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How the Mathematical Objects Determine the Mathematical Principles[1] [2]

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Abstract: In our description of Brouwer's universe we have discussed a few basic principles which have unusual consequences in practical mathematics.
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Almost a century ago, Brouwer launched his first intuitionistic programme for mathematics. He did so in his dissertation of 1907, where he formulated the basic act of creation of mathematical objects, known as the *ur-intuition* of mathematics. Mathematics, in Brouwer's view, was an intellectual activity of men (of the *subject*), independent of language and logic. The objects of mathematics come first in the process of human cognition, and description and systematization (in particular logic) follow later. The formulation of the ur-intuition is somewhat hermetic, but in view of its fundamental role, let us reproduce it here.

> Ur-intuition of mathematics (and every intellectual activity) as the substratum, divested of all quality, of any perception of change, a unity of continuity and discreteness, a possibility of thinking together several entities, connected by a 'between' that by the interpolation of new entities never gets exhausted.

As we see, Brouwer sees the ur-intuition as the genesis of both the discrete part of mathematics, let us say, the natural numbers, and of the continuous part, i.e. the continuum. Neither of these can be reduced to the other.

A more refined analysis was given in the Vienna lectures (although it is foreshadowed in the so-called 'rejected parts' of the thesis), where the notion of the falling apart of a moment of life is introduced. In the final presentation, *Consciousness, Philosophy and Mathematics* (CPM), [Brouwer 1949a], this phenomenon is described as the *move of time*: 'By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains

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the former one as a past sensation and moreover, through this distinction between present and past, recedes from both and from stillness and becomes mind.' Thus the subject has created a 'twoity' of a past and present sensation. The process evidently can be iterated, and complexes and strings of sensation become the object of attention. The sensation complexes form a bewildering mixture, in which a certain order is introduced by the *causal attention.* This carries out a process of *identification.* One may think of the identification of 'similar' complexes, or of *abstraction.*

In CPM the notion of causal sequence is further refined: 'An iterative complex of sensations whose elements have an invariable order of succession in time, whilst if one of its elements occurs, all following elements are expected to occur likewise, in the right order of succession, is called a causal sequence'. It might be tempting to explain these, let us say 'strongly causal sequences', *scs*, by a causality, independent of the will of the subject. This, however, is rejected by Brouwer. On the contrary, causality is explained by the notion of strong causal sequence. A scs can be put to use by the subject in order to realize events that are not immediately obtainable. One only has to realize the first event of a scs, or an intermediate one, in order to obtain the final event. The procedure of realizing the final (and desirable) event by realizing a preceding event was called the '*jump from end to means*', and later the *mathematical* or *cunning act.* The jump from end to means, is a useful and convenient tool for the subject to dominate nature and for the protection of his personal sphere.

Assuming that in a scs $a_0, \ldots, a_k, \ldots, a_n$ the realization of all stages is indeed of a fixed determined nature, one may recognize in the jump from end to means the germ of the constructive implication. The transition from, say, a_k to a_n is completely lawlike and thus the proof interpretation of $A \rightarrow B$ is foreshadowed by the automatic and algorithmic transition from (the building for) A to (the building for) B. Of course, the subject may and will add much more regularity to causal sequences than the primitive spontaneous sequence of sensations offers.

By abstracting from all accidental features of twoities, the *empty twoity* is obtained. In other words, by identifying all twoities one obtains the object where only order and distinction are recognized. This empty twoity then can take the place of the number 2. From there it is not difficult to generalize to the individual natural numbers, and the next step — the recognition of the iteration of the 'next number' step as a legitimate mental construction, together with the corollary, the (potentially infinite) set of natural numbers — is mentioned in passing by Brouwer . He speaks of 'unlimited unfolding' (CPM, p.1237).

Thus the basic material of 'discrete mathematics' is at the disposition of the subject. This part of the process of creating is later called *the first act of intuitionism.* We should note that the aspect of simultaneous creation of discrete and continuous, is played down, but as late as the Vienna lectures (1928) Brouwer pointed out that both acts of intuitionism are grounded in the ur-intuition. The continuum is given in the move-of-time act as the 'between'. In his Rome lecture (1908) Brouwer explicitly points out that 'the first and the second are thus kept together, and the intuition of the continuous (continere = keeping together) consists of this keeping together'. And he adds: 'This mathematical ur-intuition is nothing but the contentless abstraction of the sensation (experience) of time'. Time is thus created by the subject through the 'move of time', together with the continuum and the natural numbers. *The second act of intuitionism* is the creation of 'more or less freely proceeding infinite sequences of mathematical entities previously acquired' and of 'species', i.e. 'properties supposable for mathematical entities previously acquired'.

