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Constructive Analysis of Iterated Rational Functions[1]

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Abstract: We develop the elementary theory of iterated rational functions over the Riemann sphere $\mathbb{C}_\infty$ in a constructive setting. We use Bishop-style constructive proof methods throughout. Starting from the development of constructive complex analysis presented in [Bishop and Bridges 1985], we give constructive proofs of Montel's Theorem along with necessary generalisations, and use them to prove elementary facts concerning the Julia set of a general continuous rational function with complex coefficients. We finish with a construction of repelling cycles for these maps, thereby showing that Julia sets are always inhabited.
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1 Preliminaries

We are interested in the behaviour of analytic functions on $\mathbb{C}_\infty$, the Riemann sphere. Following [Bishop and Bridges 1985] p. 190, we shall define this domain to be the unit sphere $\left\{x \in \mathbb{R}^3 : \|x\| = 1\right\}$ along with the embedding of the complex plane $i_0 : \mathbb{C} \to \mathbb{C}_\infty$ given by

$$i_0(z) = \left(\frac{2\ \mathrm{Re}\, z}{|z|^2+1}, \frac{2\ \mathrm{Im}\, z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \tag{1}$$

We shall let j_0 denote the inverse map $j_0 : \mathbb{C}_\infty {\sim} \{\infty\} \to \mathbb{C}$ such that $j_0 \circ i_0 = \mathrm{id}_{\mathbb{C}}$. The complex plane then inherits the **chordal metric** σ defined for points in $\mathbb{C}_\infty$ by

$$\sigma(z_1, z_2) = \frac{1}{2}\|z_1 - z_2\| \quad (z_1, z_2 \in \mathbb{C}_\infty).$$

For points z_1, z_2 in $\mathbb{C}$ we have

$$\sigma(i_0(z_1), i_0(z_2)) = (1 + |z_1|^2)^{-\frac{1}{2}}(1 + |z_2|^2)^{-\frac{1}{2}}|z_1 - z_2|$$

We shall extend the notation $\sigma(z_1, z_2)$ in two useful ways: for a finite sequence of points we shall write

$$\sigma(z_1, \dots, z_n) = \min\{\sigma(z_i, z_j) \ : \ 1 \le i < j \le n\}$$

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

and for compact sets $K_1, \ldots, K_n$ we shall write

$$\sigma(K_1, \ldots, K_n) = \inf\{\sigma(z_1, \ldots, z_n) : z_1 \in K_1, \ldots z_n \in K_n\}$$

Let (X, d) be a metric space. We write

$$\begin{aligned} B_X(z_0, r) &= \{z \in X : d(z, r) < r\} \\ \overline{B}_X(z_0, r) &= \{z \in X : d(z, r) \leq r\} \\ S_X(z_0, r) &= \{z \in X : d(z, r) = r\} \end{aligned}$$

We shall omit the subscript when the metric space is $\mathbb{C}$, and use the subscript σ when the metric space is $(\mathbb{C}_\infty, \sigma)$

Suppose that $z_1, z_2 \in \mathbb{C}_\infty$ and $\sigma(z_1, z_2) < 1$. Then there exists a unique line segment $[z_1, z_2]$ connecting z_1 and z_2, namely the geodesic of arc length $< \pi/2$. Also note that if $z_0 \in \mathbb{C}_\infty$, $0 < r < 2^{-1/2}$ and $z_1, z_2 \in \overline{B}_\sigma(z_0, r)$ then $[z_1, z_2] \subset \overline{B}_\sigma(z_0, r)$ (i.e. spheres in $\mathbb{C}_\infty$ are convex if they are sufficiently small).

We shall use the definition inherited from constructive complex analysis on $\mathbb{C}$ of the relation $K \Subset U$, which holds when K is compact and U is open and there exists $\epsilon > 0$ such that $K_\epsilon \subset U$, where

$$K_\epsilon = \{z \in \mathbb{C}_\infty : \sigma(z, K) \leq \epsilon\}.$$

For any set S in a metric space we will also write $S_{<\epsilon}$ for the open set

$$S_{<\epsilon} = \bigcup \{B(z, \epsilon) : z \in S\}.$$

We shall say that a set in a metric space is **compact-or-empty** if it is either compact (in Bishop's sense - which implies that it is nonempty) or empty. Let K be compact in $\mathbb{C}_\infty$ and let $f : K \to \mathbb{C}_\infty$ be a function. Suppose that f is (uniformly) continuous on K. Then by Theorem 4.4.9 of [Bishop and Bridges 1985] we can find reals $0 < R_0, R_\infty < 1$ with $R_0 + R_\infty > 1$ such that the sets

