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Constructive Results on Operator Algebras[1]

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Abstract: We present a to following results in the constructive theory of operator algebras. A representation theorem for finite dimensional von Neumann-algebras. A representation theorem for normal functionals. The spectral measure is independent of the choice of the basis of the underlying Hilbert space. Finally, the double commutant theorem for finite von Neumann algebras and for Abelian von Neumann algebras.
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1 Operator algebras

The theory of operator algebras is an infinite-dimensional generalization of matrix algebra theory and can be viewed as a non-commutative version of topology or measure theory. As such it plays an important role in the representation theory of groups and in the mathematical underpinnings of quantum mechanics. There does not seem to be a definitive constructive theory of operator algebras, although quite a few results have been obtained by Bridges and Dediu [10] [9]. We will continue these developments, mainly considering von Neumann algebras. For the classical theory we used [15] [16] [17] [18] [22] [19] [12]. A quick introduction to the subject with an eye on physical applications can be found for instance in [18] or [17].

This paper is organized as follows. The first section includes a very short discussion of non-commutative measure theory. The next two sections contain preliminary result. Section 4 contains a representation theorem for von Neumann algebras on a finite dimensional Hilbert space. Section 5 contains a representation theorem for normal functionals. In section 6 we discuss faithful states and use them to prove that the measure in the spectral theorem does not depend on the choice of the basis. We also use faithful states to obtain some connections between the weak and the strong operator topology. In section 8 we prove a representation theorem for Abelian von Neumann algebras. In section 9 we show that the classical double commutant theorem can not be proved constructively,

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but we prove this theorem for Abelian von Neumann algebras with a weakly totally bounded unit ball.

All results are constructive in the sense of Bishop. We will also use [1] as the general reference text. In this paper all operators are (total and) bounded, unless stated otherwise. The letter H will denote a (separable) Hilbert space.

2 Non-commutative measure theory

The theory of C*-algebras is sometimes, for example in [18], called non-commutative topology for the following reason. Let X be a compact space. The set $C(X, \mathbf{C})$ of continuous complex-valued functions on X is an Abelian C*-algebra. On the other hand, every Abelian C*-algebra $\mathcal{A}$ is isomorphic with $C(\Sigma)$, where Σ is the spectrum of $\mathcal{A}$ which is compact. In [1] this was proved constructively for C*-algebras of operators on a Hilbert space. See [8] for a more general result which does not require the norm to be computable. See Takamura [27] for an introduction to the constructive theory of C*-algebras.

Many questions about the space X can be translated into questions about $C(X)$. So the theory of Abelian C*-algebras 'is' topology. Hence the theory of general C*-algebras may be called non-commutative topology.

Similarly, we can translate questions about a measure space (X, μ) into questions about the *-algebra $\{M_h : h \in L_\infty(\mu)\}$. Here M_h denotes the multiplication operator M_h defined by $M_h f := h \cdot f$, for all $f \in L_2$. In fact, this algebra is a von Neumann algebra, that is a *-algebra of operators on a Hilbert space satisfying certain closedness properties. We will give a precise definition and examples in section 4. Classically, every Abelian von Neumann algebra is isomorphic to such a von Neumann algebra of multiplication operators. In this sense, the theory of von Neumann algebras may be called non-commutative measure theory. In section 8 we give a constructive result partially justifying this viewpoint.

Let us give a very short dictionary for non-commutative measure theory. A non-commutative set or event is a projection. A non-commutative function or random variable is a normal operator. A non-commutative integral or expectation is a positive linear functional (or state).

In the algebraic approach to quantum mechanics, the observables form a C*-algebra or von Neumann algebra and the states are positive linear functionals on this algebra, see for instance [13].

3 Preliminaries

Let $B(H)$ be the quasi-normed space[3] of bounded operators on the Hilbert space H. The unit ball of H is denoted by H_1. Let $(e_n)_{n \in \mathbf{N}}$ be a basis for H and let A

[3] Any quasi-normed space may be considered as a normed-space where the norm is a generalized real number in the sense of Richman [21].

be an operator on H. The trace $\mathrm{Tr}(A)$ is defined as $\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle$, whenever this limit exists. If A and B are bounded operators such that $\mathrm{Tr}(BA)$ and $\mathrm{Tr}(B)$ exist, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ and $\mathrm{Tr}(B^*) = \overline{\mathrm{Tr}(B)}$.

We denote by $\mathcal{L}_1$ the space of, necessarily compact, operators A such that $\mathrm{Tr}(|A|)$ exists. Then $A \mapsto \mathrm{Tr}(|A|)$ defines a norm on $\mathcal{L}_1$ The space $\mathcal{L}_1$ is called the space of *trace class operators* . One can prove that for all $A \in \mathcal{L}_1$, $\mathrm{Tr}(A)$ does not depend on the choice of the basis. The space $\mathcal{L}_2$ of *Hilbert-Schmidt operators* consists of those compact operators A such that $\mathrm{Tr}(A^*A)$ exists. On $\mathcal{L}_2$ we define an inner product by $\langle A, B\rangle := \mathrm{Tr}(B^*A)$. The space $\mathcal{L}_1$ is a Banach space and the space $\mathcal{L}_2$ is a Hilbert space. Remark that $\mathcal{L}_1 \subset \mathcal{L}_2$. See [3] for more on the constructive theory of these classes of operators.

A subset B of $B(H)$ is said to be *bounded by* M if for all $A \in B$ and $x \in H$, $\|Ax\| \leq M\|x\|$. The *unit ball* $B(H)_1$ of $B(H)$ is the set of operators bounded by 1.

We state the definitions of and some results on several topologies on $B(H)$.

Definition 1. We define four topologies on $B(H)$ as follows.

1. The *uniform (or norm) topology* is the topology generated by the 'operator norm', that is the topology of the quasi-normed space $B(H)$.
2. The *strong operator topology* is the uniform topology generated by the collection of semi-norms $p_x(A) := \|Ax\|$, for all $x \in H$.
3. The *weak operator topology* is the uniform topology generated by the collection of semi-norms $p_{x,y}(A) := |\langle Ax, y\rangle|$, for all $x, y \in H$.
4. The *normal* (or ultra-weak) *topology* is the uniform topology generated by the semi-norms $p_B(A) := |\mathrm{Tr}(BA)|$, for all trace-class operators B.

