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# New Bounds for Positive Roots of Polynomials[1]

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**Abstract:** We consider a nonconstant polynomial $P$ with real coefficients that has at least one negative coefficient and derive new upper bounds for the real roots of $P$. We compare our bounds with those obtained by other methods.

**Key Words:** Roots of polynomial equations, Bounds for positive roots

**Category:** G.1.5, F.2.1

## 1 Introduction

For the computation of real roots of univariate polynomials with real coefficients it is important to have good approximations of the intervals containing the real roots. Bounds estimates for real roots are useful for the location of the roots, in particular for estimating the roots of hyperbolic polynomials [11]. There exist several criteria for a polynomial to have only real roots, for example a theorem of J. Eve [4].

The algorithms for the computation of real roots of polynomials over $\mathbb{R}$ are based on an initial over–estimate of the modulus of the largest root (see D. Lester et al. [9], C. K. Yap [12]). The effective computation of the real roots of univariate polynomials with real coefficients is a basic problem and it has deep connections with constructive and computational approaches in analysis (see E. Bishop—D. Bridges [1], D. Bridges [2], Y. N. Moschovakis [10].

The effective computation of positive roots of univariate polynomials with real coefficients is also relevant to iterative numerical processes (J. Herzberger [6], N. Kjurkchiev [8]).

In this paper we derive new bounds for the positive roots of a polynomial $P$ with real coefficients. If

$$P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots + \pm a_d\,, \quad \text{with} \quad a_i \geq 0$$

and we denote by $A$ the greatest absolute value of the negative coefficients, our bounds are given as functions of $A$, some of the positive coefficients $a_0, \ldots, a_m$, the degree $d$ and $s$, where $s \leq m$. Our results give better upper bounds for the positive roots than the estimates of J. B. Kioustelidis [7]. We also obtain bounds for superunitary roots and compare them with results of Lagrange and Longchamp.

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

## 2 Bounds for all positive roots

In this section we derive upper limits for the real roots of a polynomial $P$ as functions of the size of the negative coefficients.

**Theorem 1.** *Let*

$$P(X) \;=\; x^d - b_1 X^{d-m_1} - \cdots - b_k X^{d-m_k} + \sum_{j \neq m_1,\ldots,m_k} a_j X^{d-j}\,,$$

*with* $b_1, \ldots, b_k > 0$ *and* $a_j \geq 0$ *for all* $j \notin \{b_1, \ldots, b_k\}$.
*The number*

$$B_1(P) \;=\; \max\left\{(kb_1)^{1/m_1}, \ldots, (kb_k)^{1/m_k}\right\}$$

*is an upper bound for the positive roots of* $P$.

**Proof.** Suppose $x > 0$. We have

$$\begin{aligned} |P(x)| &\geq x^d - b_1 x^{d-m_1} - \cdots - b_k x^{d-m_k} \\ &= \frac{1}{k}(x^d - kb_1 x^{d-m_1}) + \cdots + \frac{1}{k}(x^d - kb_1 x^{d-m_k}) \\ &= \frac{x^{d-m_1}}{k}(x^{m_1} - kb_1) + \cdots + \frac{x^{d-m_k}}{k}(x^{m_k} - kb_k) \end{aligned}$$

The parantheses in the last row are strictly positive as soon as

$$x \;>\; (kb_1)^{1/m_1}, \ldots, (kb_k)^{1/m_k}\,,$$

hence the number

$$B_1(P) \;=\; \max\left\{(kb_1)^{1/m_1}, \ldots, (kb_k)^{1/m_k}\right\}$$

is an upper bound for the positive roots. □

**Remark 1** *J. B. Kioustelidis [7] gives the following upper bound for the positive real roots:*

$$B_2(P) \;=\; 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}\,.$$

*Like Theorem 1, Kioustelidis' method also returns subunitary bounds, if they exist. The bound* $B_2$ *is obtained through the estimation of the unique positive root of the associated polynomial*

$$P_{ass}(X) = X^d - \sum_{j=1}^{k} b_j\, X^{d-m_j}\,.$$

*The estimation of the unique positive root of polynomials with negative coefficients excepting the dominant one have important applications in financial mathematics [6].*

For polynomials with an even number of variations of sign, a different bound is given by the following

**Theorem 2.** *Let $P(X) \in \mathbb{R}[X]$ be such that the number of variations of signs of its coefficients is even. If*

$$P(X) = c_1X^{d_1} - b_1X^{m_1} + c_2X^{d_2} - b_2X^{m_2} + \cdots + c_kX^{d_k} - b_kX^{m_k} + g(X)\,,$$

*with $g(X) \in \mathbb{R}_+[X]$, $c_i > 0$, $b_i > 0$, $d_i > m_i > d_{i+1}$ for all $i$, the number*

$$B_3(P) = \max\left\{\left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)}\right\}$$

*is an upper bound for the positive roots of the polynomial $P$ for any choice of $c_1, \ldots, c_k$.*