$$K_{\alpha\beta} = \left\{z \in K : z \in \overline{B}_\sigma(\alpha, R_\alpha) \wedge f(z) \in \overline{B}_\sigma(\beta, R_\beta)\right\}$$

are compact-or-empty for all $\alpha, \beta \in \{0, \infty\}$. Let $f_{\alpha\beta}$ be the function f restricted to $K_{\alpha\beta}$. Let ρ_0 be the identity function on $\mathbb{C}_\infty$ and let ρ_∞ be the reciprocal function $z \mapsto 1/z$ completed to the set $\mathbb{C}_\infty$ (which preserves the metric σ). Then whenever $K_{\alpha\beta}$ is nonempty we can translate the function $f_{\alpha\beta}$ to $\mathbb{C}$ as follows:

$$\tilde{f}_{\alpha\beta} = j_0 \circ \rho_\beta^{-1} \circ f_{\alpha\beta} \circ \rho_\alpha \circ i_0.$$

Then $\tilde{f}_{\alpha\beta}$ will be a function from a compact subset of $\mathbb{C}$ to $\mathbb{C}$. We shall say that f is **differentiable** if, for all choices of R_0, R_∞ making the sets $K_{\alpha\beta}$ compact-or-empty, and for any $\alpha, \beta \in \{0, \infty\}$ such that $K_{\alpha\beta}$ is nonempty, the function

$\tilde{f}_{\alpha\beta}$ is differentiable on its compact domain in $\mathbb{C}$. If f is a function on some open set $U \subset \mathbb{C}_\infty$ and f is differentiable on every $K \Subset U$. Then we shall say that f is *analytic*.

We state without proof a version of the open mapping theorem (theorem 5.5.17, [Bishop and Bridges 1985]) for functions on $\mathbb{C}_\infty$ which we will often need to use in what follows:

Theorem 1. *Let f be analytic on an open set $U \subset \mathbb{C}_\infty$. Let $K \Subset U$ be a compact set. Then $f(K)$ is compact and $f(K) \Subset f(U)$.*

The spherical derivative

Given an analytic function $f : U \subset \mathbb{C}_\infty \to \mathbb{C}_\infty$ we can define the spherical derivative on f, which measures in absolute value the rate of change of the function f according to σ, taken in any direction away from a given point z:

$$f^\sharp(z) = \lim_{w \to z} \frac{\sigma(f(w), f(z))}{\sigma(w, z)}$$

One can then show that when $z, f(z) \neq \infty$ and $\hat{f} := i_0 \circ f \circ j_0$ (locally to $x = j_0(z)$), we have

$$f^\sharp(z) = |\hat{f}'(x)| \frac{1 + |\hat{f}(x)|^2}{1 + |x|^2} \tag{2}$$

We state the following for future reference. The proof is left to the reader.

Proposition 2 (Cauchy's inequality on $\mathbb{C}_\infty$). *If f is analytic on $\overline{B}_\sigma(z_0, r)$ then*

$$f^\sharp(z_0) \leq r^{-1} \sup\{\Theta(\sigma(f(z_0), f(z))) \ : \ \sigma(z_0, z) = r\} \tag{3}$$

where Θ is the increasing function on $[0, 1)$ given by $\Theta(t) = t(1 - t^2)^{-\frac{1}{2}}$.

A note on changing metrics

Often we will need to move between the metrics on $\mathbb{C}$ and $\mathbb{C}_\infty$ in proofs and this can cause awkwardness, so we state the inequalities which prove that these metrics are equivalent, for later reference:

$$\sigma(i_0(z_1), i_0(z_2)) \leq |z_1 - z_2| \qquad (z_1, z_2 \in \mathbb{C})$$

$$\sigma(i_0(z_1), i_0(z_2)) \leq \Delta(z_1, \epsilon) \quad \Rightarrow \quad |z_1 - z_2| \leq \epsilon \qquad (z_1, z_2 \in \mathbb{C}, \epsilon > 0)$$

where Δ is the function defined by

$$\Delta(z, \epsilon) := \epsilon(1 + |z|^2)^{-\frac{1}{2}}((1 + |z|^2)^{\frac{1}{2}} + \epsilon)^{-1} \qquad (z \in \mathbb{C}, \epsilon > 0) \tag{4}$$

Möbius functions

Consider the analytic function

$$f(z) = \frac{az+b}{cz+d} \qquad (a,b,c,d \in \mathbb{C}, ad-bc \neq 0).$$

defined on the open set $\{z \in \mathbb{C} : cz+d \neq 0\}$. This function extends uniquely to an analytic function from $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$, which is called a **Möbius function**. In the case where $b = -\overline{c}$, $d = \overline{a}$ and $|a|^2 + |c|^2 = 1$ then the function f preserves the metric σ. We shall call such a metric a **Möbius isometry**.