The space $B(H)$ equipped with one of these last three topologies is a uniform space, see [11] for a constructive theory of uniform spaces. Note that the quasi-normed space $(B(H), \{p_x : x \in H_1\})$ has a different topology than the uniform space $(B(H), \{p_x : x \in H_1\})$. The former has the norm topology, the latter the strong topology. On finite dimensional Hilbert spaces all these topologies coincide. We will now consider the general case. The following facts are well-known classically and the usual proofs are constructive.

The uniform topology is the strongest of the four, the weak topology is the weakest. The strong and normal topology are incomparable, but both are between the other two. On bounded sets the normal and the weak topologies coincide. The map $T \to T^*$ is not strongly continuous, but it is continuous in the normal topology and weakly continuous. Constructively this map is only partially defined. The maps $A, B \mapsto BA$ and $A, B \mapsto AB$ are strongly continuous from $B(H) \times B(H)_1 \to B(H)$. The unit ball $B(H)_1$ is totally bounded in the normal topology, and hence in the weak topology. On bounded sets the strong, the weak and the normal topology are metrizable and separable, so a sequentially closed set is closed.

4 Von Neumann algebras

In this section we introduce the notion of a von Neumann algebra and provide some examples.

It is not clear yet what the 'right' constructive definition of a von Neumann algebra is. A good definition would satisfy at least two demands. First, it should be general enough to contain the important examples. Second, it should give a flexible theory. Thus we will use the following definition which seems to be the weakest definition that is still powerful enough to prove interesting theorems.

Definition 2. A *von Neumann algebra* (or *vN-algebra*) $\mathcal{A}$ is an algebra of operators on a Hilbert space H such that $I \in \mathcal{A}$, $\mathcal{A}$ is closed in the strong-operator topology and $\mathcal{A}$ is self-adjoint — that is, if $T \in \mathcal{A}$ and T^* exists, then $T^* \in A$.

A von Neumann algebra is supports all constructions in the spectral theorem. That is, if a Hermitian operator is in a vN-algebra, then all its spectral projections are also in the von Neumann algebra.

When $\mathcal{A}$ is a von Neumann algebra, we denote by $\mathcal{A}_1$ its unit ball, that is the set $\mathcal{A} \cap B(H)_1$. Bridges proposed to look at von Neumann algebras with a weakly totally bounded unit ball. Classically, all von Neumann algebras satisfy this condition. However, constructively this is not the case. Let P be any proposition. Then, constructively, the von Neumann algebra

$$\{cI : c \in \mathbf{C}\} \cup \{A \in B(H) : P\}$$

can not be proved to have a located unit ball. Although it might be tempting to *define* a vN-algebra to have a weakly totally bounded unit ball, we will not do this.

We give some examples of von Neumann algebras.

- The matrix algebra of operators on $\mathbf{C}^n$, which we denote by M_n, is a von Neumann algebra. Every closed *sub-algebra of M_n is a von Neumann algebra.
- The space $B(H)$ is a von Neumann algebra.
- Let (X, μ) be a measure space. For $h \in L_\infty(\mu)$ we define the multiplication operator M_h on $L_2(\mu)$ by $M_h f = h \cdot f$. The space $\{M_h : h \in L_\infty\}$ is a von Neumann algebra. To see that it is closed let $(h_n)_{n\in\mathbf{N}}$ be a sequence in L_∞ such that the sequence $(M_{h_n})_{n\in\mathbf{N}}$ converges strongly to $A \in B(H)$. Let

m be a bound for A. Let $g_n := h_n \chi_{[|h_n| \leq m]}$. Then the sequence $(M_{g_n})_{n \in \mathbf{N}}$ also converges strongly to A. Consequently, we may assume that the sequence $(h_n)_{n \in \mathbf{N}}$ is bounded. Finally we remark that strong convergence of the bounded sequence $(M_{h_n})_{n \in \mathbf{N}}$ implies convergence in measure of the sequence $(h_n)_{n \in \mathbf{N}}$.

- Let G be a locally compact group. Define for each $f \in L_1(G)$ the convolution operator $T(f)$ on $L_2(G)$ by $T(f)g := f * g$, for all $g \in L_2(G)$. Define for each $f \in L_1(G)$ the involution $\tilde{f}$ of f by $\tilde{f}(x) := \overline{f(x^{-1})}$, for almost all $x \in G$. Then $T(f)^* = T(\tilde{f})$ for all $f \in L_1$. The strong closure of $\{T(f) : f \in L_1\}$ is a von Neumann algebra.

- Finally, the observables in a quantum system can be modeled by the Hermitian elements of a von Neumann algebra. By translating the Heisenberg picture and the Schringer picture of quantum mechanics into the language of operator algebras von Neumann was able to show that these views are equivalent, see [13] [23].

Classically, one can prove that every vN-algebra is weakly closed. We do not know how to prove this constructively, but Corollary 12 contains a result in this direction.

5 Finite dimensional von Neumann algebras

We prove a representation theorem for von Neumann algebras with a totally bounded unit ball and which act on a finite dimensional Hilbert space. To do this we use the Peter-Weyl representation theorem for compact groups.

Let H be a finite dimensional Hilbert space. Let $\mathcal{A}$ be a vN-algebra with a weakly totally bounded unit ball. On a finite dimensional Hilbert space all the topologies we considered on H and on $B(H)$ coincide. Consequently, $\mathcal{A}_1$ is totally bounded and even compact. We will show that the set of unitary operators in $B(H)$ is closed and conclude that they form a compact group. Indeed, if U_n is a sequence of unitary operators converging to an operator U, then U is an isometry[4]. Since H is finite-dimensional, U is unitary. We conclude that the set of unitary operators is closed.

We claim that the set of unitary operators in $\mathcal{A}_1$ is located. Since projecting on the real part is a uniformly continuous operation, $\mathcal{A}_1^{sa} := \{A \in A_1 : A = A^*\}$ is totally bounded. The map $H \mapsto e^{i2\pi H}$ from $\mathcal{A}^{sa}$ to the set of unitary operators in $\mathcal{A}$ is uniformly continuous and surjective. To see that it is surjective, let U be a unitary operator. Let μ_U be the spectral measure for U, then μ_U is concentrated

[4] To see this we remark that U is unitary if and only if U and U^* are isometries. If U_n converges to U, then U_n^* converges to U^*. Finally, we observe that the isometries are closed.

on the unit circle in $\mathbf{C}$. For all but countably many points z in the unit circle, $\{z\}$ is μ_U-measurable and $\mu_U(\{z\}) = 0$, see [30]. We may assume that -1 is one such point. Let log be a holomorphic function on $\mathbf{C} - (-\infty, 0]$ such that $\exp \circ \log = \mathrm{id}$. Then $e^{i2\pi(\frac{1}{i2\pi}\log U)} = U$.

