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Formal Topology and Constructive Mathematics: the Gelfand and Stone-Yosida Representation Theorems[1]

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Abstract: We present a constructive proof of the Stone-Yosida representation theorem for Riesz spaces motivated by considerations from formal topology. This theorem is used to derive a representation theorem for f-algebras. In turn, this theorem implies the Gelfand representation theorem for C*-algebras of operators on Hilbert spaces as formulated by Bishop and Bridges. Our proof is shorter, clearer, and we avoid the use of approximate eigenvalues.

Key Words: Formal topology, constructive mathematics, Riesz space, f-algebra, axiom of choice
Category: F.1.1, F.4.1, G.1

1 Introduction

This paper illustrates the relevance of locale theory for constructive mathematics. We present a constructive proof of the Stone-Yosida representation theorem for Riesz spaces motivated by considerations from formal topology. This theorem is used to derive a representation theorem for f-algebras. In turn, this theorem implies the Gelfand representation theorem for C*-algebras of operators on Hilbert spaces as formulated by Bishop and Bridges [BB85]. Our proof is shorter, clearer and we avoid the use of approximate eigenvalues.

The article is organized as follows. After dealing with some preliminaries we prove a pointfree Stone-Yosida representation theorem for Riesz spaces. In the next section this is used to obtain a representation theorem for f-algebras, which in turn is used to prove the Gelfand representation theorem. Next we discuss the similarity between Bishop's notion of compactness, i.e. complete and totally bounded, and compact overt spaces in formal topology. Finally we show that the axiom of dependent choice is needed to construct points in the formal spectrum.

We would like to stress that the present theory needs few foundational commitments, we work within Bishop-style mathematics. Moreover, the mathemat-

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

ics is predicative, even finitary, and we will not use the axiom of choice, even countable choice, unless explicitly stated.

2 Riesz spaces

We present a Stone-Yosida representation theorem for Riesz spaces. This theorem states that a Riesz space with a strong unit may be represented as a Riesz space of functions. In fact, one can even start with an l-group, or lattice ordered group, with a strong unit and construct a Riesz space from this, see [Coq05](sec. 3).

2.1 General definitions

Definition 1. A *Riesz space* is a $\mathbb{Q}$ vector space with a binary sup operation.

As usual we define $a^+ := a \vee 0$, $a^- := (-a) \vee 0$, $|a| := a^+ + a^-$ and $a \wedge b := -(-a \wedge -b)$. One can prove that a Riesz space is a lattice and that the lattice operations are compatible with the vector space operations, see [Bir67] [LZ71] [Bou64].

Definition 2. A *strong unit 1* in an ordered vector space R is a positive element such that for all $a \in R$ there exists a natural number n such that $a \leq n1$.

We will now consider a Riesz space R with a strong unit.

Definition 3. A *representation* of R is a linear map $\sigma : R \to \mathbb{R}$ such that $\sigma(1) = 1$ and $\sigma(a \vee b) = \sigma(a) \vee \sigma(b)$.

Such a representation automatically preserves all the Riesz space structure.

Example 1. If X is a compact space, then $C(X)$, its space of continuous functions, is a Riesz space where the supremum is taken pointwise. Each point of x defines a representation $\sigma_x(f) := f(x)$.

In Example 2 we show that a complete commutative algebra of Hermitian operators on a Hilbert space is a Riesz space.

2.2 Spectrum

Let P denote the set of positive elements of a Riesz space R. For a, b in P we define $a \preccurlyeq b$ to mean that there exists n such that $a \leq nb$. The following proposition is proved in [Coq05] and involves only elementary considerations on Riesz spaces.

Proposition 4. *$L(R) := (P, \vee, \wedge, 1, 0, \preccurlyeq)$ is a distributive lattice. In fact, if we define $D : R \to L(R)$ by $D(a) := a^+$, then $L(R)$ is the free lattice generated by $\{D(a) | a \in R\}$ subject to the following relations:*

1. *$D(a) = 0$, if $a \leq 0$;*
2. *$D(1) = 1$;*
3. *$D(a) \wedge D(-a) = 0$;*
4. *$D(a + b) \leqslant D(a) \vee D(b)$;*
5. *$D(a \vee b) = D(a) \vee D(b)$.*

We have $D(a) \leqslant D(b)$ if and only if $a^+ \preccurlyeq b^+$ and $D(a) = 0$ if and only if $a \leqslant 0$.