The following will be useful later.

Lemma 3. *Let $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation. Let $m > 0$. Suppose that $\sigma(g(0), g(1), g(\infty)) \geq m$. Then g satisfies the uniform Lipschitz conditions*

$$\sigma(g(w), g(z)) \leq (\pi/m^3)\sigma(w,z) \tag{5}$$

$$\sigma(w,z) \leq (\pi/m^3)\sigma(g(w), g(z)) \tag{6}$$

for all $w, z \in \mathbb{C}_\infty$,

Proof. The inequality (5) is proved exactly as in the classical case: see Theorem 2.3.3 of [Beardon 1991]. For the inequality (6), observe that $\|g^{-1}\| = \|g\|$ (with notation taken from [Beardon 1991], p. 33), so that the proof of Theorem 2.3.3 in [Beardon 1991] also yields $\|g^{-1}\| \leq 2/m^3$, whence (via 2.1.3 and 2.3.2 of [Beardon 1991]),

$$\sigma(g^{-1}(w), g^{-1}(z)) \leq (\pi/m^3)\sigma(w,z)$$

from which we easily get (6) by substituting $w \to g(w)$ and $z \to g(z)$.

Lemma 4. *Let m be a positive number, let a_1, a_2, a_3 be points in $\mathbb{C}_\infty$ such that $\sigma(a_1, a_2, a_3) \geq m$, and let g be the unique Möbius function taking $(0, 1, \infty)$ to (a_1, a_2, a_3). Then the following inequalities hold for all $z \in \mathbb{C}_\infty$*

$$m^3/\pi \leq g^\sharp(z) \leq \pi/m^3$$

$$m^3/\pi \leq (g^{-1})^\sharp(z) \leq \pi/m^3$$

Proof. Apply Lemma 3 to g and take limits.

Lemma 5. *Let m be a positive number and let a_1, a_2, a_3 and b_1, b_2, b_3 be points in $\mathbb{C}_\infty$ such that $\sigma(a_1, a_2, a_3) \geq m$ and $\sigma(b_1, b_2, b_3) \geq m$. Let g be the unique Möbius transformation of $\mathbb{C}_\infty$ which takes a_i to b_i $(1 \leq i \leq 3)$. Then*

$$m^6/\pi^2 \leq g^\sharp(z) \leq \pi^2/m^6 \tag{7}$$

Proof. Let $g = g_2^{-1} \circ g_1$ where g_1 maps (a_i) to $(0, 1, \infty)$ and g_2 maps (b_i) to $(0, 1, \infty)$. Then apply Lemma 3 to g_1, g_2 to get the result.

Rational functions

We say that a polynomial of the form $p(z) = a_0 + \ldots + a_n z^n$ is of degree at most n. If $a_n \neq 0$ then we say that p is of degree n. We wish to define a rational function R to be a quotient of polynomials P/Q with coefficients in C, extended to a continuous function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. So we need conditions on P and Q which tell us precisely when such a continuous extension exists.

Lemma 6. *Let p be a nonzero polynomial of degree at most d with coefficients in $\mathbb{C}$. Then p can be expressed as*

$$p(z) = k(z - z_1) \cdots (z - z_a)(1 - w_1 z) \cdots (1 - w_{d-a} z) \qquad (k \neq 0, z_i, w_i \in \mathbb{C}) \tag{8}$$

Moreover if we set

$$Z(p) := \mathrm{cl}\,\{i(z_1), \ldots, i(z_a), i(w_1)^{-1}, \ldots, i(w_{d-a})^{-1}\} \subset \mathbb{C}_\infty,$$

then for all $\delta \in (0, 1)$ and $z \in \mathbb{C}$, if $\sigma(i(z), Z) \geq \delta$ then $|p(z)| \geq k(\delta/2)^d$.

Proof. We omit the proof, which is fairly straightforward: apply Theorem 5.5.13 of [Bishop and Bridges 1985], to get $z_1, \ldots, z_a$ and then apply it again with ∞ relocated to 0 to get $w_1, \ldots, w_{d-a}$.