We will now use the notation and the results from [7]. We see that the set of unitary operators in $\mathcal{A}$ forms a compact group G which forms its own representation π. By recalling the construction of the extension of the representation of the group to a representation of group algebra, we see that for all $f \in C(G)$, $\pi(f) \in \mathcal{A}$. Moreover, each $A \in \mathcal{A}$ can be approximated by a finite linear combination of projections in $\mathcal{A}$. However, instead of approximating a given A in $\mathcal{A}$ by projections we may approximate it by unitary operators: for any projection P, the operator $2P - I$ is unitary. Consequently, $\pi(C(G)) = \mathcal{A}$.

The Peter-Weyl theorem supplies a sequence $(H_i)_{i\in\mathbf{N}}$ of finite-dimensional subspaces of H such that $H = \oplus_{i=1}^{\infty} H_i$ and $\pi|_{H_i} = B(H_i)$. Since H is finite-dimensional, the sequence is actually finite. Finally, a linear subset of a finite dimensional Hilbert space is finite dimensional if and only if its unit ball is located, see [1] (Section 7.2). We have proved the following theorem.

Theorem 3. *Let $n \in \mathbf{N}$ and let $H := \mathbf{C}^n$. Let $\mathcal{A}$ be a self-adjoint located subalgebra of $B(H)$ which contains the identity. Then there is a finite sequence $H_1, \dots, H_M$ of subspaces of H such that $H = \oplus_{i=1}^{M} H_i$ and $\mathcal{A}P_i \cong B(H_i)$, where P_i is the projection on H_i.*

In Takesaki [28] (p.50) this theorem is proved along the same lines, but more directly. A problem with a constructive interpretation of his proof is that in order to apply the Gelfand representation theorem to the center of the algebra we need to prove that it is separable.

Before we go deeper into the theory of general vN-algebras, we will first study linear functionals on the von Neumann algebra $B(H)$.

6 Normal functionals

In this section we give a representation of the normal functionals on $B(H)$. We simplify a result by Bridges and Dudley Ward.

Following Bridges and Dudley Ward we define a normal functional as a functional which is uniformly continuous with respect to the normal topology on bounded sets.

In the proof of the implication $1 \Rightarrow 2$ in the following Theorem we use the Riemann permutation theorem, see [29] (p.96):

Theorem 4. [CP] *Let $(x_i)_{i\in\mathbf{N}}$ be a sequence of real numbers such that for every permutation π of $\mathbf{N}$, $\sum_{i=0}^{\infty} x_{\pi(i)}$ exists, then $\sum_{i=0}^{\infty} |x_i|$ exists.*

This theorem depends on Brouwer's continuity principle (CP) which holds in the intuitionistic interpretation of Bishop's mathematics, but not in classical logic, and thus CP not available in the context of Bishop-style mathematics. Since the Riemann permutation theorem holds in the three standard models of Bishop's mathematics it would be interesting to investigate its precise logical status.

Theorem 5. *Let ϕ be a linear functional on $B(H)$. Then the following conditions are equivalent:*

1. *ϕ is normal ;*
2. *there is a $C \in \mathcal{L}_1$ such that for all $A \in B(H)$, $\phi(A) = \mathrm{Tr}(CA)$;*
3. *there are sequences x_n, y_n in H such that $\sum_{n=1}^{\infty} \|x_n\|^2$ and $\sum_{n=1}^{\infty} \|y_n\|^2$ exist and $\phi(A) = \sum_{n=1}^{\infty} \langle Ax_n, y_n \rangle$, for all $A \in B(H)$.*

Proof. $1 \Rightarrow 2$ (using CP): We assume that ϕ is normal. The unit ball of $\mathcal{L}_2$ is totally bounded in the weak topology and thus in the normal topology. Since ϕ is uniformly continuous on the unit ball, ϕ is normable as an $\mathcal{L}_2$-functional. By the Riesz representation theorem there is an operator C in $\mathcal{L}_2$ such that $\phi(A) = \mathrm{Tr}(C^*A)$, whenever $A \in \mathcal{L}_2$. We claim that $C \in \mathcal{L}_1$. Indeed, let π be a permutation of $\mathbf{N}$ and let P_n be the projection on $\mathrm{span}\{e_{\pi(i)} : i \leq n\}$. Then the sequence $(P_n)_{n\in\mathbf{N}}$ converges weakly to the identity operator I,so $\mathrm{Tr}(C^*P_n) = \phi(P_n) \to \phi(I)$. We see that $\sum_{i=0}^{\infty} \left\langle Ce_{\pi(n)}, e_{\pi(n)} \right\rangle$ exists for all permutations π of $\mathbf{N}$. Consequently, by the Riemann permutation theorem, $\sum_{i=0}^{\infty} |\left\langle Ce_n, e_n \right\rangle|$ exists — that is, C is in $\mathcal{L}_1$. It follows that the functional $A \mapsto \mathrm{Tr}(C^*A)$ may be extended to a normal functional ψ on $B(H)$. Since, ψ agrees with ϕ on the unit ball of $\mathcal{L}_2$, which is dense in $B(H)_1$ with the normal topology, $\phi(A) = \mathrm{Tr}(C^*A)$, whenever $A \in B(H)$.

A proof which does not use CP may be found in [5].

$2 \Rightarrow 3$: Let C in $\mathcal{L}_1$ be such that $\phi(A) = \mathrm{Tr}(CA)$, whenever $A \in B(H)$. Let $\Re C$ and $\Im C$ denote the real and the imaginary part of the operator C. The operator $\Re C$ is selfadjoint, so we can apply the spectral theorem to it. Let f, g be the total continuous functions on $\mathbf{R}$ such that $f(x) := \sqrt{|x|}$ and $g(x) := x/f(x)$, whenever $x \in \mathbf{R} - \{0\}$. Then $\Re C = f(\Re C)g(\Re C)$. So for all $A \in B(H)$,

$$\mathrm{Tr}((\Re C)A) = \mathrm{Tr}(f(\Re C)Ag(\Re C)) = \sum_{n=1}^{\infty} \langle Ag(\Re C)e_n, f(\Re C)e_n \rangle .$$

We define sequences $(x_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}}$ in H such that $x_{2n} := g(\Re C)e_n$, $x_{2n+1} := ig(\Im C)e_n$, $y_n := f(\Re C)e_n$ and $y_{2n+1} := f(\Im C)e_n$. These sequences satisfy the conditions in 3.