We write $a \in (p, q) := (a - p) \wedge (q - a)$. Notice that this is an element of R.

Lemma 5. *$a \in (p, q) \preccurlyeq a \in (p, s) \vee a \in (t, q)$, whenever $t < s$. Since our Riesz space has a strong unit, for each a, there exists p and q such that $p < q$ and $a \in (p, q) = 1$. Moreover, if $I_0, \ldots, I_n$ are open intervals covering (p, q), then $\bigvee a \in I_i = 1$.*

Lemma 6. *If $1 = D(b_1) \vee \ldots \vee D(b_n)$, then there exists $r > 0$ such that $D(b_1 - r) \vee \ldots \vee D(b_n - r) = 1$.*

Proof. Since $1/N \leqslant b_1^+ \vee \ldots \vee b_m^+$ we see that

$$\begin{aligned} 1/2N &\leqslant (b_1^+ \vee \ldots \vee b_m^+) - 1/2N \\ &= (b_1^+ - 1/2N) \vee \ldots \vee (b_m^+ - 1/2N) \\ &\leqslant (b_1 - 1/2N)^+ \vee \ldots \vee (b_m - 1/2N)^+. \end{aligned}$$

In [Coq05](sec. 3) the previous lemma is used to prove the following result which we will not need, but only state as motivation.

Theorem 7. *Define Σ to be the locale generated by the elements $D(a)$ and the relations in Proposition 4 together with the relation $D(a) = \bigvee_r D(a - r)$. Then Σ is a compact completely regular locale.*

Compact completely regular locales are the pointfree analogues of compact Hausdorff spaces. Moreover, the points of Σ can be identified with representations of R. In fact, if a representation σ is given, then $\sigma \in D(a)$ if and only if $\sigma(a) > 0$.

2.3 Normable elements

Upper cuts in the rational numbers may be conveniently used to deal with certain objects that would classically be real numbers, we call them *upper real* numbers, see [Ric98] [Vic03]. In general, such a cut does not have a greatest lower bound in $\mathbb{R}$. If it does the upper real number is called *located* or simply a real number. Define the upper real $U(a) := \{q \in \mathbb{Q} | \exists q' < q.a \leq q'\}$. If it is located a is said to be *normable* and the greatest lower bound is denoted by $\sup a$. Then we have $\sup a < q$ if and only if $q \in U(a)$.

Proposition 8. *If all elements of R are normable, then the predicate* $\mathrm{Pos}(a) := \sup a > 0$ *has the following properties:*

1. *If* $\mathrm{Pos}(a)$ *and* $D(a) \leqslant D(b)$*, then* $\mathrm{Pos}(b)$*;*
2. *If* $\mathrm{Pos}(a \vee b)$*, then* $\mathrm{Pos}(a)$ *or* $\mathrm{Pos}(b)$*;*
3. *If* $r > 0$*, then* $\mathrm{Pos}(a)$ *or* $D(a - r) = 0$*.*

We note that $\mathrm{Pos}(a)$ if and only if $\mathrm{Pos}(a^+)$.

In fact, in localic terms this shows that Σ is open, see [Joh84], but we will not need this.

In order to prove this proposition we first need three lemmas.

Lemma 9. $\sup(a \vee b) = \sup a \vee \sup b$*.*

Proof. $\geqslant$ Suppose that $a \vee b \leqslant q' < q$, then $a, b \leqslant q'$, so both $\sup a \leqslant q'$ and $\sup b \leqslant q'$. Thus $q' \geqslant \sup a \vee \sup b$.

$\leqslant$ If $q' \geqslant a, b$, then $q' \geqslant a \vee b$ and hence $\sup a \vee b \leqslant \sup a \vee \sup b$.

Lemma 10. *Let r be a rational number. If* $\sup b < r$ *and* $r < \sup(b \vee c)$*, then* $r < \sup c$*.*

Proof. If $\sup b < r$, then $b \leqslant r' < r$, for some rational number r'. So, using Lemma 9

$$r < \sup(b \vee c) = \sup b \vee \sup c \leqslant r' \vee \sup c.$$

Consequently, $r < \sup c$.