**Proof.** Suppose $x > 0$. We have

$$\begin{aligned}|P(x)| &\geq c_1x^{d_1} - b_1x^{m_1} + \cdots + c_kx^{d_k} - b_kx^{m_k}\\ &= x^{m_1}(c_1x^{d_1-m_1} - b_1) + \cdots + x^{m_k}(c_1x^{d_k-m_k} - b_k)\,,\end{aligned}$$

which is strictly positive for

$$x > \max\left\{\left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)}\right\}.$$

□

### Comparisons of Results

We first compare our bound $B_1(P)$ with Kioustelidis' bound $B_2(P)$.
For $k = 1$ we have $B_1 = b_1^{1/m_1} < 2b_1^{1/m_1} = B_2$.
For $k \geq 2$ and $k < 2^{m_j}$ $(1 \leq j \leq k)$ we always have $B_1(P) < B_2(P)$.
If we consider

$$P(X) = 4\,X^7 - X^6 + 0.0004\,X^5 - X^4 + 0.00004\,X^3 + 0.0000004\,X - 1$$

we obtain

$$B_1(P) = 0.96\,, \qquad B_2(P) = 1.64\,.$$

Note that the true upper bound for the postive roots of the polynomial $P$ is 0.928.

For polynomials with an even number of signs we also compare the bound $B_3(P)$ given in Theorem 2.

Let

$$\begin{aligned}Q_1(X) &= 3X^4 - X^3 + 7X^2 - 3X + 0.001\,,\\ Q_2(X) &= X^5 - 1.01\,X^4 + X^3 - 1.1\,X + 0.1\,,\\ Q_3(X) &= 3X^7 - X^6 + 7X^5 - 3X^2 + 0.001\,,\\ Q_4(X) &= 10X^9 - 17\,X^5 + 10X^4 - 13\,X + 1\,.\end{aligned}$$

We have

| | $B_1$ | $B_2$ | $B_3$ | largest positive root |
|---|---|---|---|---|
| $Q_1$ | 1.256 | 2 | 0.428 | 0.421 |
| $Q_2$ | 2.02 | 2.048 | 1.024 | 1.003 |
| $Q_3$ | 1.148 | 2 | 0.753 | 0.725 |
| $Q_4$ | 1.357 | 2.283 | 1.141 | 1.121 |

The bound $B_3(P)$ gives in many cases better results. For particular polynomials, the method used in Theorem 2 can help to derive better limits for the roots. For a given polynomial with real coefficients having at least one negative cofficient there are, in general, several ways of choosing the positive coefficients $c_1, \ldots, c_k$. If $b_j/c_j > 1$ the optimal choice is for $m_j - d_j$ maximal, while for $b_j/c_j < 1$ the optimal choice is for $m_j - d_j$ minimal.

## 3 Bounds surpassing the unity

A well known result of Lagrange (see, for example, [3]) gives an upper bound for the positive real roots of $P$ as a function of the size of the negative coefficients and the number $m$ of positive coefficients preceding the first negative one. The bound of Lagrange is

$$1 + (A/a_0)^{1/(m+1)} ,$$

where $a_0$ is the leading (positive) coefficient of $P$ and $A$ the largest abosolute value of the negative coefficients.

We obtain new bounds for the positive roots considering all positive coefficients preceding the first negative coefficients. We compare these bounds with results on superunitary roots of Lagrange and Longchamp.

**Theorem 3.** *Let $P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$, with all $a_i \geq 0$, $a_0, a_{m+1} > 0$. Denote*

$$A \; = \; \max\left\{a_i\,;\,\mathrm{coeff}\,(X^{d-i}) \; < \; 0\right\} \, .$$

*The number*

$$1 \, + \, \max\left\{ \left(\frac{A}{2(sa_0 + \cdots + +2a_{s-2} + a_{s-1})}\right)^{1/(m-s+2)} , \left(\frac{A}{2(a_0 + a_1 + \cdots + a_s)}\right)^{1/(m-s+1)} \right\}$$

*is an upper bound for the positive roots of $P$ for any $s \in \{1, 2, \ldots, m\}$.*

**Proof.** Let $x \in \mathbb{R}$, $x > 1$. We have

$$
\begin{aligned}
|P(x)| &\geq |a_0x^d + \cdots + a_m x^{d-m}| - |a_{m+1}x^{d-m-1} \mp + \ldots \mp a_d| \\
&\geq a_0 x^d + \cdots + a_s x^{d-s} - A(x^{d-m-1} + \cdots + 1) \\
&\geq (a_0\, x^s + \cdots + a_s)x^{d-s} - A\, \frac{x^{d-m} - 1}{x-1} \\
&= \frac{(a_0\, x^s + \cdots + a_s)(x-1)x^{d-s} - A}{x-1} \cdot x^{d-m} + \frac{A}{x-1}\,.
\end{aligned} \tag{1}
$$