$3 \Rightarrow 1$: This implication is clear.

This proof seems to be simpler then the proof by Bridges and Dudley Ward [5].

The previous theorem allows us to define a number of functionals that will be useful later. Let ϕ be a normal functional on $B(H)$. Constructively, the map $A \mapsto \phi(A^*)$ is not totally defined, but there is a unique continuous extension of this map to $B(H)$ because the set of operators with an adjoint is weakly dense in $B(H)$. In this chapter we will tacitly assume that this extension has been made and we will write $\phi(A^*)$ even if we are not able to compute A^*. A similar remark holds for functions we now define. Let A be an operator. Define the functional ϕ^A on $B(H)$ by $\phi^A(B) := \phi(BA)$. There is a trace class operator C such that $\phi^A(B) := \mathrm{Tr}(CBA) = \mathrm{Tr}(ACB)$. Consequently, the functional ϕ^A is normal as $AC \in \mathcal{L}_1$. The set of operators with an adjoint is weakly dense in $B(H)$, so the map $A, B \mapsto \phi(B^*A) = \phi^A(B^*)$ can be uniquely extended to a continuous map on $B(H)^2$. Finally, an operator with an adjoint has an absolute value, so the map $A \mapsto \phi(|A|)$ can be uniquely extended to a normal functional to $B(H)$.

7 Faithful states

In this section we prove that the spectral measure of a given operator is independent of the choice of the basis, simplifying a result by Bridges and Ishihara. We introduce an inner product on $B(H)$ and use it to obtain some results connecting the weak and the strong topology.

Definition 6. A linear functional ϕ on a vN-algebra $\mathcal{A}$ is called *positive* if for each A in $\mathcal{A}$ which has an adjoint, $\phi(A^*A) \geq 0$. A positive linear functional on a vN-algebra is called a *state* .

Let $\mathcal{A}$ be a von Neumann algebra. We define for a moment a state ϕ to be *accurate* if for all $A \in \mathcal{A}$, $\phi(A^*A) = 0$ implies $A = 0$. For such ϕ, define the norm $d_\phi(A) := \phi(|A|)$ on $\mathcal{A}$. Let us call the normal state ϕ *faithful* if it is accurate and the identity map from $(\mathcal{A}_1, d_\phi)$ to the uniform space $\mathcal{A}_1$ with the weak operator topology is uniformly continuous. The inverse of this map is continuous for any normal state ϕ. One can prove classically that an accurate normal state is faithful. Usually one defines a state to be faithful if and only if it is accurate. We chose to use the present definition of faithfulness because it seems to correspond naturally with the constructive definition of absolute continuity in [1]. To see this consider $[0,1]$ with Lebesgue measure μ and define the von Neumann algebra $\mathcal{A} := \{M_h : h \in L_\infty[0,1]\}$. Suppose that ν is a measure on $[0,1]$ such that all $f \in L_\infty(\mu)$ are ν-integrable and if $f \in L_\infty(\mu)$ and $\nu(f) = 0$, then $\mu(f) = 0$. Then ν is an accurate state on the von Neumann algebra $L_\infty(\mu)$. If μ is absolutely continuous relative to ν, then ν is a faithful state.

Any spectral measure is derived from a faithful normal state. Indeed, define the trace class operator C such that for all $n \in \mathbf{N}$, $Ce_n := 2^{-n}e_n$ and define the state $\phi(A) := \mathrm{Tr}(CA)$, for all $A \in B(H)$. It is easy to check that ϕ is accurate. We show that ϕ is a faithful normal state. Let $\varepsilon > 0$ and $x, y \in H_1 \cap \mathrm{span}\{e_1, \dots, e_m\}$ be given. Let A be an operator with an adjoint in $\mathcal{A}_1$ such that $\phi(|A|) < 2^{-m}\varepsilon$. Let B be the positive square root of $|A|$. Then $\phi(|A|) = \mathrm{Tr}(CB^2) < 2^{-m}\varepsilon$ and for all $n \leq m$, $\|Be_n\| < \varepsilon$. So $|\langle Be_n, Bx\rangle| \leq \|Be_n\| < \varepsilon$,for all $x \in H_1$ and $n \leq m$. By Theorem 1.1 in [4] for each $x, y \in H_1$, there is a $z \in H_1$ such that $|\langle Ay, x\rangle - \langle |A|y, z\rangle| < \varepsilon$. Consequently, for all $n \leq m$, $|\langle Ae_n, x\rangle| < 2\varepsilon$, because $\langle |A|e_n, x\rangle = \langle Be_n, Bx\rangle$. Finally, remark that the operators with an adjoint are dense.

Theorem 7. *Any two spectral measures are equivalent.*

Proof. Let A be a Hermitian operator bounded by 1. Let ϕ be as defined above. Remark that the measure μ_ϕ defined by $\mu_\phi(f) := \phi(f(A))$, for all $f \in C[-1, 1]$, is a spectral measure for A. Suppose that ψ is defined in a similar way as ϕ, but with respect to a different basis. Then the identity map from $(B(H)_1, d_\psi)$ to $(B(H)_1, d_\phi)$ is uniformly continuous, because ψ is a faithful normal state. It follows that the measures μ_ϕ and μ_ψ are equivalent.

This is the main result obtained by Bridges and Ishihara in [2].

The key fact in the proof of the spectral theorem is that if A is a Hermitian operator bounded by 1, μ is the spectral measure for A and $(p_n)_{n\in\mathbf{N}}$ is a sequence of polynomials and $\mu(|p_n|^2) \to 0$, then the sequence $(p_n(A))_{n\in\mathbf{N}}$ converges strongly. In fact, such a statement holds more generally. When ψ is a faithful normal state and $(A_n)_{n\in\mathbf{N}}$ is a bounded sequence of operators such that $\psi(A_n^* A_n) \to 0$, then $A_n^* A_n \to^w 0$. Consequently, $\|A_n x\|^2 = \langle A_n^* A_n x, x\rangle \to 0$, whenever $x \in H$. That is, $A_n \to^s 0$.