Lemma 11. *If* $0 < \sup(b_1 \vee b_2)$*, then* $\sup b_1 > 0$ *or* $\sup b_2 > 0$*.*

Proof. Suppose that $0 < r < \sup(b_1 \vee b_2)$. Either $\sup b_1 < r$ or $\sup b_1 > 0$. In latter case we are done. In the former case $\sup b_2 > r$ by Lemma 10.

Proof. **[of Proposition 8]**

Property 1 is clear.

Property 2 is Lemma 11.

Finally, to prove property 3 we decide whether $\sup a > 0$ or $\sup a < r$. In the former case $\mathrm{Pos}(a)$. In the latter case $D(a - r) = 0$.

Corollary 12. *If* $1 = D(a_1) \vee \ldots \vee D(a_n)$*, we can find* $i_1 < \ldots < i_k$ *such that* $1 = D(a_{i_1}) \vee \ldots \vee D(a_{i_k})$ *and* $\mathrm{Pos}(a_{i_1}), \ldots , \mathrm{Pos}(a_{i_k})$*.*

Proof. By Lemma 6 there exists $r > 0$ such that $1 = D(a_1 - r) \vee \ldots \vee D(a_n - r)$. By Proposition 8 for all i, $D(a_i - r) = 0$ or $\mathrm{Pos}(a_i)$. From this the result follows.

Lemma 13. *Let* $I := (p, q)$*,* $J := (r, s)$*. Define* $I + J := (p + r, q + s)$ *and* $I \vee J := (p \vee r, q \vee s)$*. If* $|a + b - c| \leqslant \varepsilon$ *and* $\mathrm{Pos}(a \in I \wedge b \in J \wedge c \in K)$*, then the distance between* $I + J$ *and* K *is bounded by* ε*. If* $|a \vee b - c| \leqslant \varepsilon$ *and* $\mathrm{Pos}(a \in I \wedge b \in J \wedge c \in K)$*, then the distance between* $I \vee J$ *and* K *is bounded by* ε*.*

Finally, if $|a - b| \leqslant \varepsilon$ *and* $\mathrm{Pos}(a \in I \wedge b \in J)$*, then the distance between* I *and* J *is bounded by* ε*.*

Proof. We only prove the last fact. If the distance between I and J is bigger than ε, than $r - q > \varepsilon$ or $p - s > \varepsilon$. Consequently, $D(q - a \wedge b - r) = 0$ or $D(a - p \wedge s - b) = 0$. Both cases imply that $D(a \in I \wedge b \in J) = 0$.

Definition 14. A Riesz space R is *separable* if there exists a sequence a_n such that for all a and $\varepsilon > 0$, there exists n such that $|a - a_n| \leqslant \varepsilon$.

Theorem 15. [DC] *Let* R *be a separable Riesz space all elements of which are normable. Assume that* $\mathrm{Pos}(a)$*, then there exists a representation* σ *such that* $\sigma(a) > 0$*.*

Proof. We write $\varepsilon_n := 1/2^n$. Using dependent choice and Lemma 5 we define a sequence of rationals such that

$$\mathrm{Pos}(a \wedge a_0 \in (q_0 - \varepsilon_0, q_0 + \varepsilon_0) \wedge \ldots \wedge a_n \in (q_n - \varepsilon_n, q_n + \varepsilon_n)).$$

If $b \in R$, we can find a sequence of elements a_{n_k} such that for any $\varepsilon > 0$ we have $|b - a_{n_k}| \leq \varepsilon$ when k is large enough. Then q_{n_k} is a Cauchy sequence and we define $\sigma(b) := \lim_k q_{n_k}$. By Lemma 13 this definition does not depend on the choice of the sequence a_{n_k}. The map σ is a representation such that $\sigma(a) > 0$ and $\sigma(a_n) \in (q_n - \varepsilon_n, q_n + \varepsilon_n)$ for all n.

A suggestive way to state that $\sigma(a) > 0$ is to say that σ is a *point* in $D(a)$.