Suppose that P and Q are two nonzero polynomials of degree at most d with the additional property that the zero sets $Z(P)$ and $Z(Q)$ are a positive distance from one another. Then either $\sigma(\infty, Z(P)) > 0$ or $\sigma(\infty, Z(Q)) > 0$ and so either $\deg P = d$ or $\deg Q = d$, and conversely. In this case the function $R : z \mapsto P(z)/Q(z)$ can be extended to a continuous function on $\mathbb{C}_\infty$ which can be shown to be analytic. In fact one can prove that any analytic function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ can be expressed in this form. We shall call such a function a **rational map**, and we shall call the unique integer d the **degree** of R. We shall also sometimes write $\mathcal{R}_d$ for the set of rational maps of degree d and $\mathcal{R}$ for the set $\cup_{d \in \mathbb{Z}^+} \mathcal{R}_d$.

Since we never have any reason to take the pointwise product of rational maps, we can safely write RS for the concatenation of two rational maps: $RS : z \mapsto R(S(z))$. The iteration R^n is defined similarly. Given a rational map $R = P/Q$ one can show that there is a finite set of points $C \subset \mathbb{C}_\infty$ (in fact the critical values, which we shall return to later) such that if $\sigma(z_0, C) > 0$ then the equation $R(z) = z_0$ has d distinct roots. From this one can deduce that

$$\deg(RS) = \deg(R) \deg(S)$$

and

$$\deg(R^n) = \deg(R)^n$$

for all $R, S \in \mathcal{R}$.

Fixed points

Let R be a rational function. We say that z is a **fixed point** of R if $R(z) = z$. Let $\mathrm{fp}(R)$ denote the set of such points. Suppose now that R is **non-identity**, meaning that there is some $z \in \mathbb{C}_\infty$ such that $R(z) \neq z$. By conjugating with a Möbius isometry μ we can assume that $z = \infty$. Then the fixed points of R will lie in $\mathbb{C}$ and can be obtained by solving $P(z) - zQ(z) = 0$ where $R = P/Q$. If we obtain a complete multiset of zeros $\{z_1, \ldots, z_n\}$ for this polynomial then we shall clearly have $\mathrm{fp}(R) = \overline{\{z_1, \ldots, z_n\}}$. In such a case we shall say that there are n fixed points of R counting multiplicities, although this statement cannot be taken as literally as in the classical case. More generally we shall talk about the n roots of an equation involving rational functions and points in $\mathbb{C}_\infty$ when such an equation can be reduced by whatever means to a polynomial of degree n over $\mathbb{C}$.

Theorem 7. *Let R be a non-identity rational function of degree d. Then R has $d + 1$ fixed points counting multiplicities.*

Proof. As above, assume that $R(\infty) \neq \infty$ and express R as P/Q in lowest terms. We are looking for roots of $P(z) - zQ(z) = 0$. The number of such roots will of course be the degree of this polynomial, so we need to show that $\deg Q = d$. But we know that either $\deg P = d$ or $\deg Q = d$, and since $R(\infty) \neq \infty$ we must have $\deg Q = d$.

2 Normal and abnormal classes of functions

Let $\mathcal{F}$ be a family of analytic functions on some compact set K. We say that $\mathcal{F}$ is **normal** on K if there is a bound B such that $f^\sharp(z) \leq B$ for all $f \in \mathcal{F}$ and all $z \in K$. We say that $\mathcal{F}$ is **abnormal** on K if for any $C > 0$ there is $f \in \mathcal{F}$ and $z \in K$ such that $f^\sharp(z) > C$. Now suppose that $\mathcal{F}$ is a family of analytic functions on an open set U. Then we say that $\mathcal{F}$ is normal on U if $\mathcal{F}$ is normal on K for every $K \Subset U$, and that $\mathcal{F}$ is abnormal on U if there exists $K \Subset U$ such that $\mathcal{F}$ is abnormal on K. Given $z \in U$ we shall say that $\mathcal{F}$ is normal at z if there is an open neighbourhood W of z such that $\mathcal{F}$ is normal on W, and that $\mathcal{F}$ is abnormal at z if for every open neighbourhood W of z, $\mathcal{F}$ is abnormal on W. Let $\mathrm{diam}_\sigma(K)$ denote the σ-diameter of any compact $K \subset \mathbb{C}_\infty$.