In CPM the two acts are tacitly lumped together under the act of 'unlimited unfolding'. The process of creation of causal sequences and complexes does extend beyond the realm of mathematics; indeed the physical world, as well as the social one is made up of those objects. If we look for a minute at the physical phenomena, then we can see the role of mathematics as follows. The objects of the physical world are obtained by abstraction from sensation complexes, a further abstraction gets the subject to mathematical objects and structures. And hence there is a natural connection between the physical universe and the mathematical, something like a projection. Although this does not explain the success of mathematics in full, it shows that the connections do not come out of the blue.

By and large, the above sketches the genesis of Brouwer's mathematical universe. In the dissertation Brouwer goes to great lengths to determine the possible sets in mathematics on the basis that there are no sets but those we can create ourselves. After the introduction of choice sequences (cf. the second act) he revised his views. The extent of the mathematical universe is modest compared to the traditional Cantorian universe, from a classical point of view, Brouwer's universe does not get beyond ω_1. But what it lacks in 'height' is compensated by the extra fine structure which is inherent to the intuitionistic approach (and its logic).

The most spectacular part of the universe is the second-order part, let us say second-order arithmetic with sequences, species, or both. Where the first-order part yields more-or-less a subtheory of classical arithmetic, but the second-order part has certain specific properties that are incompatible with classical mathematics.

We will look at a few of these principles. The first and most striking principle was introduced by Brouwer in his courses on pointset theory of 1915-1917. The principle appeared in print in 1918, in modern formulation it reads 'A mapping F from choice sequences to natural numbers has the property that each $F(\alpha)$ is determined by an initial segment $\overline{\alpha}k (= (\alpha 0, \alpha 1, \alpha 2, \ldots, \alpha(k-1))$

Formalized: $\forall\alpha\exists x\forall\beta(\overline{\alpha}x = \overline{\beta}x \to F(\alpha) = F(\beta)$

The principle finds a more general form in the *Principle of weak continuity*

$$WC \qquad \forall\alpha\exists x A(\alpha, x) \to \forall\alpha\exists x\exists y\forall\beta(\overline{\alpha}y = \overline{\beta}y \to A(\beta, x)$$

Brouwer formulated his functional version in a proof, giving no argument for it. A first attempt at a justification could run as follows: in order to compute the natural number $F(\alpha)$ a finite number of steps is required, when the computation is finished only finitely many members of the sequence α have been generated, and so only this initial segment enters into the computation. Hence any sequence β with the same initial segment yields the same value under F. This argument only works in the case that only numerical information of α is used. In general, however, information of a different kind may be used.

Here is an example, formulated as a game (Brouwer introduced game formulations in his Groningen Lectures, 1930). There are two players, I and II. I provides successively information about α and II has an algorithm for computing $F(\alpha)$. At each step II may ask for more information or show the output. In our example II simply takes $F(\alpha) = 100$

	I	II
0	7	?
1	2	?
2	301	?
$\vdots$	$\vdots$	$\vdots$
13	5 and α becomes stationary	$F(\alpha) = 5$

Note that I may (and perhaps *must*) give more information than just the numerical values of α. Indeed, if one accepts the idea of mathematics as a solitary play of the subject, then I and II are no more than puppets controlled by the subject. Thus the availability of full information is obvious.