Let Σ be the set of representations of R. We call Σ the *spectrum* of R. Each element a of R defines a pseudo norm $\rho_a(\sigma) := |\sigma(a)|$. If the algebra is separable and a_n is a dense sequence, we can collect, like Bishop, all the pseudo-norms into one metric $\sum 2^{-n}(1+\|a_n\|)^{-1}|\sigma(a_n)|$.

We have the following Stone-Yosida representation theorem, see [Sto41] [Yos42].

Theorem 16. [DC] *Let R is a separable Riesz space all elements of which are normable. The spectrum Σ is a complete totally bounded metric space. For a in R, we define $\hat{a} : \Sigma \to \mathbb{R}$ by $\hat{a}(\sigma) := \sigma(a)$ and $\|a\| := \sup(|a|)$. Then $\sup_\sigma |\hat{a}(\sigma)| = \|a\|$. Finally, the set of functions $\hat{a}$ is dense in $C(\Sigma)$.*

Proof. We define $U_{ars} := D(a \in (r, s))$ as an element of the lattice $L(R)$. Intuitively, such an element may be thought of as the set $\{\sigma : \sigma(a) \in (r, s)\} = \{\sigma : \rho_a(\sigma) < s - r\}$. Let $\varepsilon > 0$. Using Lemma 5 one may find finitely many U_{ars} such that $s - r < \varepsilon$, the supremum of which equals 1 in the lattice $L(R)$. By Corollary 12 one can assume all these elements to be positive. Each of them contains a point by Theorem 15. The collection of these points form an ε-net. Consequently, Σ is totally bounded.

It is straightforward to show that Σ is also complete as a uniform space.

For each $\sigma \in \Sigma$ we have $|\sigma(a)| \leqslant \|a\|$. To see this suppose that $\sigma(a) > \|a\|$. Then there exists $\varepsilon > 0$ such that $\sigma(a) - a \geqslant \varepsilon 1$, however $\sigma(\sigma(a) - a) = 0$. If $r < \|a\|$, then by Theorem 15, there exists σ such that $r < |\sigma(a)|$.

Finally, the density follows from Proposition 3.1 in [Coq05]. Its proof involves only elementary properties of Riesz spaces.

Notice the interplay between the *pointwise* and *pointfree* framework. From a formal covering, in the lattice $L(R)$, it is possible to deduce that Σ, a metric space, is totally bounded. This is remarkable since in a recursive interpretation of Bishop's mathematics Σ does not have enough points.

3 f-algebras

In this section we apply the results of the previous section to f-algebras.

Definition 17. An *f-algebra* is a Riesz space with a strong unit and a commutative[3] multiplication such that $0 \leqslant ab$, whenever $0 \leqslant a$ and $0 \leqslant b$.

3.1 f-algebra of operators

Example 2. If R is a complete commutative algebra of normable self-adjoint operators on a Hilbert space H, then R is a Riesz space with the order $\leqslant$ defined by $0 \leqslant A$ if and only if $(Au, u) \geqslant 0$ for all u in H.

[3] One can prove classically that the commutativity requirement follows form the other properties of an f-algebra. We intend to provide a constructive proof of this separately using the pointfree description of the spectrum.

In the rest of this subsection we prove that if $A, B \geqslant 0$, then $AB \geqslant 0$.

We have now defined two notions of boundedness on the algebra of operators. One as a bounded operator: A is bounded by a if for all x, $\|Ax\|^2 \leqslant a\|x\|^2$. The other from the ordering: A is bounded by a if $A \leqslant aI$, where I is the identity operator.

Lemma 18. *The two notions of boundedness coincide — that is, for all x, $(Ax, x) \leqslant (ax, x)$ if and only if for all x, $\|Ax\|^2 \leqslant a\|x\|^2$. Consequently, $\|A^2\| = \|A\|^2$.*

Proof. The usual proof, for instance in [Lan83] using the polarization identity, is constructive.

Since $(AB^2x, x) = (A(Bx), (Bx))$, we see that $AB^2 \geqslant 0$, whenever $A \geqslant 0$. This suffices to prove that R is an ordered ring.