Theorem 8. *Let $\mathcal{F}$ be a class of functions on an open set $U \subset \mathbb{C}_\infty$ which contains a point z_0. Then $\mathcal{F}$ is normal at z_0 if and only if*

$$\forall\, \epsilon > 0\ \exists\, r > 0\ \forall\, f \in \mathcal{F} \quad \mathrm{diam}_\sigma\, f(\overline{B}_\sigma(z_0, r)) \leq \epsilon. \tag{9}$$

and $\mathcal{F}$ is abnormal at z_0 if and only if

$$\exists\, \epsilon > 0\ \forall\, r > 0\ \exists\, f \in \mathcal{F} \quad \mathrm{diam}_\sigma\, f(\overline{B}_\sigma(z_0, r)) \geq \epsilon. \tag{10}$$

Proof. First suppose that $\mathcal{F}$ is normal at z_0. Choose $R > 0$ and $B > 0$ such that $f^\sharp$ is bounded by B on $\overline{B}_\sigma(z_0, R)$ for all $f \in \mathcal{F}$. Then for any $r \in (0, R]$ and $z_1, z_2 \in \overline{B}_\sigma(z_0, r)$ a simple path-integral argument gives

$$\sigma(f(z_1), f(z_2)) \leq B \cdot \sigma(z_1, z_2) \leq B \cdot 2r.$$

Thus $\operatorname{diam} f(\overline{B}_\sigma(z_0, r)) \leq 2Br$. So for given $\epsilon > 0$, $r = \min\{R, \epsilon/2B\}$ satisfies (9).

Conversely, suppose that (9) is satisfied for $\mathcal{F}$ at z_0. We want to give a bound for $f^\sharp$ on some $\overline{B}_\sigma(z_0, R)$. Apply equation (9) with $\epsilon = 1/2$ to obtain $r > 0$. Let $R = r/2$. Suppose that $f \in \mathcal{F}$ and $z \in \overline{B}_\sigma(z_0, R)$. Then $\operatorname{diam} f(\overline{B}_\sigma(z, R)) \leq 1/2$ and so by Proposition 2, $f^\sharp(z) \leq R^{-1}\Theta(1/2)$. Thus we have the desired bound.

Now for the second part of the theorem. Suppose first that $\mathcal{F}$ is abnormal at z_0. We prove that (10) is satisfied with $\epsilon = 1/2$. Let $r > 0$. The set $\mathcal{F}$ is abnormal on $\overline{B}_\sigma(z_0, r/2)$ and so there is $z \in \overline{B}_\sigma(z_0, r/2)$ and $f \in \mathcal{F}$ such that

$$f^\sharp(z) \geq 4r^{-1}\Theta(1/2)$$

(where Θ is the function defined in Proposition 2). By Proposition 2, we also have

$$f^\sharp(z) \leq (r/2)^{-1} \sup\{\Theta(\sigma(f(z), f(w))) \,:\, \sigma(w, z) = r/2\}$$

And so we can find w such that $\sigma(w, z) = r/2$ and

$$\Theta(\sigma(f(z), f(w)) \geq (r/4) f^\sharp(z) \geq \Theta(1/2)$$

so that (since Θ is increasing) $\sigma(f(z), f(w)) \geq 1/2$. Since $z, w \in \overline{B}_\sigma(z_0, r)$ we have established that $\operatorname{diam} f(\overline{B}_\sigma(z, r)) \geq 1/2 = \epsilon$, as required.

Conversely suppose that condition (10) holds for some $\epsilon > 0$. Fix $r > 0$. We aim to show that $\mathcal{F}$ is abnormal on $\overline{B}(z_0, r)$. We may suppose that $r < 2^{-1/2}$. There is an $f \in \mathcal{F}$ such that $\operatorname{diam} f(\overline{B}(z_0, r)) \geq \epsilon$. So there exist $z_1, z_2 \in \overline{B}(z_0, r)$ such that $\sigma(f(z_1), f(z_2)) \geq \epsilon/2$. Since $\sigma(z_1, z_2) \leq 2r$, there exists a point w on the line segment $[z_1, z_2]$ such that $f^\sharp(w) \geq 1/2(\epsilon/2)(2r)^{-1} = \epsilon/8r$. Since $w \in \overline{B}(z_0, r)$ and r was arbitrarily small, we can find arbitrarily high values of $f^\sharp(z)$ arbitrarily close to z_0. So $\mathcal{F}$ is abnormal at z_0.

3 Montel's theorem

We shall say that a class $\mathcal{F}$ of functions on $S \subset \mathbb{C}_\infty$ omits a value $z \in \mathbb{C}_\infty$ if for all $w \in S$ and all $f \in \mathcal{F}$, $f(w) \neq z$. We shall also say that $\mathcal{F}$ attains a value $z \in \mathbb{C}_\infty$ if there exists $f \in \mathcal{F}$ and $w \in S$ such that $f(w) = z$.