Now there obviously are β's with the same initial segment $\overline{\beta}14 = \overline{\alpha}14 = (7, 2, 301, \ldots, 5)$ with $F(\beta) \neq 5$. This failure of the simple argument is caused by the fact that suddenly a condition of a higher order is put on α. And higher order condition cannot be avoided, if only because one wants to allow lawlike sequences (think of the difference between the decimals of π and those determined by flipping a coin). Hence a better argument is required. One was provided by Mark van Atten in a setting which slightly, but justifiably, extended Brouwer's framework. Brouwer demanded that once one has introduced a condition on future choices (of values or conditions), one sticks to it. However, it is fairly clear that his main stipulation was that each finite sequence of choices has at least one immediate successor. By allowing higher order conditions to be repealed, the extendibility condition is observed, and the extra flexibility certainly does not restrict the practical aspects of choice sequences. Now the possible ephemeral nature of higher order conditions, disqualifies them for use in the computation of the output of F on input α, see [van Atten – van Dalen 2002]. The analysis lays down certain conditions on the class of sequences for the validity of the continuity principle. The principle is in fact justified for the holistic universe, but we can see that there is a new problem for research: for which universes does WC hold? A simple example of a universes that violates the continuity principle is the one in which each sequence eventually becomes constant. The function F assigns this constant value to α; F is obviously not continuous. There is a rich literature on the continuity principle, see for example [van Dalen–Troelstra 1988a, van Dalen–Troelstra 1988b]. The continuity principle has striking consequences in everyday mathematics e.g. *Brouwer's continuity theorem - all real functions are continuous* and the *indecomposability of the continuum* - $\mathbb{R}$ *cannot be split into two non-empty parts*. Both results confirm the above mentioned incompatibility, in particular the latter shows that the principle of the excluded middle is false: $\neg\forall x \in \mathbb{R}(x = 0 \vee x \neq 0)$

Weyl, in his basic paper, *On the new foundational crisis in mathematics*, [Weyl 1921], adopted Brouwer's intuitionistic programme, adding his own interpretations to it. In particular Weyl did not give the same status to choice sequences Brouwer did. For Weyl choice sequences did not belong to mathematics proper; all he accepted was the real status of initial segments. As a consequence arbitrary reals were replaced by generating intervals. Such an interval, say (a, b) for rational a and b, represents for Weyl the open horizon of 'the reals that are potentially given by the interval'. Concrete real numbers are given by lawlike sequences of intervals, and arbitrary ones by choice sequences, in the representing interpretation. Hence there is on Weyl's approach a fundamental distinction between existential quantification (over lawlike reals), and universal quantification (over choice reals). Apart from everything else, this destroys the hope of salvaging the principle of the excluded middle. Here Brouwer's and Weyl's roads separated. For Weyl quantified statements were 'judgement abstracts', not to be taken for real judgements, whereas Brouwer recognized quantified statements as ordinary statements with ordinary proof conditions. Hence for Weyl the continuity of all real functions was an obvious consequence of the notion of arbitrary real number (approximations follow from approximations), whereas for Brouwer there was a hard theorem to be proved. For more on the Brouwer-Weyl views, see [van Atten–van Dalen–Tieszen 2002].

A further analysis, making use of transfinite principles (*the principle of Bar Induction*, established the *bar theorem, the fan theorem*, and the *locally uniform continuity theorems* (real functions on intuitionistically compact subsets of $\mathbb{R}$ are uniformly continuous). For the practical consequences of these properties of Brouwer's universe see [van Dalen–Troelstra 1988a, van Dalen–Troelstra 1988b].

So far the treatment of the universe was completely uniform, but in the twenties Brouwer started to make the distinction between the lawlike and the full continuum. Equivalently, between the set of lawlike sequences and the set of (all) choice sequences. Historically speaking, there was a perfect reason to do so. When dealing with infinite processes algorithms are the first things that come to mind, for the law is the thing that guarantees infinite continuation. The first Brouwerian counterexamples, were, not surprisingly, based on an algorithm: the decimal expansion of π. However, once choice sequences were recognized by him as legitimate objects (the subject is free to make choices), it was natural to look for a counterpart of the (lawlike) Brouwerian counter examples where one uses a decidable property of a lawlike sequence, which has neither been proved, nor rejected. One should fully exploit the choice-character of sequences in the hope of exploiting the properties of the full Brouwerian universe. In 1927 there are the first signs of the new method, which was published some twenty years later, and which goes by the name of the 'creating subject'. The underlying idea is that the subject investigates some particular property, while he carries out a convenient bookkeeping at the same time: if at moment n A has not yet been established, put down a 0, otherwise a 1. Brouwer uses the expression 'the creating subject experiences the truth of A'. Here it is tacitly assumed that 'the creating subject experiences the truth or he does not', the simple argument being that 'in doubt, one does not experience the truth'. A reasonable assumption. In view of the fact that the ur-intuition, in its function as a time-measuring and -introducing principle, provides the subject with a sequence of moments ordered like the natural numbers, the time parameter n is a natural one. The effect of the activity of the creating subject is that a choice sequence α is in the following way associated to a proposition A:

