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Constructive Suprema[1]

Marian Alexandru Baroni
(CFR High School, Galati, Romania
marianbaroni@yahoo.com)

Abstract: Partially ordered sets are investigated from the point of view of Bishop's constructive mathematics, which can be viewed as the constructive core of mathematics and whose theorems can be translated into many formal systems of computable mathematics. The relationship between two classically equivalent notions of supremum is examined in detail. Whereas the classical least upper bound is based on the negative concept of partial order, the other supremum is based on the positive notion of excess relation. Equivalent conditions of existence are obtained for both suprema in the general case of a partially ordered set; other equivalent conditions are obtained for subsets of a lattice and, in particular, for subsets of $\mathbf{R}^n$.

Key Words: Constructive mathematics, partially ordered set, supremum
Category: F.4.1

1 Introduction

The aim of this paper is a constructive examination of the notions of supremum and infimum. Our setting is Bishop's constructive mathematics (see [Bishop 1967] or [Bishop and Bridges 1985]), mathematics developed with intuitionistic logic, a logic based on the strict interpretation of "existence" as "computability". The use of intuitionistic logic allows the interpretation of the results in a wide variety of models, including Brouwer's intuitionism, recursive mathematics, and even classical mathematics. Instead of going into details about the varieties of constructivism, we direct the reader to [Beeson 1985, Bridges and Richman 1987, Troelstra and van Dalen 1988].

The notions of supremum and infimum are almost ubiquitous in the theory of partially ordered sets and in that of the algebraic ordered structures, such as ordered groups, ordered vector spaces, or ordered algebras. In the classical theory, the supremum is defined as the least upper bound. In $\mathbf{R}$ we have an alternative definition, which is based on the strict order relation. Classically, the two definitions are equivalent but this does not hold constructively, the latter supremum being stronger than the least upper bound [Mandelkern 1983]. This illustrates a main feature of constructive mathematics: classically equivalent definitions or theorems are no longer equivalent under constructive scrutiny.

The first major problem that arises in the constructive development of a theory is to obtain appropriate counterparts of the classical notions. The definition

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

of the supremum as the least upper bound can be used in a general context: the supremum of a subset of an arbitrary partially ordered set can be defined exactly in the same way. However, in many cases this supremum, which is based on the negative concept of partial order, is too weak a notion. For subsets of the real line, the other supremum is more useful: it enables one to prove stronger results. To obtain a generalization of the stronger supremum, one needs an affirmative relation. By using the excess relation [von Plato 2001], we present such a generalization in Section 3. Following von Plato, we consider a partially ordered set as a set endowed with an excess relation, whose negation is the partial order relation. (See Section 2.)

In Section 3 we examine the relationship between supremum and weak supremum (the least upper bound). As a main result, we prove that the supremum of a subset S exists if and only if S has a weak supremum and it satisfies a certain condition of order locatedness. Suprema of subsets of a lattice are characterized in Section 4. Various equivalent conditions for the existence of supremum and weak supremum of a subset of $\mathbf{R}^n$ are given in Section 5.

2 Partially ordered sets

Although the linear order has been investigated in detail (see, for example, [Bridges 1994, Bridges 1999, Bridges and Reeves 1999, Greenleaf 1978]), a constructive study of "positive" partial order relations has begun only recently [Negri 1999, von Plato 2001].

From a constructive point of view, the partial order is a negative concept and, consequently, its role as a primary relation should be played by an affirmative relation. As shown in [von Plato 2001], an excess relation, a generalization of the linear order, can be used to define a partially ordered set in a constructive manner.

Let X be a nonempty[2] set. A binary relation $\nleq$ on X is called an **excess relation** if it satisfies the following axioms:

E1 $\neg(x \nleq x)$,

E2 $x \nleq y \Rightarrow \forall z \in X\ (x \nleq z \vee z \nleq y)$.

We say that x **exceeds** y whenever $x \nleq y$ and we also denote this by $y \ngeq x$. As pointed out in [von Plato 2001], we obtain an apartness relation $\neq$ and a partial order $\leq$ on X by the following definitions:

$$x \neq y \Leftrightarrow (x \nleq y \vee y \nleq x)\,,$$
$$x \leq y \Leftrightarrow \neg\,(x \nleq y)\,.$$

[2] By "nonempty" we mean "inhabited"; we can construct an element of the set.

