain (U_1, U_2, U_3) of entourages such that $S \times T \subset \sim U_1$, $U_{i+1}^2 \subset U_i$, and $X \times X = U_i \cup \sim U_{i+1}$, $i = 1, 2$. Let

$$E = \{x \in X : \exists s \in S\, ((s, x) \in U_2)\}\,.$$

Consider $s \in S$ and $y \in\sim E$. Then either $(s, y) \in U_2$,, or $(s, y) \in \sim U_3$. In the first case, the definition of E shows that $y \in E$—a contradiction. So the second must be the case, that is $S \times \sim E \subset \sim U_3$ and therefore $S \bowtie\sim E$. On the other hand, if $t \in T, x \in E$, then either $(x, t) \in U_2$, or $(x, t) \in\sim U_3$. In the first case, as $x \in E$, there exists $s \in S$ such that $(s, x) \in U_2$, and hence $(s, t) \in U_2^2 \subset U_1$, which is absurd. Hence $E \times T \subset\sim U_3$. Thus $E \bowtie T$.

The following lemma provides us with yet another very useful property of pre–apartness spaces.

Lemma 5. *If X satisfies the Efremovič condition, then for all $S, T, A \subset X$ we have*

B4$_s$ $S \bowtie T \wedge -T \subset\sim A \Rightarrow S \bowtie A.$

Proof. Let $S \bowtie T$ and $-T \subset\sim A$. Using symmetry and the Efremovič property, there exists E such that $S \bowtie\sim E$ and $E \bowtie T$. Then $E \subset -T$ and therefore $\sim -T \subset\sim E$; so

$$A \subset\sim\sim A \subset\sim -T \subset\sim E.$$

Since $S \bowtie\sim E$, we conclude that $S \bowtie A$.

[3] The strange labelling of this condition comes from the system of axioms for a point-set apartness. See, for instance, [5].

Note that if we rewrite axiom B4 as

$$x \in -S \wedge -S \subset \sim T \Rightarrow x \in -T,$$

then the property in the above lemma is a generalisation of B4, and this is why we refer to it as the **B4–strong** condition.

Proposition 6. *If X satisfies the decision condition A5, then $S \bowtie T$ implies*

(i) $\overline{T} \subset \sim S$.
(ii) $T \subset \mathrm{Int}(\sim S)$.

If X also satisfies the B_{4s} condition, then we have

(iii) $S \bowtie T \Leftrightarrow \overline{S} \bowtie \overline{T}$.

Proof. Let $x \in \overline{T}$ and let $y \in S$. As $S \subset -T$, by A5, either $x \neq y$ or $x \in -T$. The latter alternative is ruled out, so $x \in \sim S$.
To prove statement (ii), first note that for any set S we have $\mathrm{Int}(\sim S) = -S$. Indeed, as $-S \subset \sim S$, it is clear that $-S \subset \mathrm{Int}(\sim S)$. Conversely, if $x \in \mathrm{Int}(\sim S)$, then there exists $U \subset X$ such that $x \in -U \subset \sim S$. A direct application of B4 now shows that $x \in -S$. Using symmetry, we have $T \bowtie S$, so $T \subset -S = \mathrm{Int}(\sim S)$.
The implication from right to left in (iii) is clear. Conversely, since $S \bowtie T$ and, by Lemma 3, $-T = -\overline{T}$, $\mathrm{B4}_s$ immediately implies that $S \bowtie \overline{T}$. Another application of Lemma 3 and $\mathrm{B4}_s$ gives us the desired conclusion.

The Efremovič condition on a space is a very powerful tool. ¿From a constructive point of view though, because this is a strong existential statement, we prefer to avoid it and use, wherever possible, the properties derived from it. $\mathrm{B4}_s$ and its consequences have numerous applications in the development of our theory (see [4, 14, 16], to quote only a few). For instance, since we have seen that $x \bowtie S \Leftrightarrow x \bowtie \sim\sim S$ (Lemma 1), we wonder if the same property holds for pairs of sets as well[4]. There does not seem to be much hope in proving it without additional conditions, but it is immediate from $\mathrm{B4}_s$: for if $S \bowtie T$, then since $-T \subset \sim T = \sim\sim\sim T$, we see from $\mathrm{B4}_s$ that $S \bowtie \sim\sim T$.

We conclude this note with a result presenting various forms of the Efremovič property.

Proposition 7. *The following statements are equivalent in a symmetric pre–apartness space X that satisfies A5.*

(i) $S \bowtie T \Rightarrow \exists E(S \bowtie E \wedge \sim E \bowtie T \wedge - \sim E \cap \neg E = \emptyset)$.
(ii) $S \bowtie T \Rightarrow \exists E(S \bowtie E \wedge \neg E \bowtie T)$.
(iii) $S \bowtie T \Rightarrow \exists E(S \bowtie E \wedge \sim E \bowtie T)$.

Proof. To prove that (i) implies (ii), let E be as in (i) and get F such that $F \bowtie \sim E$ and $\sim F \bowtie T$. Now, for each $x \in \neg E$ and each $z \in F \subset - \sim E$, by A5 we have either $z \neq x$ or $x \in - \sim E$. Last case is ruled out, so $x \in \sim F$, that is $\neg E \subset F \bowtie T$.

Since $\sim E \subset \neg E$, it is immediate that (ii) implies (iii).

[4] Pre–apartness spaces satisfying a similar property $S \bowtie T \Leftrightarrow S \bowtie \neg\neg T$ are called **firm**. See [6].

Now consider E as in (iii). Keeping in mind that the Efremovič condition (via $\mathrm{B4}_s$) implies $S \bowtie E \Rightarrow S \bowtie \sim\sim E$, take $F = \sim\sim E$, and so we get $S \bowtie F$ and $T \bowtie \sim\sim\sim E = \sim F$. Since $- \sim E \subset \sim\sim E$ we have $\neg F = \neg \sim\sim E \subset \neg - \sim E$, and hence

$$\neg F \cap - \sim F \subset \neg - \sim E \cap - \sim (\sim\sim E) = \neg - \sim E \cap - \sim E = \emptyset,$$

and the implication from (iii) to (i) obtains.

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