Lemma 19. [Rie32] (p33, footnote 9) *Let R be a as above. Then every positive element is the uniform limit of a sum of squares.*

Proof. We can assume that $0 \leqslant A \leqslant 1$. Define $A_0 := A$ and $A_{n+1} := A_n - A_n^2$. Then $0 \leqslant A_{n+1} \leqslant 1$, since $A_{n+1} = A_n(1-A_n)^2 + (1-A_n)A_n^2 \geqslant 0$ and $1 - A_{n+1} = 1 - A_n + A_n^2$. Moreover, $A_{n+1} = A_n - A_n^2 \leqslant A_n$. Since $A = A_1^2 + \cdots + A_n^2 + A_{n+1}$, we have $A_n^2 \leqslant 1/n \to 0$. By lemma 18 this implies that $A_n \to 0$.

Corollary 20. *$AB \geqslant 0$, whenever $A, B \geqslant 0$.*

Proof. If $A, B \geqslant 0$, then $(ABx, x) = \sum(AB_nx, B_nx) \geqslant 0$, where B_n is a sequence such that ΣB_n^2 converges to B.

The following lemma shows that one can construct the square root[4] when R is complete and thus one can define the absolute value as $|A| := \sqrt{A^2}$. From the absolute value one first defines $A^+ := (|A|+A)/2$ and then $A \vee B := A + (B-A)^+$. Consequently, the algebra is a Riesz space and an f-algebra.

Lemma 21. *For all $A \geqslant 0$ we can build a Cauchy sequence (A_n) of positive elements such that $A_n^2 \to A$.*

Proof. We can assume $0 \leq A \leq I$. We define the two sequences $A_n \in [0, I]$ and $r_n \in [0, 1]$ defined by $A_0 = 0$ and $r_0 = 0$ and

$$A_{n+1} = 1/2(1 - A + A_n^2) \qquad r_{n+1} = 1/2(1 + r_n^2)$$

[4] This is the usual lemma that R admits square root of positive elements if R is complete. Notice that the proof is directly constructive, and it corresponds to the usual Taylor expansion of $(1 - x)^{1/2}$.

Clearly, we have $A_n \leq r_n$ for all n.

We claim that we have for all n

$$A_n \leq A_{n+1} \qquad r_n \leq r_{n+1} \qquad A_{n+1} - A_n \leq r_{n+1} - r_n$$

This is proved by induction from the equalities

$$A_{n+1} - A_n = 1/2(A_n + A_{n-1})(A_n - A_{n-1})$$
$$r_{n+1} - r_n = 1/2(r_n + r_{n-1})(r_n - r_{n-1})$$

It follows that we have

$$(I - A_n)^2 - A = 2(A_{n+1} - A_n) \leq 2(r_{n+1} - r_n)$$

In order to conclude, all is left is to show that (r_n) has limit 1. We know that $0 \leq r_n \leq r_{n+1} \leq 1$ and we have

$$1 - r_{n+1} = (1 - r_n)1/2(1 + r_n) \leq (1 - r_n)(1 - \epsilon/2)$$

if $r_n \leq 1 - \epsilon$. This shows that if $(1 - \epsilon/2)^N \leq \epsilon$ we have $1 - r_n \leq \epsilon$ for all $n \geq N$.

3.2 Gelfand representation

Any f-algebra is a Riesz space so we have a Gelfand representation of the f-algebra qua Riesz space, see Theorem 16.

Theorem 22. *The Gelfand transform* $\hat{\cdot}$ *preserves multiplication.*

Proof. Since $2ab = (a+b)^2 - a^2 - b^2$. We need to prove that $\hat{\cdot}$ preserves squares — that is $\sigma(a^2) = \sigma(a)^2$. For this we first prove: $\sigma(ab) > 0$, whenever $\sigma(a), \sigma(b) > 0$.

If $\sigma(a) \geqslant r > 0$, then $\sigma(a - r) > 0$. By Lemma 6.3 in [Coq05], $(a-r)^+ \wedge b^+ \leqslant \frac{1}{r}(ab)^+$, so $\sigma(ab)^+ \geqslant r(\sigma(b)^+ \wedge \sigma(a-r)^+) > 0$, which was to be proved.