An equality $=$ and a strict partial order $<$ can be obtained from the relations $\neq$ and $\leq$ in the standard way:

$$x = y \Leftrightarrow \neg (x \neq y)\,,$$
$$x < y \Leftrightarrow (x \leq y \wedge x \neq y)\,.$$

If an apartness and a partial order are considered as basic relations, the transitivity of strict order cannot be obtained. (A proof using Kripke models is given in [Greenleaf 1978].) In contrast, an excess relation as a primary relation enables us to prove this property. Moreover, it is straightforward to see that

$$(x \leq y \wedge y < z) \vee (x < y \wedge y \leq z) \Rightarrow x < z.$$

Throughout this paper, a **partially ordered set** will be a nonempty set endowed with a partial order relation induced, as above, by an excess relation. Let us note that the statement

$$\neg(x \leq y) \Rightarrow x \nleq y$$

does not hold in general. For real numbers, it is equivalent to **Markov's principle**:

> *if* (a_n) *is a binary sequence such that* $\neg\forall n(a_n) = 0$*, then there exists* n *such that* $a_n = 1$*.*

Although this principle is accepted in the recursive constructive mathematics developed by A.A. Markov, it is rejected in Bishop's constructivism. For further information on Markov's principle, the reader is directed to [Bridges and Richman 1987] and [Troelstra and van Dalen 1988].

To end this section, let us consider an example. Let X be a set of real–valued functions defined on a nonempty set S, and let $\nleq$ be the relation on X defined by $f \nleq g$ if there exists x in S such that $g(x) < f(x)$. Clearly, this is an excess relation whose corresponding partial order relation is the pointwise ordering of X. When $S = \{1, 2, \ldots, n\}$, we may view the set of all real–valued functions on S as the Cartesian product $\mathbf{R}^n$. In this case,

$$(x_1, x_2, \ldots, x_n) \nleq (y_1, y_2, \ldots, y_n) \Leftrightarrow \exists i \in \{1, 2, \ldots, n\}\ (x_i > y_i).$$

The natural apartness, equality, partial order, and strict partial order on $\mathbf{R}^n$ are obtained from this, as follows:

$$(x_1, x_2, \ldots, x_n) \neq (y_1, y_2, \ldots, y_n) \Leftrightarrow \exists i \in \{1, 2, \ldots, n\}\ (x_i \neq y_i),$$
$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \Leftrightarrow \forall i \in \{1, 2, \ldots, n\}\ (x_i = y_i),$$
$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \Leftrightarrow \forall i \in \{1, 2, \ldots, n\}\ (x_i \leq y_i),$$

$$(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n) \Leftrightarrow \forall i \in \{1, 2, \ldots, n\}\ (x_i \leq y_i)\ \wedge \\ \exists j \in \{1, 2, \ldots, n\}\ (x_j < y_j).$$

For $n = 1$, the above excess relation is nothing else than the linear order $>$ on the real number set $\mathbf{R}$.

3 Suprema and infima

As in the classical case, a nonempty subset S of a partially ordered set X is said to be **bounded above** if there exists an element b of X such that $a \leq b$ for all a in S. In this case, b is called an **upper bound** for S. A **bounded below** subset and a **lower bound** are defined similarly, as expected. It is said that S is **order bounded** if it is bounded above and below. The classical least upper bound will be called **weak supremum**. In other words, an upper bound w of S is a weak supremum of S if

$$(\forall a \in S\ (a \leq b)) \Rightarrow w \leq b.$$

The definition of join of two elements of a lattice [von Plato 2001] can be easily extended to a general definition of the supremum [Baroni 2003]. Consider an excess relation $\nleq$ on X, a nonempty subset S of X, and $s \in X$, an upper bound for S. We say that s is a **supremum** of S if

$$(x \in X \wedge s \nleq x) \Rightarrow \exists a \in S\ (a \nleq x).$$

If S has a (weak) supremum, then that (weak) supremum is unique. We denote by $\sup S$ and w-sup S the supremum and the weak supremum of S, respectively, when they exist. The **infimum** $\inf S$ and the **weak infimum** w-inf S are defined similarly. A lower bound m for S is called the

- **infimum** of S if $(x \in X \wedge x \nleq m) \Rightarrow \exists a \in S\ (x \nleq a)$;
- **weak infimum** of S if $(\ \forall a \in S\ (b \leq a)) \Rightarrow b \leq m$.