7.1 Reformulating the spectral theorem

Theorem 7 suggests that instead of using a non-canonical measure in the spectral theorem one could use Chan's measurable spaces [6]. Intuitively, these measurable spaces are to measure spaces what uniform spaces are to metric spaces. Thus instead of using one measure we have a family of them. In the context of the spectral theorem we will use the family $\mu_x(A) := \langle Ax, x\rangle$, where x ranges over the unit sphere. We will now state the spectral theorem without reference to a basis.

Let (X, L, μ_i) be a measurable space. Like Chan we assume that all the μ_i are probability measures. Each μ_i defines a pseudo-metric $d_i(f, g) := \mu_i(|f - g| \wedge 1)$. On bounded sets this collections pseudometrics define the same topology as the uniform topology considered by the norms. Convergence with respect to these pseudo-metrics we call convergence in measure. Let L_∞ be the completion of the bounded sets with respect to this topology. It is proved in [26] that this is actually equivalent to the ordinary definition of convergence in measure. Chan defines measurable mappings using the notion of transition function. It seems that one can instead use the notion of a uniformly continuous mapping between uniform spaces with the pseudo-metrics defined above. This works well in our case, but whether it can replace all Chan's constructions in probabilty theory requires further exploration.

Thus by analogy with the treatment in [25] we obtain the following theorem. The formal space mentioned in the following theorem is a space in the sense of formal, or pointfree, topology, see [25] and the references therein for details. Alternatively, assuming Bishop's Gelfand representation theorem and assuming that the algebra $\mathcal{A}$ satisfies the conditions needed to apply it, we could use an ordinary space.

Theorem 8. *Let $\mathcal{A}$ be an Abelian algebra of operators on a Hilbert space. There is an embedding ι of the algebra $\mathcal{A}$ into a space of continuous functions on a formal compact space, its spectrum. The algebra $C(\Sigma)$ may be equipped with a measurable structure in such a way that the topology of convergence in measure coincides with the weak operator topology on $\mathcal{A}$. Consequently, if A_n is a bounded sequence and $\iota(A_n) \to \varphi$ in measure, then A_n converges strongly.*

As already observed by Chan, we can not expect the profile theorem to apply to measurable spaces in its full generality. However, there are a number of ways to avoid this [6] [25]. Finally, one can always choose a faithful state, but this is better postponed until it is really needed.

7.2 An inner product on $B(H)$

Let ϕ be a faithful normal state on $B(H)$. Define the inner product $\langle\ ,\rangle_\phi$ on $B(H)$ by $\langle A, B\rangle_\phi = \phi(B^*A)$, whenever $A, B \in B(H)$. In general the space $B(H)$ will not be complete with this inner product. We let $\mathcal{H}_\phi$ denote its completion. When ψ is another faithful normal state, then the inner products $\langle\ ,\rangle_\phi$ and $\langle\ ,\rangle_\psi$ are equivalent on bounded sets.

Lemma 9. *Let ϕ be a faithful normal state on $B(H)$. Let $(A_n)_{n\in\mathbf{N}}$ be a bounded sequence in $B(H)$ and let $A \in B(H)$. Then $A_n \to A$ in the weak (strong) operator topology if and only if $A_n \to A$ in the weak (strong) sense in $\mathcal{H}_\phi$.*

Proof. Let $(e_n)_{n\in\mathbf{N}}$ be a basis for H and define C as the operator such that $Ce_n = 2^{-n}e_n$, whenever $n \in \mathbf{N}$. The state ψ defined by $\psi(A) := \mathrm{Tr}(CA)$, whenever $A \in B(H)$, is faithful and normal. Because $\langle\ ,\rangle_\phi$ and $\langle\ ,\rangle_\psi$ are equivalent on bounded sets, we may assume that $\phi = \psi$.

Define the trace-class operator U_{nm} by $U_{nm}x = \langle e_m, x\rangle\, e_n$, whenever $n, m \in \mathbf{N}$ and $x \in H$. On bounded sets the weak operator topology is determined by the seminorms $A \mapsto |\langle Ae_m, e_n\rangle|$, where $n, m \in \mathbf{N}$. Since $\langle Ae_m, e_n\rangle = 2^n \mathrm{Tr}(CU^*_{nm}A)$, we see that convergence in the weak topology implies weak convergence in $\mathcal{H}_\phi$. Conversely, the functional $A \mapsto \mathrm{Tr}(CB^*A)$ is normal, for all $B \in B(H)$. Consequently, if the sequence $(A_n)_{n\in\mathbf{N}}$ converges weakly to A, then $\mathrm{Tr}(CB^*A_n) \to \mathrm{Tr}(CB^*A)$.

On bounded sets, the strong operator topology is determined by the norm $A \mapsto \phi(A^*A) = \sum_{n=1}^{\infty} 2^{-n}\|Ae_n\|^2$, so strong convergence in $\mathcal{H}_\phi$ is equivalent to convergence in the strong operator topology.

Corollary 10. *A bounded convex inhabited subset of $B(H)$ is weakly totally bounded if and only if it is located in the strong topology.*

Proof. This follows from Lemma 9 and Corollary 5 in [14].

In order to proceed we need a lemma. This lemma is sometimes proved classically by deriving a contradiction from the assumption that a point in the weak closure is in the complement of the strong closure. Such an argument only seems to work constructively when the set is located. Classically, the hypothesis that the set is bounded is superfluous. It is not clear to me if this hypothesis is necessary in constructive mathematics.

Lemma 11. *The weak closure and the strong closure of a bounded inhabited convex subset of a Hilbert space are equal.*

Proof. Let B be a bounded, inhabited and convex set. Suppose that $(x_n)_{n\in\mathbf{N}}$ is a sequence in B converging weakly to $x \in H$. By considering the bounded inhabited convex set $\{y - x : y \in B\}$, we may suppose that $x = 0$. Moreover, we may assume that B is bounded by 1. For each $i \in \mathbf{N}$, $\langle x_i, x_n\rangle \to 0$, so we can choose a subsequence $(y_n)_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that for all $i \in \mathbf{N}$ and all $j > i$, $|\langle y_i, y_j\rangle| \leq 2^{-i-j}$. Then

$$\begin{aligned}\|y_1 + y_2 + \ldots + y_n\|^2 &\leq \sum_{i=1}^{n} \|y_i\|^2 + 2\sum_{j=1}^{n}\sum_{i=j+1}^{n} |\langle y_i, y_j\rangle| \\ &\leq n + 2\end{aligned}$$

We see that $\frac{1}{n}(y_1 + \cdots + y_n) \to^s 0$.