Suppose that $|\sigma(a)| < q$. Then $q > \sigma(a)$ and so $\sigma(q - a) > 0$. Similarly, $\sigma(q + a) > 0$. Consequently, $\sigma(q^2 - a^2) = \sigma((q - a)(q + a)) > 0$. By a similar argument we see that if $|\sigma(a)| < q$, then $|\sigma(a^2)| < q^2$. We conclude that $\sigma(a^2) = \sigma(a)^2$.

We have proved the following representation theorem for f-algebras which explains the name f-algebra: an f-algebra is an abstract function algebra.

Theorem 23. [DC]*Let $\mathcal{A}$ be a separable f-algebra of normable elements, then the spectrum Σ is a compact metric space and there exists an f-algebra embedding of $\mathcal{A}$ into $C(\Sigma)$.*

We now specialize this theorem to the f-algebra in Example 2 and obtain Bishop's version of the Gelfand representation theorem. In fact, like Bishop we first prove the theorem for Hermitian operators. As a corollary we obtain the Gelfand duality theorem for a separable Abelian C*-algebra, exactly as stated by Bishop [BB85] Cor.8.28 by considering its self-adjoint part which is an f-algebra.

Corollary 24. [DC]*Let $\mathcal{A}$ be a separable f-algebra of normable Hermitian operators on a Hilbert space, then the spectrum is a compact metric space and there exists an f-algebra embedding of $\mathcal{A}$ into $C(\Sigma)$.*

Theorem 25. [DC]*Let R be an Abelian C*-algebra of operators on a Hilbert space. Then there exists a C*-algebra embedding $\varphi$of R into $C(\Sigma, \mathbb{C})$, where Σ is a compact metric space. Moreover, $\varphi(1) = 1$ and R is norm-dense.*

Bishop's Gelfand representation theorem states that for any commutative algebra of normable operators on a separable Hilbert space there exists a norm-preserving isomorphism to the algebra of continuous functions on its spectrum. To prove that this map is norm-preserving Bishop proves that certain ε eigenvectors can be computed. In fact, the computational information of the ε eigenvectors is used only to prove the non-computational statement that the map is norm-preserving. In contrast, we work directly on the approximations so that we can avoid these unused computational steps.

3.3 Peter-Weyl

For a typical application, we let G be a compact group and R be the algebra of operators over $L_2(G)$ generated by the unit operator and the operators $T(f)(g) := f * g$, where $*$ denotes the convolution product. Each operator $T(f)$ is compact and hence normable. The non-trivial representations of R are then exactly the characters of the group G. This gives a reduction of the Peter-Weyl theorem to the Gelfand representation theorem, see [CS05].

4 Compact overt locales

Bishop defines a metric space to be compact if it is complete and totally bounded and proves that all uniformly continuous functions defined on such a metric space are normable. In contrast, in the framework of locale theory it is not true in general that all functions on a compact regular locale are normable, i.e. the norm is only defined as an upper real which may not be a located.

However, the locale considered in Theorem 7 is not only compact completely regular, but also overt as shown by Proposition 8. In this case all the continuous functions are normable. This is a general fact.

Theorem 26. *If X is a compact completely regular locale, then X is overt if and only if for any $f \in C(X)$ there exists* $\sup f \in \mathbb{R}$ *such that* $\sup f < s$ *if and only if* $f^{-1}(-\infty, s) = X$.

Proof. We prove only the 'only if' part. If X is overt and $f \in C(X)$, then an approximation of the supremum can be found by considering a finite covering of X by positive opens of the form $f^{-1}(r, s)$, where $s - r$ is small.

The previous facts suggest a similarity between Bishop's compact metric spaces and the compact overt spaces in formal topology.

$$\text{Bishop compact} \Leftrightarrow \text{compact overt}$$

Clearly this requires further developments building on ideas in [ML70] [Joh84]. However, we postpone this to further work.

We would like to conclude this discussion with the following comparison between the three following frameworks: classical mathematics with the axiom of choice, Bishop's mathematics and our framework, predicative constructive mathematics without dependent choice[5]. Using classical logic and the axiom of choice one can show that the spectrum defined in Theorem 7 has enough points [Joh82]. Thus in this setting the pointfree and pointwise description of the space coincide. In a recursive interpretation of Bishop's framework these descriptions differ. However, using dependent choice, normability and separability assumptions, we have shown how to deduce that Σ is totally bounded from the pointfree description of Σ.