Since each (weak) infimum with respect to the excess relation $\nleq$ is a (weak) supremum with respect to the dual relation $\ngeq$, we will obtain dual properties for (weak) supremum and (weak) infimum. The results will be given only for the suprema, without mentioning the corresponding counterparts for infima.

For subsets of $\mathbf{R}$, we obtain the standard constructive supremum. An upper bound s of S is the supremum of S if

$$s > x \Rightarrow \exists a \in S\ (a > x).$$

Classically, according to the least–upper–bound principle, each nonempty subset of $\mathbf{R}$ that is bounded above has a supremum. This is not valid from the constructive point of view, either in the stronger form (with supremum), or in the weaker form (with weak supremum). The stronger version entails the **limited principle of omniscience**[3] (**LPO**):

> *for every binary sequence (a_n), either $a_n = 0$ for all n, or else $a_n = 1$ for some n.*

This principle is false in both the intuitionistic and the recursive models of constructive mathematics, and is regarded as highly nonconstructive in Bishop's style mathematics. Similarly, the weaker version entails another nonconstructive principle, the **weak limited principle of omniscience** (**WLPO**):

> *for every binary sequence (a_n), either $a_n = 0$ for all n, or it is contradictory that $a_n = 0$ for all n.*

Nevertheless, there are appropriate constructive substitutes of the least–upper–bound principle for both suprema [Bishop and Bridges 1985, Mandelkern 1983], so that **R** is order complete, even from the constructive standpoint. If S is a nonempty subset of **R**, then $\sup S$ exists if and only if S is bounded above and for all real numbers α, β with $\alpha < \beta$, either β is an upper bound of S or else there exists $a \in S$ such that $a > \alpha$ (Proposition 4.3 in Chapter 2 of [Bishop and Bridges 1985]). This equivalent condition can be extended to a general definition of order locatedness which, in turn, leads us to a constructive definition of order completeness [Baroni 2004a,b]. A nonempty subset S of the partially ordered set X is said to be **upper located** if for each pair x, y of elements of X, with $y \nleq x$, either there exists an element a of S with $a \nleq x$ or else there exists an upper bound b of S with $y \nleq b$.

Proposition 3.1 *Let S be a nonempty subset of the partially ordered set X. Then S has a supremum if and only if it is upper located and its weak supremum exists.*

Proof. Let s be the supremum of S and x, y, a pair of elements of X such that y exceeds x. Then either $y \nleq s$ or $s \nleq x$. In the former case, y exceeds the upper bound s of S and in the latter one, according to the definition of supremum, there exists an element of S that exceeds x.

Conversely, assume that S is upper located and let w be the weak supremum of S. We will prove that $w = \sup S$. To this end, let x be an element of X such that $w \nleq x$. If b is an upper bound of S, then the condition $w \nleq b$ is contradictory to the definition of weak supremum. As S is upper located, there exists a in S that exceeds x. By the definition of supremum, it follows that $w = \sup S$. □

The next proposition provides us equivalent conditions for the existence of the weak supremum.

[3] This implication was explained in detail in the first chapter of [Bishop 1967].

Proposition 3.2 *For an upper bound s of S, the following conditions are equivalent.*

(1) $s = w\text{–}\sup S$.

(2) $\neg(s \leq x) \Rightarrow \neg\,(\forall a \in S\ (a \leq x))$.

(3) $s \nleq x \Rightarrow \neg\,(\forall a \in S\ (a \leq x))$.

