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Constructive Equivalents of the Uniform Continuity Theorem[1]

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Abstract: For the purpose of constructive reverse mathematics, we show the equivalence of the uniform continuity theorem to a series of propositions; this illuminates the relationship between Brouwer's fan theorem and the uniform continuity theorem.
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Working in the system **EL**, we investigate how the following axioms are related to each other:

- Every pointwise continuous function $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.
- Every pointwise continuous function $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is bounded.
- For every pointwise continuous function $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ it is decidable whether it is constant or not.
- The fan theorem: every detachable bar is uniform.

An introduction to the formal system **EL** can be found in Chapter 3 of [7]. Every object is based on natural numbers and functions $\alpha \in \mathbb{N} \to \mathbb{N}$. There is a canonical bijection between the set $\{0,1\}^*$ of finite binary sequences and $\mathbb{N}$, by setting

$$u_0 = (),\ u_1 = (0),\ u_2 = (1),\ u_3 = (00),\ u_4 = (01), u_5 = (10),\ u_6 = (11), \ldots$$

Therefore, we can work with functions $g \in \{0,1\}^* \to \mathbb{N}$ as well. A function $\alpha \in \mathbb{N} \to \mathbb{N}$ is a binary sequence if

$$bin(\alpha) \equiv \forall n \in \mathbb{N}\,(\alpha(n) = 0 \vee \alpha(n) = 1)\,.$$

For any formula A we write

$$\forall \alpha \in \{0,1\}^{\mathbb{N}} A$$

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

as an abbreviation of

$$\forall \alpha \in \mathbb{N} \to \mathbb{N} \; (bin(\alpha) \to A) \, .$$

Let $\overline{\alpha}n$ denote the restriction of (finite or infinite) sequences to their first n components. Concatenation of finite sequences u, v is denoted by $u * v$. Finally, let $|u|$ denote the length of a finite binary sequence u. These operations are definable in **EL**. We are interested in continuous functions $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$. Under the compact metric

$$d(\alpha, \beta) = \inf \left\{ 2^{-n} \mid \overline{\alpha}n = \overline{\beta}n \right\}$$

on $\{0,1\}^{\mathbb{N}}$, pointwise continuity reads as

$$\forall \alpha \in \{0,1\}^{\mathbb{N}} \exists n \in \mathbb{N} \, \forall \beta \in \{0,1\}^{\mathbb{N}} \left(\overline{\alpha}n = \overline{\beta}n \to (F(\alpha) = F(\beta)) \right) \tag{1}$$

and uniform continuity reads as

$$\exists n \in \mathbb{N} \, \forall \alpha, \beta \in \{0,1\}^{\mathbb{N}} \left(\overline{\alpha}n = \overline{\beta}n \to (F(\alpha) = F(\beta)) \right) .$$

For the sake of working within **EL**, we use a concept of continuity which is based on functions $f \in \{0,1\}^* \to \mathbb{N}$ rather than on functions $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$. A function $f \in \{0,1\}^* \to \mathbb{N}$ pointwise continuous if

$$pc(f) \equiv \forall \alpha \in \{0,1\}^{\mathbb{N}} \, \exists n \in \mathbb{N} \, \forall u \in \{0,1\}^* \, (f(\overline{\alpha}n) = f(\overline{\alpha}n * u)) \, .$$

A function $f \in \{0,1\}^* \to \mathbb{N}$ is uniformly continuous if

$$uc(f) \equiv \exists n \in \mathbb{N} \, \forall \alpha \in \{0,1\}^{\mathbb{N}} \, \forall u \in \{0,1\}^* \, (f(\overline{\alpha}n) = f(\overline{\alpha}n * u)) \, .$$

The uniform continuity theorem reads as

$$\mathbf{UC} \equiv \forall f \in \{0,1\}^* \to \mathbb{N} \, (pc(f) \to uc(f)) \, .$$

Note that **UC** is equivalent to the statement: each pointwise continuous function $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous; thus the investigation of the constructive content of the uniform continuity theorem is not biased by representing continuous functions as type 1 objects. For $f \in \{0,1\}^* \to \mathbb{N}$ it can be decided whether it is constant or not if

$$dc(f) \equiv \forall u, v \in \{0,1\}^* \, (f(u) = f(v)) \vee \exists u, v \in \{0,1\}^* \, (f(u) \neq f(v)) \, .$$