Theorem 9 Montel's theorem. *A family $\mathcal{F}$ of analytic fuctions on an open set $U \subset \mathbb{C}_\infty$ which omits three distinct values is normal.*

Proof. Let $K \Subset U$. We construct a bound on the spherical derivative $f^\sharp(z)$, for $f \in \mathcal{F}$ and $z \in K$. Suppose that each $f \in \mathcal{F}$ omits distinct a, b, c. Let h be the unique Möbius transformation which maps a, b, c to $0, 1, \infty$. If we prove the theorem for the class $\{f \circ h : f \in \mathcal{F}\}$ then since $(f \circ h)^\sharp(z) = f^\sharp(z) h^\sharp(f(z))$, and since the spherical derivative of h is bounded away from 0 on $\mathbb{C}_\infty$ by Lemma 5, we will have proved the theorem for $\mathcal{F}$ as well. So we can assume that $a, b, c = 0, 1, \infty$. We will find a uniform bound on $f^\sharp$ over K which works for any function f which omits the values $0, 1, \infty$, as this includes all the functions in $\mathcal{F}$. So let f be such a function. Choose $\epsilon > 0$ such that $K_\epsilon \subset U$. Fix $z_0 \in K$. Without loss of generality we can assume that $z_0 = 0$ (by applying a suitable Möbius isometry to the domain of f). Let i_0 be the injection of $\mathbb{C}$ into $\mathbb{C}_\infty$ given by equation (1). Since f omits the value ∞, we can consider the map $j_0 \circ (f \mid B_\sigma(0, \epsilon)) \circ i_0$. Let $\hat{f}$ be this map restricted to $B(0, \epsilon)$. (Note that $i_0(B(0, \epsilon)) \subset B_\sigma(0, \epsilon)$.) From equation (2), we see that

$$f^\sharp(0) = |\hat{f}'(0)| \left(1 + |\hat{f}(0)|^2\right).$$

We can assume that $|\hat{f}(0)| < 2$, because if necessary we can replace f with $1/f$ which also omits $0, 1$. So our problem is reduced to that of finding an upper bound for $|\hat{f}'(0)|$ which works for all z_0 and $f \in \mathcal{F}$. Consider now the map $g : B(0, 1) \to \mathbb{C}$ given by $g(z) = \hat{f}(\epsilon z)$. Then $g'(0) = \epsilon \hat{f}'(0)$. So we just need to find an upper bound for $|g'(0)|$. Now g is an analytic function on $B(0, 1)$ which omits the values 0 and 1 – otherwise known as a Picard function – so by the constructive version of Schottky's theorem (Theorem 5.6.19, [Bishop and Bridges 1985]), g satisfies

$$|g(z)| \leq \Phi(\alpha, |z|) \quad (z \in B(0, 1))$$

so long as $|g(0)| < \alpha$, where

$$\Phi(\alpha, r) = 1 + \exp(2^{18} e^{18} (\alpha + 3)^8 (1 + r^2)^4 (1 - r)^{-8}). \tag{11}$$

But $|g(0)| = |\hat{f}(0)| < 2$, so by combining the above with Cauchy's inequalities on $\mathbb{C}$ (5.4.13.1, [Bishop and Bridges 1985]), we obtain

$$g'(0) \leq 2 \, \|g\|_{S(0,1/2)} \leq 2 \, \Phi(2, 1/2).$$

This bound on $g'(0)$ in turn gives us a bound of $10\epsilon^{-1} \Phi(2, 1/2)$ on $f^\sharp(z_0)$. Since z_0 is a general point in K, and f a general function on U omitting $0, 1$ and ∞, we have proven normality of $\mathcal{F}$ on K. Since K was a general compact set well-inside U, we have shown that $\mathcal{F}$ is normal on U.

Theorem 10. *Let $\mathcal{F}$ be an abnormal family of functions on an open set $U \subset \mathbb{C}_\infty$, and let a_1, a_2, a_3 be three distinct points in $\mathbb{C}_\infty$. Then there is $f \in \mathcal{F}$, and $z \in U$ such that $f(z) \in \{a_1, a_2, a_3\}$.*

Proof. Let $K \Subset U$ be such that $\mathcal{F}$ is abnormal on K and choose $r > 0$ such that $K_r \subset U$. Then for any $f \in \mathcal{F}$ and any a_i, either there is $z \in K_r$ such that $f(z) = a_i$ or $\sigma(f(K_{r/2}), a_i) > 0$. (This can be derived from Theorem 1, which implies that there is a $\delta > 0$ such that $f(K)_\delta \subset f(K_{r/2})$.) If the latter case holds for each of a_1, a_2, a_3, then f belongs to the class of functions which omit these three values on the open set $K_{<r/2}$. This class is normal by Theorem 9, so there is a uniform bound k on $f^\sharp(z)$ for all $z \in K$. But since $\mathcal{F}$ is abnormal on K, we can choose $f \in \mathcal{F}$ which exceeds this bound. For this f we obtain a contradiction if the above case $\sigma(f(K_{r/2}), a_i) > 0$ is satisfied for $i = 1, 2, 3$, so for some i we must have $z \in K_r \subset U$ such that $f(z) = a_i$, as desired.