Proof. The condition (2) is a direct consequence of the definition of weak supremum; whence (1) implies (2). Since $s \nleq x$ entails $\neg(s \leq x)$, (2) implies (3). To prove that (1) follows from (3), take an upper bound b of S and suppose that $s \nleq b$. Then, according to (3), b is not an upper bound for S, a contradiction. Therefore $\neg(s \nleq b)$, that is, $s \leq b$. □

As shown in [Mandelkern 1983] (Proposition 4.13), a subset S of $\mathbf{R}$ has a weak supremum s not only when $s{=}\sup S$ but also when s is the supremum of the set

$$\neg\neg S = \{a \in X : \neg\neg(a \in S)\}.$$

We will extend these results to the general case.

Proposition 3.3 *Let X be a partially ordered set, S a subset of X, and s an element of X. Then*

$$s = \sup S \ \Rightarrow\ s = \sup(\neg\neg S) \ \Rightarrow\ s = w\text{–}\sup(\neg\neg S) \ \Leftrightarrow\ s = w\text{–}\sup S.$$

Proof. To prove the leftmost implication, it suffices to prove that each upper bound of S is an upper bound for $\neg\neg S$ too. Let a be an arbitrary element of $\neg\neg S$, b an upper bound for S, and assume that $a \nleq b$. If $a \in S$, then $a \leq b$, a contradiction. Therefore $\neg(a \in S)$, but this is contradictory to $a \in \neg\neg S$.

It follows from Proposition 3.1 that each supremum is also a weak supremum, so that the second implication is proved. Since each upper bound of $\neg\neg S$ is an upper bound for S and vice versa, s is the least upper bound of $\neg\neg S$ if and only if it is the least upper bound of S. □

We cannot expect to prove constructively that the existence of $\sup(\neg\neg S)$ entails the existence of $\sup S$. If the supremum of each subset of $\mathbf{R}$ exists whenever $\sup(\neg\neg S)$ exists, then LPO holds [Mandelkern 1983]. We will give a Brouwerian example, that is more direct than the one given in [Mandelkern 1983]. Let (a_n) be an arbitrary binary sequence and consider the set

$$S = \{a_n + 1 : n \in \mathbf{N}\} \cup \{x \in \mathbf{R} : x = 2 \text{ if } \forall n(a_n = 0)\}.$$

Assuming that $2 \notin S$ we see that $a_n = 0$ for all n, a contradiction. It follows that $\neg\neg(2 \in S)$; that is, $2 \in (\neg\neg S)$ and therefore $2 = \sup(\neg\neg S)$. If $\sup S$ exists, then, according to Proposition 3.3, $\sup S{=}2$. We can observe that in this case $2 \in S$ and either $a_n{+}1{=}2$ for some n, or $a_n = 0$ for all n.

An open problem raised in [Mandelkern 1983] requires a Brouwerian example for the implication $s = \text{w-sup}\, S \Rightarrow s = \sup(\neg\neg S)$ in the real case. This problem is still unsolved. However, we can show that for arbitrary partially ordered sets, this implication entails a nonconstructive principle.

Proposition 3.4 *If for each partially ordered set X and each subset S of X, the supremum of $\neg\neg S$ exists whenever the weak supremum of S exists, then WLPO holds.*

Proof. Consider $X = \mathbf{R^2}$. Let (a_n) be an arbitrary binary sequence and $S = \{(0,2)\} \cup \{x \in \mathbf{R^2} : x = (2,0) \text{ if } \exists n(a_n = 1) \wedge x = (2,1) \text{ if } \forall n(a_n = 0)\}$. It is easily to prove that $\text{w-sup}\, S = (2,2)$. If we assume that $(2,2) = \sup(\neg\neg S)$, then there exists $x = (x_1, x_2) \in \neg\neg S$ such that $(x_1, x_2) \not\leq (1,2)$, therefore $1 < x_1$. If $x_1 \neq 2$, then $\neg(x \in S)$, a contradiction. It follows that $x_1 = 2$ and, as a consequence, $\neg\neg((2, x_2) \in S)$. Either $x_2 < 1$ or $x_2 > 0$. In the former case, suppose that $x_2 \neq 0$. Then $\neg((2, x_2) \in S)$, which is contradictory to $\neg\neg((2, x_2) \in S)$. Therefore $x_2 = 0$; that is, $(2,0) \in \neg\neg S$. The latter case is handled in a similar manner; we obtain the condition $(2,1) \in \neg\neg S$. Consequently,