$$\exists\alpha(A \leftrightarrow \exists x(\alpha x \neq 0))$$

This formalization of Brouwer's argument is due to Kripke and is called *Kripke's Schema, KS* Note that KS is an extra condition on the richness of the Brouwerian universe. It asserts the existence of particular sequences, compare the role of the axiom of choice. Thus it is not automatically seen that the old principles still hold. It has in fact been shown that KS is consistent with most principles. Kreisel formulated an interesting 'tensed modal' extension of the existing theories which captures the properties of the creating subject, and which is equivalent to the extension by KS [Kreisel 1967], [van Dalen 1978]).

The classically inclined logician will note that KS is a very weak comprehension principle, which is provable in the classical setting. So whatever strength one can expect from KS, it has to come from suitable extra principles, such as the continuity principle.

We will now proceed to show a number of consequences of KS in practical mathematics, consequences which are not mere curiosities, but which make manifest certain features of the universe one would expect, and some unexpected phenomena to boot. The proofs are carried out under the assumption of the continuity principle and Kripke's Schema. It turns out to be convenient to reformulate Kripke's Schema, such that there is at most one 1 in the sequence $\alpha : \forall x(\sum_{y \leq x} \alpha(y) \leq 1)$. Let us call such a sequence satisfying $A \leftrightarrow \exists x(\alpha x = 1)$, a *Kripke sequence for A.*

(1) $\neg\forall xy \in \mathbb{R}(x \neq y \to x\#y)$

(2) $\neg\forall xy \in \mathbb{R}(\neg\neg x < y \to x \ < y)$

(2) was shown by Brouwer in [Brouwer 1949b], and (1) follows by a completely similar argument.

(3) *The Principle of* $\forall\alpha\exists\beta$*-continuity fails*, [Myhill 1966].

Proof: consider the statement $r \in \mathbb{R}$. We apply KS to $\forall x(\alpha(x) = 0)$:

$$\exists\beta(\forall x(\alpha(x) = 0 \leftrightarrow \exists y(\beta(y) = 1))$$

Hence $\forall\alpha\exists\beta(\ldots)$; by $\forall\alpha\exists\beta$-continuity there should be a continuous functional G: $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\forall\alpha((\forall x(\alpha(x) = 0 \leftrightarrow \exists y(G(\alpha)(y) = 1))$. Hence we have a continuous functional G testing if an α is the zero-sequence $\mathbf{0}$. I.e. G is $\mathbf{0}$ on all sequences distinct from $\mathbf{0}$, and non-zero on $\mathbf{0}$. This functional is clearly discontinuous.

Note that therefore there is a real foundational choice to be made here: adopt KS or $\forall\alpha\exists\beta$-continuity, but not both.

(4) *All negative dense subsets of* $\mathbb{R}$ *are indecomposable.*

By a negative subset X we mean one for which $X = X^{cc}$ (in particular the complement of a set is negative).

Proof. This theorem follows from two lemma's. Let X be negative and dense in $\mathbb{R}$.

(4.1) If $X = A \cup B$, with $A \cap B = \emptyset$, then converging sequences (a_i) and (b_i) in respectively A and B cannot have the same limit.

Assume $\forall k\exists n\forall m(|a_{m+n} - b_{m+n}| < 2^{-k})$. We consider the Kripke sequences α for $r \in \mathbb{Q}$ and β for $r \notin \mathbb{Q}$, where r is an arbitrary real number.

We define new sequences γ and c_i by

$$\begin{cases} \gamma(2n) = \alpha(n) \\ \gamma(2n+1) = \beta(n) \end{cases} \quad \text{and} \quad \begin{cases} c_{2n} = a_n \\ c_{2n+1} = b_n \end{cases}$$

Now we introduce a new sequence (d_i)

$$d_n = \begin{cases} c_n \text{ if } \forall k \leq n(\gamma(k) = 0) \\ c_k \text{ if } k \leq n \text{ and } \gamma(k) = 1 \end{cases}$$

Claim: $d \in X$.
If $d \notin X$, then $d \notin A$; hence (d_n) does not become stationary in A. So $\alpha(n) = 0$ for all n. And by the definition of Kripke sequence we get $r \notin \mathbb{Q}$. Similarly $d \notin B$; hence (d_n) does not become stationary in B. Therefore $\beta(n) = 0$ for all n, and thus $r \notin \mathbb{Q}$. Contradiction.
So $\neg\neg d \in X$. But since X is negative, we find $d \in X$.