Thus we define

$$\mathbf{DC} \equiv \forall f \in \{0,1\}^* \to \mathbb{N} \, (pc(f) \to dc(f)) \, .$$

A function $f \in \{0,1\}^* \to \mathbb{N}$ is bounded if

$$bo(f) \equiv \exists n \in \mathbb{N} \, \forall u \in \{0,1\}^* \, (f(u) \leq n) \, .$$

Let us define

$$\mathbf{PB} \equiv \forall f \in \{0,1\}^* \to \mathbb{N}\,(pc(f) \to bo(f))\,.$$

Now it remains to formalise Brouwer's fan theorem. This is done similarly as in [5]. A function $f \in \{0,1\}^* \to \mathbb{N}$ is a bar if

$$bar(f) \equiv$$

$$\forall u, v \in \{0,1\}^*\,(f(u) = 0 \to f(u * v) = 0)\,\&\,\big(\forall \alpha \in \{0,1\}^{\mathbb{N}} \exists n\,(f(\overline{\alpha}n) = 0)\big)\,.$$

A function $f \in \{0,1\}^* \to \mathbb{N}$ is a uniform bar if

$$ubar(f) \equiv \exists n \in \mathbb{N}\,\forall \alpha \in \{0,1\}^{\mathbb{N}}\,(f(\overline{\alpha}n) = 0)\,.$$

Now we can define

$$\mathrm{FT} \equiv \forall f \in \{0,1\}^* \to \mathbb{N}\,(bar(f) \to ubar(f))\,.$$

For a function $f \in \{0,1\}^* \to \mathbb{N}$ and $n \in \mathbb{N}$ we define

$$\sup(f, n) \equiv \exists u \in \{0,1\}^*\,(f(u) = n)\,\& \forall u \in \{0,1\}^*\,(f(u) \leq n)\,.$$

Thus $\sup(f, n)$ just says that n is the supremum of f.[2] Its existence is guaranteed at least in the case of uniform continuity:[3]

Lemma 1. $\boldsymbol{EL} \vdash \forall f \in \{0,1\}^* \to \mathbb{N}\,(uc(f) \to \exists n \in \mathbb{N}\,\sup(f, n))$

Proof. Fix a uniformly continuous f. Then there is $m \in \mathbb{N}$ such that

$$\forall \alpha \in \{0,1\}^{\mathbb{N}}\,\forall u \in \{0,1\}^*\,(f(\overline{\alpha}m) = f(\overline{\alpha}m * u))\,.$$

Thus we have

$$\sup(f, \max\{f(u) \mid u \in \{0,1\}^* \text{ and } |u| \leq m\}).$$

□

At some stage we shall use the following version of the axiom of choice

$$\mathbf{AC}^* \equiv$$

$$\forall u \in \{0,1\}^*\,(A(u) \vee \neg A(u)) \to$$

$$\exists g \in \{0,1\}^* \to \mathbb{N}\,(\forall u \in \{0,1\}^*\,(g(u) = 0 \leftrightarrow \neg A(u)))\,,$$

where $A(u)$ is a Σ^0_1–formula.

[2] The infimum is treated analogously.
[3] See Corollary 4.3 in Chapter 4 of [2] for an informal proof of this fact.

Proposition 2.

$$\boldsymbol{EL} \vdash \boldsymbol{UC} \leftrightarrow \boldsymbol{PB}$$
$$\boldsymbol{EL} \vdash \boldsymbol{UC} \rightarrow \boldsymbol{FT} + \boldsymbol{DC}$$
$$\boldsymbol{EL} + \boldsymbol{AC}^{*} \vdash \boldsymbol{FT} + \boldsymbol{DC} \rightarrow \boldsymbol{UC}$$

Proof. ($\mathbf{EL} \vdash \mathbf{UC} \rightarrow \mathbf{PB}$) This follows from Lemma 1.