$$\neg\neg(\exists n(a_n = 1)) \ \vee \ \neg\neg(\forall n(a_n = 0))$$

and this, in turn, entails WLPO. □

4 Lattices

Linear order in lattices was investigated constructively in [Greenleaf 1978] and [von Plato 2001]. The general case, when the lattice operations are compatible with a partial order relation, was investigated by von Plato. The following definition of a lattice is the positive one introduced in [von Plato 2001]. Let L be a nonempty set endowed with an excess relation $\not\leq$ and two binary operations, **meet** and **join**, denoted by $\wedge$ and $\vee$. It is said that L is a **lattice** if the following axioms are satisfied for all a, b, c in L:

M1 $a \wedge b \leq a$ and $a \wedge b \leq b$,

M2 $c \not\leq a \wedge b \Rightarrow (c \not\leq a$ or $c \not\leq b)$,

J1 $a \leq a \vee b$ and $b \leq a \vee b$,

J2 $a \vee b \not\leq c \Rightarrow (a \not\leq c$ or $b \not\leq c)$.

In other words, taking into account the definitions of supremum and infimum, a partially ordered set L is a lattice if for all a and b in L, $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ exist. As a consequence, for each pair x, y of elements of a lattice, we may also write $a \vee b$ and $a \wedge b$ for $\sup\{a, b\}$ and $\inf\{a, b\}$, respectively. In a lattice the conditions $a \not\leq b$, $a \wedge b < a$ and $b < a \vee b$ are equivalent.

For instance, the set $\mathbf{R}$ is a lattice with respect to the operations $\vee$ and $\wedge$ given by $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$, as defined in [Bishop 1967].

The next proposition provides us characterizations of supremum and weak supremum in lattices.

Proposition 4.1 *Let S be a nonempty subset of a lattice L and s an upper bound of S.*

(i) *The element s is the supremum of S if and only if for all x in L with $x < s$ there exists a in S with $a \not\leq x$.*

(ii) *The following conditions are equivalent.*

(1) $s = \text{w-sup}\, S$.
(2) $x \in L \wedge \neg\neg(x < s) \Rightarrow \neg(\forall a \in S\ (a \leq x))$.
(3) $x \in L \wedge x < s \Rightarrow \neg(\forall a \in S\ (a \leq x))$.

Proof. (i) Assume that $s = \sup S$. Since $s \not\leq x$ whenever $x < s$, the existence of a in S with $a \not\leq x$ is guaranteed by the definition of supremum. To prove the converse implication, let x be an element of L such that $s \not\leq x$. Therefore $s \wedge x < s$ and, according to the hypothesis, there exists an element a of S that exceeds $s \wedge x$. The last condition is equivalent to $a \wedge (s \wedge x) < a$ and, as $a \wedge s = a$, to $a \wedge x < a$. Consequently, there exists an element a of S such that $a \not\leq x$; whence $s = \sup S$.

(ii) It follows from Proposition 3.2 and the implication $\neg\neg(x < s) \Rightarrow \neg\ (s \leq x)$ that (1) entails (2). Clearly, (3) is a consequence of (2). To prove that (3) implies (1), assume that b is an upper bound of S and s exceeds b. Then $s \wedge b < s$ and, as a consequence, it is contradictory for $s \wedge b$ to be an upper bound of S. If a is an arbitrary element of S, then $a \leq s$ and $a \leq b$, therefore $a \leq s \wedge b$, a contradiction. Consequently, if b is an upper bound of S, then $\neg(s \not\leq b)$, that is, $s \leq b$. In other words, s is the weak supremum of S. □

5 Suprema in $\mathbf{R}^n$

We investigate a specific example: the Cartesian product $\mathbf{R}^n$ of n copies of $\mathbf{R}$. For each i, $1 \leq i \leq n$, let us consider the projection π_i of $\mathbf{R}^n$ onto $\mathbf{R}$, defined by

$$\pi_i(x_1, x_2, \ldots, x_n) = x_i.$$

The next result enables us to calculate the (weak) supremum of a subset S of $\mathbf{R}^n$ by computing the (weak) suprema of the projections $\pi_i(S)$, and vice versa.