Theorem 11. *There exists a function $k_0 : \mathbb{R}^+ \times (0,1) \to \mathbb{R}^+$ such that for any $m > 0$ and $r \in (0,1)$, for any three points $a_1, a_2, a_3 \in \mathbb{C}_\infty$ satisfying $\sigma(a_1, a_2, a_3) \geq m$, and for any analytic f on $B_\sigma(0,r)$ satisfying $f^\sharp(0) \geq k_0(r,m)$, f takes one of the values a_1, a_2, a_3 on $B_\sigma(0,r)$.*

Proof. First we prove that there is a limit k_1 so that any f defined on $\overline{B}_\sigma(0,1/2)$ with $f^\sharp(0) > k_1$ will attain one of the values $0, 1, \infty$ on $B_\sigma(0,1/2)$. This is not difficult: for any $z \in \mathbb{C}_\infty$ either f attains z on $B_\sigma(0,1/2)$ or $\sigma(f(\overline{B}_\sigma(0,1/4)), z) > 0$ (by Theorem 1 – see proof of Theorem 10 above), but if the latter holds for each value of z from $\{0, 1, \infty\}$ then f belongs to a normal class of functions on $B(0, r/2)$ by Theorem 9 and so there is a bound k_1 on $f^\sharp(0)$ for all f in this class. If f exceeds this bound then the only remaining option is that f attains one of $0, 1, \infty$.

Now fix $m > 0$ and $r \in (0,1)$. Let us define the scaling factor

$$\theta(r) = r(1 - r^2)^{-\frac{1}{2}}.$$

This function is chosen so that j_0 maps $B_\sigma(0,r)$ to $B(0, \theta(r))$. Then we can define a scaling function $g : B_\sigma(0,1/2) \to B_\sigma(0,r)$ by

$$g(z) = i_0(\theta(r)\, \theta(1/2)^{-1}\, j_0(z)).$$

Define the function k_0 by

$$k_0(r,m) = \pi m^{-3}\, \theta(1/2)\, \theta(r)^{-1} k_1.$$

Suppose that f is an analytic function on $B_\sigma(0,r)$ such that $f^\sharp(0) > k_0(r,m)$. Let $a_1, a_2, a_3 \in \mathbb{C}_\infty$ be such that $\sigma(a_1, a_2, a_3) > m$. Let h be the unique Möbius function taking (a_1, a_2, a_3) to $(0, 1, \infty)$. Consider the function $F = h \circ f \circ g$. Then F is analytic on $B(0,1/2)$ and $F^\sharp(0) > k_1$ by Lemma 4 so F attains one of the values $0, 1, \infty$, whence f attains one of the values a_1, a_2, a_3, as desired.

4 Generalising Montel's Theorem

Theorem 12. *Let U be an open set in $\mathbb{C}_\infty$ and let $\varphi_1, \varphi_2, \varphi_3$ be analytic functions on U such that*

$$\varphi_i(z) \neq \varphi_j(z) \quad (z \in K, 1 \leq i < j \leq 3). \tag{12}$$

Let $\mathcal{F}$ be a family of analytic functions on U, such that for every $z \in U$, every $f \in \mathcal{F}$ and every $i \in \{1, 2, 3\}$, we have $f(z) \neq \varphi_i(z)$. Then $\mathcal{F}$ is normal in U.

Theorem 13. *Let U be open and let $\varphi_1, \varphi_2, \varphi_3$ be analytic functions on U satisfying equation (12) above. Let $\mathcal{F}$ be an abnormal family of analytic functions on U. Then there is $f \in \mathcal{F}$, $i \in \{1, 2, 3\}$ and $z \in U$ such that $f(z) = \varphi_i(z)$.*

In proving these theorems we shall make use of the following:

Proposition 14. *Let φ be an analytic function on an open set $U \subset \mathbb{C}_\infty$, and suppose $\varphi(z) \neq 0$ for all $z \in U$. Let $K \Subset U$. Then $m(\varphi, K) > 0$.*

Proof. Use Theorem 1. Since $K \Subset U$ we have $\varphi(K) \Subset \varphi(U)$ and $0 \notin \varphi(U)$, from which the result follows.