Corollary 12. *Let C be convex inhabited bounded subset of $B(H)$, then C is weakly closed if and only if it is strongly closed.*

Proof. This follows from Lemma 9 and Lemma 11.

From this lemma we would be able to prove that a von Neumann algebra is weakly closed *if* we would be able to prove that every weakly converging sequence is bounded.

8 Representation of Abelian von Neumann algebras.

In this section we prove the following theorem.

Theorem 13. *Let $\mathcal{A}$ be an Abelian von Neumann algebra containing a strong-operator-dense C*-algebra of operators $\mathcal{B}$. Then there is a compact space Σ and a measure μ on Σ such that $\mathcal{A}$ is isomorphic with $\{M_h : h \in L_\infty(\mu)\}$.*

Proof. By the Gelfand representation theorem there is a compact set Σ such that $\mathcal{B}$ is C*-isomorphic to $C(\Sigma)$. Let $\iota : C(\Sigma) \to \mathcal{B}$ be this isomorphism and let ϕ be a faithful normal state on $B(H)$. Define a linear functional μ on $C(\Sigma)$ by $\mu(f) := \phi(\iota(f))$, whenever $f \in C(\Sigma)$. It is clear that μ is bounded by 1. We claim that μ is positive. To prove this we observe that $f \in C(\Sigma)$ is positive if and only if $f = |f| = (\bar{f}f)^{1/2}$. Now suppose that $f \geq 0$. Then $\iota(f)$ is a positive operator, since ι is an isomorphism and $\iota(|f|) = |\iota(f)|$. The functional μ is positive, since ϕ is a state. Finally, since $1 \in C(\Sigma)$, the measure μ is finite.

Let $(f_n)_{n\in\mathbf{N}}$ be a bounded sequence in L_∞ which converges in measure to $f \in L_\infty$. We claim that $\iota(f_n) \to^s \iota(f)$. Since μ is finite, f_n converges to f in L_2,so $\phi\iota(|f - f_n|^2)$ converges to 0. Consequently $\iota(f - f_n)$ converges strongly to 0 by Remark **??**. We have proved that $\iota(f_n) \to^s \iota(f)$.

We claim that $\iota(L_\infty)$ is strong operator closed. Suppose that $\iota(f_n)$ converges strongly to an operator A. Let b be a bound for A. Define for all $n \in \mathbf{N}$, $g_n := f_n\chi_{[|f_n|\leq b]}$. The sequence $(\iota(g_n))_{n\in\mathbf{N}}$ also converges to A, since $\mathcal{A}$ is Abelian. Each operator $\iota(g_n)$ is normal, so $\iota(g_n)^*$ converges strongly to A and thus $\iota(|g_n - g_m|^2)$ converges to 0. Because ϕ is faithful, $\phi\iota(|g_n - g_m|^2)$ converges to 0. We see that the sequence $(g_n)_{n\in\mathbf{N}}$ converges in L_2. We let g be its limit. Since the sequence $(\iota(g_n))_{n\in\mathbf{N}}$ converges strongly to both $\iota(g)$ and A, they must be equal.

If an Abelian von Neumann algebra contains a dense set of operators with an adjoint, then all elements have adjoints. This follows directly from the identity $\|Ax\| = \|A^*x\|$ for normal operators A.

Classically, every von Neumann algebra on a separable Hilbert space contains a separable C*-algebra which is strong-operator-dense [16] (Lemma 14.1.17). Constructively, the examples we mentioned in the first section contain such a C*-algebra.

- For matrix algebras the norm topology coincides with the weak-operator topology, hence the von Neumann algebra of matrix operators is itself a C*-algebra.

- When μ is a measure on a locally compact space X, then $\{M_h : h \in C(X)\}$ is strongly dense in $\{M_h : h \in L_\infty(\mu)\}$. The operators $\{g \mapsto fg : f \in L_1\}$ form a C*-algebra.

- The algebra $B(H)$ is the strong closure of the space of compact operators.
- Suppose that $\mathcal{A}$ is an Abelian von Neumann algebra with a weakly totally bounded unit ball. We claim that $\mathcal{A}$ contains a strongly dense separable C*-algebra.

  The unit ball is located in the strong operator topology, by Corollary 10. Consequently, there is a dense sequence of operators in $\mathcal{A}$. All operators in $\mathcal{A}$ are normal, so by the spectral theorem we may construct a sequence of projections with a dense span. By adding projections we may also make sure that the sequence also contains all 'finite intersections'. Because the strong operator topology is metrizable on $B(H)_1$ we may make sure that if P and Q are in the sequence, then either $P = Q$ or $P \neq Q$. Now let $\mathcal{C}$ be the norm closure of all simple functions, i.e. all the operators of the form $\sum_{i=1}^{n} c_i P_i$, where $c_1, \dots, c_n$ are in $\mathbf{C}$ and $P_1, \dots, P_n$ are projections in the dense sequence. The C*-algebra $\mathcal{C}$ is dense in the strong operator topology.

The presence of the strongly dense separable C*-algebra was needed in order to apply Bishop's Gelfand representation theorem. By using the pointfree Gelfand representation theorem we can avoid this hypothesis, but still obtain a similar result. However, instead of a compact space we would obtain a compact formal space, see [8].

9 The Double Commutant Theorem

In this section we discuss the double commutant theorem. Subsection 1 contains a short discussion of this classical theorem. Subsections 2 and 3 contain proofs for the finite-dimensional and the Abelian case under the assumption that the unit ball of the von Neumann algebra is weakly totally bounded.

Definition 14. Let $\mathcal{A}$ be an algebra of operators on a Hilbert space. The *commutant* $\mathcal{A}'$ of $\mathcal{A}$ is the set $\{B \in B(H) : \forall A \in \mathcal{A}[AB = BA]\}$.

Let $\mathcal{A}$ be an algebra of operators. Then $\mathcal{A}'$ is a weakly closed algebra and $\mathcal{A} \subset \mathcal{A}''$. If $\mathcal{B}$ is another algebra of operators and $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{B}' \subset \mathcal{A}'$. If we take $\mathcal{B} = \mathcal{A}''$, we see that $\mathcal{A}''' \subset \mathcal{A}'$. But on the other hand $\mathcal{A}' \subset (\mathcal{A}')''$. So $\mathcal{A}' = \mathcal{A}'''$.