Our conclusion is that the best formulation of the representation theorem is the pointfree one, since, besides being neutral on the use of the axiom of choice and classical logic, it implies the usual formulations both in Bishop's framework and in classical mathematics.

5 Choice

5.1 No points

As mentioned before, when all the elements of the algebra are normable, one can construct points in the spectrum using dependent choice. We claim that dependent choice is needed for this. In fact, it is known that there exist compact overt locales for which we need countable choice to construct a point. Thus it suffices to consider the space of continuous functions on such a locale. We think that a nice example can be extracted from [Ric00].

Richman [Ric00](p.5) gave an informal argument that indicates that one can not construct the zeroes of the complex polynomial $X^2 - a$ unless one knows

[5] This framework is related to Richman's proposal to develop constructive mathematics without using countable choice, see [Ric00].

whether $a = 0$ or not. To a in $\mathbb{C}$ we can associate the locale Y_a of roots of $X^2 - a$. The existence of a point in Y_a requires dependent choice. On the other hand using results from [Vic03] it should be possible to show that Y_a is compact overt as an element of the completion of the metric space n-multisets in $\mathbb{C}$. The metric on this space is the usual Hausdorff metric on compact subsets of $\mathbb{C}$.

Finally, we remark that this leaves open the question whether it is possible to construct the points of the spectrum of a discrete countable Riesz space over the rationals, or more generally, to construct the points of the spectrum of a separable Riesz space.

5.2 Spreads

It is interesting to note that Richman [Ric02] proposes to use spreads to avoid dependent choice. We suggest to use formal spaces instead. One motivation of formal topology [ML70] [Sam87] was precisely to give a direct treatment of Brouwer's spreads by working with trees of finite sequences. Formal topology may be seen as a predicative and constructive version of locale theory. Johnstone [Joh82] stresses that one may avoid the use of the axiom of choice in topology by using locale theory and dealing directly with the opens. In this light it may not be so surprising that Richman uses spreads to avoid choice.

Richman's definition of spread differs in two respects from Heyting's definition. The branching of the tree is arbitrary, i.e. not necessarily indexed by the natural numbers, and it is not decidable whether or not a branch can be continued. This may be compared to the present situation where we study the maximal spectrum Σ. When Σ has a countable base, we may define it as a finitely branching tree. When furthermore Σ is overt, every positive branch can be continued in a positive way. One difference between Richman's spreads and our approach is that Richman requires the infinite branches to be elements of a metric space.

6 Conclusion

We gave a constructive proof of the Stone-Yosida representation theorem for Riesz spaces. This theorem was used to prove a representation theorem for f-algebras, from which we derived the Gelfand representation theorem for commutative C*-algebras of operators on a Hilbert space. This constructive theorem generalizes the one by Bishop and Bridges. In a similar way one may prove a generalization of Bishop's spectral theorem, see [Spi05].

It should be noted that we have used normability and separability hypothesis in the statements of the main theorems and used the axiom of dependent choice. In fact, without these hypothesis we can still obtain the spectrum as a compact locale. The normability is necessary to show that the spectrum is overt. The separability hypothesis is used to obtain a metric space instead of a uniform space. Finally, the axiom of dependent choice is used in Theorem 15 to construct a point in each positive open, and thus obtain a metric space in the sense of Bishop.

In this context, we would like to mention a problem for both constructive versions of the Gelfand representation theorem: can it be applied to construct the Bohr compactification of, say, the real line, like Loomis [Loo53]? Considering that the Stone-ech compactification has been successfully treated in locale theory [Joh82], one would hope that a similar treatment is possible. Since the almost periodic functions do not form an algebra constructively, we may consider the f-algebra of functions generated by them. However, in this algebra not all elements are normable. Any element of the group determines a point in the spectrum. However, it is not possible to extend the group operation to the spectrum and obtain a localic group, since every compact localic group has a positivity predicate [Wra90] and since, moreover, the spectrum is compact this would imply that all the functions in the f-algebra are normable. This, as we stated before, is not the case. See Spitters [Spied] and the references therein for a constructive theory of almost periodic functions.

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