Proposition 5.1 *Let S be a nonempty subset of $\mathbf{R}^n$ that is bounded above, and let $s = (s_1, s_2, \ldots, s_n)$ be an element of $\mathbf{R}^n$. Then, the following statements hold.*

(i) $s = \sup S \Leftrightarrow \forall i \in \{1, 2, \ldots, n\}\ (s_i = \sup \pi_i(S))$.

(ii) $s = \text{w–}\sup S \Leftrightarrow \forall i \in \{1, 2, \ldots, n\}\ (s_i = \text{w–}\sup \pi_i(S))$.

Proof. (i) Clearly, s is an upper bound for S if and only if for each i, s_i is an upper bound of $\pi_i(S)$. Assuming that $s = \sup S$, we prove that $s_1 = \sup \pi_1(S)$. For each $\alpha \in \mathbf{R}$ with $s_1 > \alpha$ we have to find an element $a_1 \in \pi_1(S)$ such that $a_1 > \alpha$. If $s_1 > \alpha$, then $s \nleq (\alpha, s_2, \ldots, s_n)$, so there exists $a = (a_1, a_2, \ldots, a_n) \in S$ with $a \nleq (\alpha, s_2, \ldots, s_n)$. It follows that either $a_1 > \alpha$ or else $a_j > s_j$ for some $j \geq 2$. Since s is an upper bound for S, the latter case is contradictory, so $a_1 > \alpha$ and $s_1 = \sup \pi_1(S)$. Similarly, $s_i = \sup \pi_i(S)$ for each $i \geq 2$.

To prove the converse implication, let us assume that for all i, $s_i = \sup \pi_i(S)$. Consider $x = (x_1, x_2, \ldots, x_n) \in S$ with $s \nleq x$—that is, $s_j > x_j$ for some j. Since $s_j = \sup \pi_j(S)$, there exists $a_j \in \pi_j(S)$ such that $a_j > x_j$. If a is an element of S with $\pi_j(a) = a_j$, then $a \nleq x$. Consequently, $s = \sup S$.

(ii) This can be proved in a similar way. □

Clearly, the corresponding properties for the infimum are also valid. As a consequence, we can define lattice operations on $\mathbf{R}^n$ in a natural way.

$$(x_1, x_2, \ldots, x_n) \vee (y_1, y_2, \ldots, y_n) = (x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n),$$
$$(x_1, x_2, \ldots, x_n) \wedge (y_1, y_2, \ldots, y_n) = (x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n).$$

As we will prove in the next proposition, upper locatedness is another equivalent condition for the existence of the supremum of a bounded above subset of $\mathbf{R}^n$.

Proposition 5.2 *If S is a nonempty subset of $\mathbf{R}^n$, then the following conditions are equivalent.*

(1) *The supremum of S exists.*

(2) *There exists an element $s \in \mathbf{R}^n$ such that s is an upper bound of S and for each $x \in \mathbf{R}^n$ with $x < s$, at least an element a of S exceeds x.*

(3) *The set S is bounded above and upper located.*

(4) *The set S is bounded above, and for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ with $x_i < y_i$ for each $i \in \{1, \ldots, n\}$, either y is an upper bound of S or there exists a in S such that $a \nleq x$.*

(5) *Each set $\pi_i(S)$ has a supremum.*

Proof. In view of Proposition 4.1(i), the conditions (1) and (2) are equivalent. If $\sup S$ exists, then S is bounded above and, according to Proposition 3.1, is upper located; hence (2) entails (3). Since (1) and(5) are equivalent (Proposition 5.1(i)), we need only prove the implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5).