One of the fundamental theorems in the classical von Neumann algebra theory is the double commutant theorem (DCT), which states that if $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{A} = \mathcal{A}''$. This theorem makes a connection between the algebraic property $\mathcal{A} = \mathcal{A}''$ and the analytical, or topological, property $\mathcal{A}$ is weakly closed. Either of these properties may be used to define a von Neumann algebra classically.

It is impossible to prove the DCT constructively. Indeed, let P be any statement and define

$$\mathcal{A} := \{A \in B(H) : P \vee \neg P\} \cup \{cI : c \in \mathbf{C}\}.$$

Let A be an element of $\mathcal{A}'$. Then $P \vee \neg P$ implies that $\exists c \in \mathbf{C}[A = cI]$ and thus So $\neg\neg\exists c \in \mathbf{C}[A = cI]$, because $\neg\neg(P \vee \neg P)$. Since if such c exists it must be equal to $\langle Ae_1, e_1\rangle$, we conclude that $\exists c \in \mathbf{C}[A = cI]$. We conclude that $\mathcal{A}' = \{cI : c \in \mathbf{C}\}$ and thus $\mathcal{A}'' = B(H)$. If $\mathcal{A} = \mathcal{A}''$, then $P \vee \neg P$. So the double commutant theorem implies the principle of excluded middle.

9.1 A classical proof

We will start with a sketch of a classical proof of the Double Commutant Theorem.

Let $\mathcal{A}$ be a von Neumann algebra and let B be an element of $\mathcal{A}''$, $x \in H$ and $\varepsilon > 0$. We want to find $A \in \mathcal{A}$ such that $\|(A - B)x\| < \varepsilon$. The space $\mathrm{cl}\mathcal{A}x$ is $\mathcal{A}$-invariant. Let P be a projection on $\mathrm{cl}\mathcal{A}x$. One can show that $P \in \mathcal{A}'$, so $PB = BP$. It follows that $Bx \in \mathrm{cl}\mathcal{A}x$, which is what we wanted to prove. Once we know how to find such an operator A for all x in H we can use a method called amplification (see for instance [20] (4.6.3,4.6.7)) to show that for all finite sequences $x_1, \dots, x_n$ and all $\varepsilon > 0$, there is $A \in \mathcal{A}$ such that for all $i \leq n$, $\|(A - B)x_i\| < \varepsilon$. That is, B is in the strongly closed set $\mathcal{A}$.

The problem one encounters constructively is that $\mathrm{cl}\mathcal{A}x$ is in general not located. Hence one can not compute the projection P above. In fact, all we need in the proof is that $\mathcal{A}x$ is located for all x in a dense set. To see that we can not hope to prove that $\mathcal{A}x$ is located, for all $x \in H$ instead of just for all x in a dense set, consider the vN-algebra

$$\mathcal{A} := \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{C}\} \subset M_2.$$

Let $c \in \mathbf{C}$ and define $x := e_1 + ce_2$. To compute the projection on $\mathcal{A}x$ we need to know whether $c = 0$ or $c \neq 0$. This example is due to Bridges.

The proof of the DCT gives us a way to find operators in the commutant. It also shows that a vN-algebra is weakly closed, since the commutant of an algebra is always weakly closed.

9.2 DCT, the finite dimensional case

Let $\mathcal{A}$ be a von Neumann algebra on a finite dimensional Hilbert space. Suppose that $\mathcal{A}$ has a totally bounded unit ball. Then we can compute $\mathcal{A}'$ explicitly using Theorem 3, see for instance [28] (p.53). It is easy to see that $\mathcal{A}'' = \mathcal{A}$.

9.3 DCT, the commutative case

Let $\mathcal{A}$ be an Abelian vN-algebra and suppose that $\mathcal{A}_1$ is weakly totally bounded. Recall that when $\mathcal{A}_1$ is weakly totally bounded, then $\langle \mathcal{A}_1 x, y\rangle$ is totally bounded, for all $x, y \in H$. Consequently, $\mathcal{A}_1 x$ is weakly totally bounded, for all x, and hence it is located, see [14]. In Lemma 17 we will see that $\mathcal{A}x$ is located for all x in a dense set. As we observed in Section 9.1 this is enough to complete a constructive proof of the DCT.

All elements in an Abelian vN-algebra are normal, so we can apply the spectral theorem to them. Let $\mathcal{A}_M := \{A \in \mathcal{A} : A \,\text{is bounded by}\, M\}$, whenever $M \in \mathbf{R}^+$.

Lemma 15. *Let $\mathcal{A}$ be an Abelian von Neumann algebra. Fix $N > 1$. Let x, y be elements of H such that $\|y\| = 1$. Let A be an operator such that $|\rho(y, \mathcal{A}_N x)^2 - \rho(y, Ax)^2| < \varepsilon$. When $M \in (1, N)$ is admissible for A, there is a projection $P \in \mathcal{A}$ with $\|x - Px\| \leq \frac{2}{M}$, $AP \in \mathcal{A}_M$ and $|\rho(y, \mathcal{A}_N Px)^2 - \rho(y, APx)^2| < \varepsilon$.*

Proof. Let P be $\chi_{\{x:|x|\leq M\}}(A)$ as defined in the spectral theorem; then $P \in \mathcal{A}$ and $AP \in \mathcal{A}_M$. Fix $\eta > 0$ such that $|\rho(y, \mathcal{A}_N x)^2 - \rho(y, Ax)^2| < \varepsilon + \eta$; then because $AP \in P\mathcal{A}_M \subset \mathcal{A}_M$, either $\rho(y, APx)^2 - \rho(y, \mathcal{A}_N x)^2 < \varepsilon$ or $\rho(y, APx)^2 \geq \rho(y, Ax)^2$. In the former case there is nothing to prove, because $AP \in \mathcal{A}_M \subset \mathcal{A}_N$, so we may assume that $\rho(y, APx)^2 \geq \rho(y, Ax)^2$, that is $\|APx - y\|^2 \geq \|Ax - y\|^2$. Hence

$$\|y\|^2 - 2\mathrm{Re}\langle APx, y\rangle + \|APx\|^2 \geq \|y\|^2 - 2\mathrm{Re}\langle Ax, y\rangle + \|Ax\|^2,$$

so $\|A(I - P)x\|^2 \leq 2|\langle A(I - P)x, y\rangle| \leq 2\|A(I - P)x\|\|y\|$. Consequently, $\|A(I - P)x\| \leq 2$, because $\|y\| = 1$. Now

$$4 \geq \|A(I - P)x\|^2 \geq M^2\|(I - P)x\|^2,$$

so $\|x - Px\|^2 \leq 4/M^2$.