The last right hand side of (1) is strictly positive provided that

$$
(a_0\, x^s + \cdots + a_s)(x-1)x^{m-s} \;\geq\; A\,. \tag{2}
$$

Now let $x = 1 + y$ and note that $x^j \geq 1 + jy$ for all $j \in \mathbb{N}$. It follows that

$$
\begin{aligned}
&(a_0\, x^s + \cdots + a_s)(x-1)x^{m-s} \\
&\qquad \geq (a_0(1+sy) + \cdots + a_{s-1}(1+y) + a_0)\; y^{m-s+1} \\
&\qquad = (sa_0 + \cdots + 2a_{s-2} + a_{s-1})\, y^{m-s+2} + (a_0 + \cdots + a_s)\, y^{m-s+1}.
\end{aligned}
$$

Therefore (2) is satisfied if

$$
\begin{aligned}
(sa_0 + \cdots + 2a_{s-2} + a_{s-1})\, y^{m-s+2} &\geq A/2\,, \\
(a_0 + \cdots + a_{s-1} + a_s)\, y^{m-s+1} &\geq A/2\,.
\end{aligned}
$$

These inequalities are satisfied as soon as

$$
y \;\geq\; \max\left\{ \left(\frac{A}{2(sa_0 + \cdots + +2a_{s-2} + a_{s-1})}\right)^{1/(m-s+2)}, \left(\frac{A}{2(a_0 + a_1 + \cdots + a_s)}\right)^{1/(m-s+1)} \right\}.
$$

This proves that

$$
1 \,+\, \max\left\{ \left(\frac{A}{2(sa_0 + \cdots + +2a_{s-2} + a_{s-1})}\right)^{1/(m-s+2)}, \left(\frac{A}{2(a_0 + a_1 + \cdots + a_s}\right)^{1/(m-s+1)} \right\}
$$

is an upper bound for the positive roots of the polynomial $P$. □

When $s = 1$, respectively $s = m$ in Theorem 3 we obtain:

**Corollary 4.** *The numbers*

$$M_1 = 1 + \max\left\{ \left(\frac{A}{2a_0}\right)^{1/(m+1)}, \left(\frac{A}{2(a_0+a_1)}\right)^{1/m} \right\},$$

$$M_2 = 1 + \max\left\{ \left(\frac{A}{2(ma_0+\cdots++2a_{m-2}+a_{m-1})}\right)^{1/2}, \frac{A}{2(a_0+a_1+\cdots+a_m)} \right\}$$

*are upper bounds for the positive roots of the polynomial $P$.*

We also note the classical bounds of Lagrange and Longchamp:

$$L_1 = 1 + \left(\frac{A}{a_0}\right)^{1/(m+1)}, \qquad L_2 = 1 + \frac{A}{a_0+\cdots++a_{m-1}} \qquad \text{(cf. L. S. Grinstein [5])}$$

and compare them with our bounds in Corollary 4.
We have, for example,

$$\begin{cases} L_1 < M_1 & \text{if } a_0 < 2^m A\,, \\ L_2 < M_2 & \text{if } (a_0+\cdots+a_m)^2 < 2(ma_0+\cdots+a_{m-1})A\,. \end{cases}$$

## Comparison with classical bounds

We compare our bounds $M_1(P)$ and $M_2(P)$ with the classical bounds of J.–L. Lagrange and M. Longchamp.

Let

$$\begin{aligned} P_1(X) &= X^5 + 10X^4 - 61X^3 + 1\,,\\ P_3(X) &= 4X^6 + X^5 + X^4 - 3 * X^3 - 4X^2 + X - 5\,,\\ P_3(X) &= X^7 + 3X^6 - 3X^4 + 2X^3 - 4X^2 + X - 2.5\,,\\ P_4(X) &= X^9 + 3X^8 + 2X^7 + X^6 - 4X^4 + X^3 - 4X^2 - 3\,. \end{aligned}$$

We have

| | $L_1$ | $L_2$ | $M_1$ | $M_2$ | largest positive root |
|---|---|---|---|---|---|
| $P_1$ | 8.81 | 62 | 6.52 | 6.52 | 4.27 |
| $P_2$ | 2.08 | 2 | 1.85 | 1.52 | 1.16 |
| $P_3$ | 2.57 | 2 | 2.25 | 1.63 | 1.12 |
| $P_4$ | 2.41 | 1.66 | 2.18 | 1.42 | 1.07 |

The following table gives the values of the bounds discussed in the previous section:

| | $B_1$ | $B_2$ | $B_3$ | largest positive root |
|---|---|---|---|---|
| $P_1$ | 7.81 | 15.62 | 6.1 | 4.27 |
| $P_2$ | 2.86 | 2.07 | — | 1.19 |
| $P_3$ | 1.81 | 2.88 | — | 1.12 |
| $P_4$ | 1.64 | 2.63 | — | 1.07 |

Notice that the bound $B_1(P)$ is also useful for the estimation of polynomials with real roots surpassing the unity.

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