To avoid cumbersome notation, we will assume that $n = 2$. First we prove that (3) entails (4). Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of $\mathbf{R}^2$ such that $x_1 < y_1$ and $x_2 < y_2$. Pick an element $a = (a_1, a_2)$ of S, and consider the elements $z = (y_1, a_2)$ and $w = (a_1, y_2)$. Both z and w exceed x; hence either there exists an element of S that exceeds x or else we can construct upper bounds (b_1, b_2) and (b'_1, b'_2) of S with $z \nleq (b_1, b_2)$ and $w \nleq (b'_1, b'_2)$. In the latter case, $b_1 < y_1$ and $b'_2 < y_2$, so y is an upper bound of S.

To prove that (4) entails (5), consider an upper bound (b_1, b_2) of S. If α and β are two real numbers with $\alpha < \beta$, set $x = (\alpha, b_2)$ and $y = (\beta, b_2+1)$. Then either y is an upper bound of S or there exists $a = (a_1, a_2)$ in S with $a \nleq x$. In the former case, β is an upper bound of $\pi_1(S)$; in the latter, $\alpha < a_1$. Consequently, $\pi_1(S)$ satisfies the equivalent condition for the existence of supremum in $\mathbf{R}$. The set $\pi_2(S)$ is proved to be upper located in a similar way. $\square$

Each excess relation $\nleq$ on a set X induces a **pseudoexcess relation** $\nleq_p$ on X, defined by

$$x \nleq_p y \Leftrightarrow \forall z \in X \; (\neg\neg(x \nleq z) \vee \neg\neg(z \nleq y)).$$

It is straightforward to observe that

$$x \nleq y \Rightarrow x \nleq_p y \Rightarrow \neg(x \leq y).$$

If x is a real number, then

$$x \nleq_p 0 \Leftrightarrow \forall z \in \mathbf{R} \; (\neg\neg(x > z) \vee \neg\neg(z > 0);$$

that is, x is **pseudopositive**. If the relations $\nleq$ and $\nleq_p$ coincide, then the **weak Markov principle**

every pseudopositive real number is positive

holds. More details about the weak Markov principle can be found in [Ishihara 2004, Mandelkern 1988].

Although the relations $\nleq$, $\nleq_p$, and the double negation of $\nleq$ are not constructively equivalent, we can prove that for a subset S of $\mathbf{R}^n$ and an upper bound s of S, $s = \sup S$ if and only if

$$s \nleq x \Rightarrow \exists a \in S \; (a \nleq_p x)$$

or, equivalently,

$$s \nleq x \Rightarrow \exists a \in S \; (\neg(a \leq x)).$$

We need only prove that $s = \sup S$ provided that the latter condition holds. To this end, assume that $s = (s_1, s_2, \ldots, s_n) \nleq x = (x_1, x_2, \ldots, x_n)$; without loss of generality we may assume that $s_1 > x_1$. Let $y_1 = (x_1 + s_1)/2$ and set $y = (y_1, s_2, \ldots, s_n)$. Then $s \nleq y$, so there exists $a = (a_1, a_2, \ldots, a_n)$ in S such that $\neg(a \leq y)$. It follows that $\neg(a_1 \leq y_1)$, which implies $a_1 > x_1$ and this, in turn, entails $a \nleq x$. Therefore $s = \sup S$.

Similar modifications in the conditions (2),(3), and (4) of Proposition 5.2 lead to other equivalent conditions for the existence of supremum in $\mathbf{R}^n$.

In the next proposition we show that for $n \geq 2$ the condition in the left–hand side of (4) (Proposition 5.2) cannot be replaced by the weaker condition $x < y$.

Proposition 5.3 *Let $n \geq 2$ be an integer, and S a nonempty subset of $\mathbf{R}^n$ that is bounded above. If for all x and y in $\mathbf{R}^n$ with $x < y$, either y is an upper bound of S or else there exists a in S such that $a \nleq x$, then LPO holds.*

Proof. If S satisfies the hypothesis, then $\sup S$ exists (in account of condition (4) of Proposition 5.2). Let $s = (s_1, \ldots, s_n)$ be the supremum of S and take an arbitrary real number α. If $x = (\alpha, s_2, \ldots, s_n)$ and $y = (\alpha, s_2 + 1, \ldots, s_n + 1)$, then $x < y$, and either y is an upper bound of S or else we can find an element $a = (a_1, \ldots, a_n)$ in S that exceeds x. In the former case, α is an upper bound of $\pi_1(S)$; whence $s_1 \leq \alpha$. In the latter case, either $\alpha < a_1$ or else $s_j < a_j$ for some $j \geq 2$. Since $s = \sup S$, the latter condition is contradictory. Consequently, for each real number α, either $\alpha \geq s_1$ or $\alpha < s_1$. This property entails LPO. □

We have corresponding results for the weak supremum. The proofs are similar and hence omitted.