Finally, remark that by the Pythagorean theorem,

$$\begin{aligned}\|y - Az\|^2 - \rho(y, \mathcal{A}_N z)^2 = {} & \|Py - PAz\|^2 - \rho(Py, P\mathcal{A}_N z)^2 \\ & + \|P^\perp y - P^\perp Az\|^2 - \rho(P^\perp y, P^\perp \mathcal{A}_N z)^2,\end{aligned}$$

whenever $z \in H$. So choosing $z := Px$, we get

$$\begin{aligned}\|y - APx\|^2 - \rho(y, \mathcal{A}_N Px)^2 & \leq \|Py - PAx\|^2 - \rho(Py, P\mathcal{A}_N x)^2 \\ & \quad + \|P^\perp y\|^2 - \|P^\perp y\|^2 \\ & < \varepsilon + 0.\end{aligned}$$

Classically, $\mathcal{A}_M x$ is a closed convex set, so for each $y \in H$ there is a point in $\mathcal{A}_M x$ which is closest to y. If Ax is an approximation to the point in $\mathcal{A}_N x$ which is closest to y, then by the previous lemma $A(Px)$ is an approximation for the point in $\mathcal{A}_N(Px)$ which is closest to y.

Lemma 16. *Let $\mathcal{A}$ be an Abelian von Neumann algebra and let $K \in \mathbf{N}$ and $x_1, y \in H$. Then there are a sequence $(P_n)_{n\in\mathbf{N}}$ of projections in $\mathcal{A}$ and a sequence $(A_n)_{n\in\mathbf{N}}$ of operators in $\mathcal{A}$ such that $x_\infty := \lim_{m\to\infty}(\Pi_{n=1}^m P_n)x_1$ exists, $\|x_\infty - x_1\| \leq 1/K$ and the sequence $(\|y - A_n x_\infty\|)_{n\in\mathbf{N}}$ converges to $\inf_{A\in\mathcal{A}} \|y - Ax_\infty\|$.*

Proof. Let $\varepsilon > 0$ and $N \in \mathbf{N}$. For $z \in H$, let us call A an ε-approximation for the distance from y to $\mathcal{A}_N z$, if $A \in \mathcal{A}_N$ and $\|y - Az\|^2 - \rho(y, \mathcal{A}_N z)^2 < \varepsilon$. Let A_1 be a 1-approximation for the distance from y to $\mathcal{A}_{2K} x_1$. Let A_2 be a 1/2-approximation for the distance from y to $\mathcal{A}_{2^2K} x_1$. Compute a projection P_2 as in Lemma 15 with $M = 2K$, $N = 2^2K$, $A = A_2$ and $x = x_1$. Set $x_2 := P_2 x_1$.

Continue in this way as follows. Let A_{n+1} be a 2^{-n}-approximation for the distance from y to $\mathcal{A}_{2^{n+1}K} x_n$ and compute P_{n+1} as in Lemma 15 with $M = 2^n K$, $N = 2^{n+1}K$, $A = A_{n+1}$ and $x = x_n$ and set $x_{n+1} := P_{n+1} x_n$.

For all $n \in \mathbf{N}$, $\|P_{n+1} x_n - x_n\| \leq 2^{-(n+1)}/K$. We define $x_\infty := \lim_{n\to\infty} x_n$. Then $\|x_\infty - x_1\| \leq \sum_{n=1}^\infty 2^{-(n+1)}/K = 1/K$. To see that the sequence $(A_n x_\infty)_{n\in\mathbf{N}}$ converges observe that

$$\begin{aligned}
\|A_{n+1}x_\infty - A_n x_\infty\|^2 &= \lim_{m\geq n+2} \|(\Pi_{i=n+2}^m P_i)(A_{n+1}x_{n+1} - A_n x_n)\|^2 \\
&\leq \|A_{n+1}x_{n+1} - A_n x_n\|^2 \\
&\leq 2^{1-n} + 2^{1-n}.
\end{aligned}$$

This last inequality holds, because for all $n \in \mathbf{N}$, $A_{n+1}x_{n+1} = A_{n+1}P_{n+1}x_n \in \mathcal{A}_{2^{n+1}K} x_n$ and the operators $A_n P_{n+1}$ and $A_{n+1}P_{n+1}$ are 2^{1-n}-approximations to for the distance from y to $\mathcal{A}_{2^nK} P_{n+1} x_n$.

Proposition 17. *Let $\mathcal{A}$ be an Abelian von Neumann algebra. The set $\mathcal{A}x$ is located for all x in a dense set.*

Proof. Let $x \in H$ and $\varepsilon > 0$. We claim that there is $z \in H$ such that $\rho(x, z) < \varepsilon$ and the projection $P_{[\mathcal{A}z]}$ exists.

Define by the previous lemma $x_1 \in H$ such that $\rho(x, x_1) < \varepsilon/2$ and the projection of e_1 on $\mathrm{cl}\mathcal{A}x_1$ exists. Continue in this way as follows. Define $x_{n+1} \in H$ such that $\rho(x_n, x_{n+1}) < \varepsilon/2^{n+1}$ and the projection of e_{n+1} on $\mathrm{cl}(\mathcal{A}x_{n+1})$ exists. Recall from the proof of Lemma 15 that for all $N \in \mathbf{R}^+$, $P \in \mathcal{A}$ and $A \in \mathcal{A}_N$:

$$\rho(y, APx) - \rho(y, \mathcal{A}_N Px) \leq \rho(y, Ax) - \rho(y, \mathcal{A}_N x).$$

We see from the construction of x_{n+1} that for all $i \leq n$, the projection of e_i on $\mathrm{cl}\mathcal{A}x_{n+1}$ exists.

Define $x_\infty := \lim_{n\to\infty} x_n$; then for all $n \in \mathbf{N}$, the projection of e_n on $\mathrm{cl}\mathcal{A}x_\infty$ exists. Because a projection is linear and contracts the norm we see that we can compute the projection of y on $\mathrm{cl}\mathcal{A}x_\infty$, for all $y \in H$.

From this last proposition we can obtain the double commutant theorem for Abelian vN-algebras with a weakly totally bounded unit-ball:

Theorem 18. *Let $\mathcal{A}$ be an Abelian von Neumann algebra with weakly totally bounded unit ball, then $\mathcal{A}'' = \mathcal{A}$.*

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