($\mathbf{EL} \vdash \mathbf{PB} \rightarrow \mathbf{UC}$) Fix a pointwise continuous $f \in \{0,1\}^* \rightarrow \mathbb{N}$. We define $g \in \{0,1\}^* \rightarrow \mathbb{N}$ by

$$g(w) = \max\left(\{k \in \{0,\ldots,|w|-1\} \mid f(\overline{w}k) \neq f(w)\} \cup \{0\}\right)$$

and show that g is pointwise continuous as well. Fix $\alpha \in \{0,1\}^{\mathbb{N}}$; there is $m \in \mathbb{N}$ such that

$$\forall u \in \{0,1\}^* \left(f(\overline{\alpha}m) = f(\overline{\alpha}m * u)\right).$$

Fix $u \in \{0,1\}^*$. Then

$$g(\overline{\alpha}m) = \max\left(\{k \in \{0,\ldots,m-1\} \mid f(\overline{\alpha}k) \neq f(\overline{\alpha}m)\} \cup \{0\}\right) =$$

$$\max\left(\left\{k \in \{0,\ldots,m+|u|-1\} \mid f(\overline{\overline{\alpha}m * u}k) \neq f(\overline{\alpha}m * u)\right\} \cup \{0\}\right) =$$

$$g(\overline{\alpha}m * u).$$

Thus g is pointwise continuous.

Now, by $\mathbf{PB}$, there is $k \in \mathbb{N}$ such that

$$\forall u \in \{0,1\}^* \left(g(u) < k\right).$$

For every $\alpha \in \{0,1\}^{\mathbb{N}}$ and for every $u \in \{0,1\}^*$ we can conclude that

$$f(\overline{\alpha}k) = f(\overline{\alpha}k * u),$$

since otherwise $g(\overline{\alpha}k * u) \geq k$, which is absurd. Thus f is uniformly continuous.

($\mathbf{EL} \vdash \mathbf{UC} \rightarrow \mathbf{FT}$) Let $f \in \{0,1\}^* \rightarrow \mathbb{N}$ be a bar. Then f is pointwise and therefore uniformly continuous. It follows that f is a uniform bar.

($\mathbf{EL} \vdash \mathbf{UC} \rightarrow \mathbf{DC}$) Let $f \in \{0,1\}^* \rightarrow \mathbb{N}$ be uniformly continuous. Comparing the supremum of f with the infimum of f yields a decision whether f is constant or not.

($\mathbf{EL} + \mathbf{AC}^* \vdash \mathbf{FT} + \mathbf{DC} \rightarrow \mathbf{UC}$) Fix a pointwise continuous function $f \in \{0,1\}^* \rightarrow \mathbb{N}$. For every $u \in \{0,1\}^*$ the assignment

$$\{0,1\}^* \ni w \mapsto u * w$$

is definable in **EL** and pointwise continuous. Thus the function

$$f_u : \{0,1\}^* \to \mathbb{N},\, w \mapsto f(u * w)$$

is pointwise continuous as well. For $u \in \{0,1\}^*$ we define

$$A(u) \equiv \exists v, w \in \{0,1\}^* \left(f(u * v) \neq f(u * w)\right).$$

Thus $A(u)$ is the Σ^0_1–statement: f_u is not constant. And $\neg A(u)$ is the statement: f_u is constant. By $dc(f_u)$ we have $A(u) \vee \neg A(u)$. Thus, by $\mathbf{AC}^*$ there is $g \in \{0,1\}^* \to \mathbb{N}$ such that

$$\forall u \in \{0,1\}^* \left(g(u) = 0 \leftrightarrow \neg A(u)\right).$$

By the definition of A it follows that

$$\forall u, v \left(g(u) = 0 \to g(u * v) = 0\right). \tag{2}$$

By (2) and the pointwise continuity of f we can see that g is a bar; by **FT**, g is a uniform bar, which implies the uniform continuity of f. $\Box$

It was Hajime Ishihara who propagated formal approaches to constructive reverse mathematics [4]. See also the work of Iris Loeb [6] and Wim Veldman [8].

It is well known that in Bishop's constructive mathematics the uniform continuity theorem implies Brouwer's fan theorem. Under continuous choice, the reverse implication holds as well; see Section 3 of Chapter 5 in [3] for proofs of these results. Continuous choice is required in order to assure that pointwise continuous functions possess a modulus of pointwise continuity. That means that the assignment $\forall\alpha\exists n \ldots$ in (1) is given by a pointwise continuous function. One can show that Brouwer's fan theorem is equivalent to the proposition: each function which possesses a modulus of pointwise continuity is uniformly continuous [1].

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