Proposition 5.4 *For a nonempty subset S of $\mathbf{R}^n$, the following conditions are equivalent.*

(1) *The weak supremum of S exists.*

(2) *There exists $s \in \mathbf{R}^n$ such that s is an upper bound of S and*

$$s \nleq x \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(3) *There exists $s \in \mathbf{R}^n$ such that s is an upper bound of S and*

$$\neg(s \leq x) \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(4) *There exists $s \in \mathbf{R}^n$ such that s is an upper bound of S and*

$$\neg\neg(\forall a \in S\ (a \leq x)) \Rightarrow (s \leq x).$$

(5) *There exists $s \in \mathbf{R}^n$ such that s is an upper bound of S and*

$$x < s \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(6) *There exists $s \in \mathbf{R}^n$ such that s is an upper bound of S and*

$$\neg\neg(x < s) \Rightarrow \neg(\forall a \in S\ (a \leq x)).$$

(7) *There exists $s \in \mathbf{R}^n$ such that s is an upper bound of S and*

$$\neg\neg(\forall a \in S\ (a \leq x)) \Rightarrow \neg(x < s).$$

(8) *The set S is bounded above and for all x, y in $\mathbf{R}^n$ with $\nleq x$, either y exceeds an upper bound of S or x is not an upper bound of S.*

(9) *The set S is bounded above, and for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$ with $x_i < y_i$ for each $i \in \{1, \ldots, n\}$, either y is an upper bound of S or else it is contradictory that x be an upper bound of S.*

(10) *Each projection $\pi_i(S)$ has a weak supremum.*

Proposition 5.5 *Let $n \geq 2$ be an integer, and S a nonempty subset of $\mathbf{R}^n$ that is bounded above. If, for all x and y in $\mathbf{R}^n$ with $x < y$, either y is an upper bound of S or else it is contradictory that x be an upper bound of S, then WLPO holds.*

For an order bounded subset of $\mathbf{R}^n$, another property of upper locatedness, which is in general weaker than the one in the definition, is equivalent to the existence of supremum. A similar result holds for weak supremum.

Proposition 5.6 *Let S be a nonempty order bounded subset of $\mathbf{R}^n$.*

(i) *The supremum of S exists if and only if, for all x and y in $\mathbf{R}^n$ with $y \nleq x$, either $y \nleq a$ for all a in S or else there exists a in S such that $a \nleq x$.*

(ii) *The weak supremum of S exists if and only if, for all x and y in $\mathbf{R}^n$ with $y \nleq x$, either $y \nleq a$ for all a in S or it is contradictory that x be an upper bound of S.*

Proof. We prove only (i), the proof of (ii) being similar. If the supremum of S exists, then S is upper located and, as a consequence, the condition in the right–hand side holds.

Conversely, let $b = (b_1, \ldots, b_n)$ an upper bound of S, and let $m = (m_1, \ldots, m_n)$ be a lower bound. If α and β are real numbers with $\alpha < \beta$, then $(\beta, m_2, \ldots, m_n) \nleq (\alpha, b_2, \ldots, b_n)$. It follows that either $(\beta, m_2, \ldots, m_n) \nleq a$ for all a in S or else there exists an element $a = (a_1, \ldots a_n)$ in S such that $(a_1, \ldots, a_n) \nleq (\alpha, b_2, \ldots, b_n)$. In the former case, β is an upper bound of $\pi_1(S)$; in the latter, there exists a_1 in $\pi_1(S)$ with $\alpha < a_1$. Consequently, we see that $\sup \pi_1(S)$ exists. Similarly, we prove that $\sup \pi_i(S)$ exists for each $i \geq 2$. This proves the existence of $\sup S$. □

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