As $X = A \cup B, d \in A \vee d \in B$. If $d \in A$ then (d_n) does not become stationary in B, hence $\forall n \beta(n) = 0$. By the definition of β this implies $\neg\neg r \in \mathbb{Q}$. A similar argument shows that $\neg r \in \mathbb{Q}$ if $d \in B$. As a result we get $\neg r \in \mathbb{Q} \vee \neg\neg r \in \mathbb{Q}$. As r was en arbitrary real, we have established $\forall r \in \mathbb{R}(\neg r \in \mathbb{Q} \vee \neg\neg r \in \mathbb{Q})$, which contradicts the indecomposability of $\mathbb{R}$. Therefore $lim(a_n) \neq lim(b_n)$.

(4.2) If the above sets A and B are inhabited (i.e. contain an element), then there are sequences in A and B converging to the same point.
The proof is a piece of elementary analysis, see [van Dalen 1999].

Conclusion: X is indecomposable.

This theorem shows that there are lots of indecomposable subsets of the continuum, for example the irrationals, $\mathbb{Q}^c$, and the not-not-rationals, $\mathbb{Q}^{cc}$. The continuum is clearly extremely 'connected'; even if we punch holes in it, it still remains indecomposable. Note that classically $\mathbb{Q}^c$ is *not* connected. It is even zero-dimensional. Intuitionistically it has dimension 1. The moral is that the intuitionistic continuum is very tight, and that its topology will offer unknown surprises and difficulties.

(5) *The powerset of* $\mathbb{N}$ *exists.*
More precisely: each subset of $\mathbb{N}$ can be represented by a suitable $0-1$ choice sequence.

The basic idea of the proof is that, given a subset X there is for each n a Kripke sequence α_n such that $n \in X \leftrightarrow \exists x(\alpha_n(x) = 1)$ All these α_n's can be glued together to form one α that tests membership for X. For the technical details, see [Dalen, D. van 1977].

(6) *If $\mathbb{R}$ is indecomposable, then there are no discontinuous functions*, ([van Dalen 2001]).
The converse is obvious, and it allows one to conclude the indecomposability on the basis of Brouwer's negative version of the continuity theorem (cf. [Brouwer 1927]).

Proof. Let f be discontinuous, say in 0. It is no restriction to assume $f(0) = 0$. Then $\exists k \forall n \exists x(|x| < 2^{-n} \wedge |f(x) > 2^{-k})$

After determining k we can find a sequence (x_n) with $|f(x_n)| > 2^{-k}$ and $|x_n| < 2^{-n}$.

Let α and β again be Kripke sequences for $r \in \mathbb{Q}$ and $r \notin \mathbb{Q}$. Put

$$\begin{cases} \gamma(2n) = \alpha(n) \\ \gamma(2n+1) = \beta(n) \end{cases} \quad \text{and} \quad c_n = \begin{cases} x_n \text{ if } \forall k \leq n(\gamma(k) = 0) \\ x_k \text{ if } k \leq n \text{ and } \gamma(k) = 1 \end{cases}$$

(c_n) converges, say to c. As $0 < 2^{-k}$, we get $f(c) < 2^{-k} \vee f(c) > 0$. If $f(c) < 2^{-k}$, then $f(c) = 0$, so $\forall p(\gamma(p) = 0)$, which is impossible. So $f(c) > 0$, and therefore $r \in \mathbb{Q} \vee r \notin \mathbb{Q}$. As before we see that this yields a non-trivial decomposition of the continuum. Contradiction.

This result establishes an equivalence between a certain characteristic of a function and the nature of its domains. Results of this kind are familiar from recursion theory and descriptive set theory.

In our description of Brouwer's universe we have discussed a few basic principles which have unusual consequences in practical mathematics. One of the challenges of constructive mathematics is to find new principles that embody certain specific phenomena that shed new and unexpected light on the universe. Markov's principle is one of those principles, but unfortunately, one cannot justify it on the basis of a strong notion of 'constructive'. Kripke's schema is a good candidate. What we need is more experience with its applications, furthermore it would be desirable to find a realistic mathematical principle equivalent to KS, in the tradition of reverse mathematics.

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