We state the following without proof.

Lemma 15. *Let $r > 0$. Then for all $\epsilon > 0$ there is $\delta \in (0, r/2)$ such that for any $a_1, a_2, a_3 \in \mathbb{C}_\infty$ with $\sigma(a_1, a_2, a_3) \geq r$ and any $b_1, b_2, b_3 \in \mathbb{C}_\infty$ with $\sigma(a_i, b_i) \leq \delta$ $(1 \leq i \leq 3)$, if H is the unique Möbius transformation taking a_i to b_i then $\sigma(H(z), z) \leq \epsilon$ for all $z \in \mathbb{C}_\infty$.*

Lemma 16. *Let K be a compact set in $\mathbb{C}_\infty$, and let $\varphi_1, \varphi_2, \varphi_3$ be differentiable functions on K. Let $r > 0$ be such that*

$$\sigma(\varphi_1(z), \varphi_2(z), \varphi_3(z)) \geq r \qquad (z \in K). \tag{13}$$

For $w, w' \in K$, let $H_{w,w'}$ be the unique Möbius transform which maps $\varphi_i(w)$ to $\varphi_i(w')$. Then the limit

$$L(z, w) = \lim_{w' \to w} \frac{\sigma(H_{w,w'}(z), z)}{\sigma(w, w')} \qquad (w \in K, z \in \mathbb{C}_\infty)$$

exists everywhere and there is a bound B such that $L(z, w) < B$ for all $z \in \mathbb{C}_\infty$ and $w \in K$.

Proof. We shall compute $L(z, w)$. To do this we need to reduce to cases where z, $H_{w,w'}(z)$ and $\varphi_i(w)$ are bounded away from ∞, so that we can map back to $\mathbb{C}$. Let μ be a Möbius isometry of $\mathbb{C}_\infty$. Then $\mu \circ H_{w,w'} \circ \mu^{-1}$ is the Möbius transformation taking $\mu \circ \varphi_i(w)$ to $\mu \circ \varphi_i(w')$, and

$$\sigma(\mu \circ H_{w,w'} \circ \mu^{-1}(\mu(z)), \mu(z)) = \sigma(H_{w,w'}(z), z).$$

We can break down K into finitely many smaller compact sets K_i such that $\varphi_k(K_i)$ is in each case of σ-diameter less than some small constant $h > 0$. We can also break down $\mathbb{C}_\infty$ into compact L_j each of diameter less than h. If h is small enough then for each i, j there will be a point $\zeta_{ij} \in \mathbb{C}_\infty$ such that $\sigma(\zeta_{ij}, L_j) \geq h$ and $\sigma(\zeta_{ij}, \varphi_k(K_i)) \geq h$ for all i, j, k. Then for each i, j we can apply a Möbius isometry to take ζ_{ij} to ∞. So we shall only need to prove our result for $w \in K$ and $z \in L$ where we can assume that $\sigma(\infty, L) > 0$ and $\sigma(\infty, \varphi_i(K)) > 0$. Similarly, we can also assume that $\sigma(\infty, K) > 0$. Finally, we can assume that there is some $R > 0$ such that $\sigma(\infty, H_{w,w'}(L)) \geq R$ for all $w, w' \in K$, by applying Lemma 15 (and breaking down K, L into smaller compact sets if necessary). With all of these assumptions in place, we are now ready to compute the limit $L(z, w)$. Let $\hat{\varphi}_i = i_0 \circ \varphi_i \circ j_0$. and define $\hat{H}_{w,w'}$ similarly (for $w', w \in K$). Since $z = H_{w,w}(z)$, we can see that $L(z, w)$ is the spherical derivative - should this be shown to exist - of the function $G_{z,w} : w' \mapsto H_{w,w'}(z)$. Since all of our values are safely bounded away from ∞, it is sufficient to find a uniform bound on

$$L_2(\zeta, w) = \lim_{w' \to w} \frac{|\hat{H}_{w,w'}(\zeta) - \zeta|}{|j_0(w) - j_0(w')|} \qquad (w \in K, \zeta \in j_0(L)) \tag{14}$$

We can now compute $\hat{H}_{w,w'}$. It is of the form

$$\hat{H}_{w,w'}(\zeta) = \frac{A\zeta + B}{C\zeta + D}$$

where A, B, C, D are constant polynomials in the six values $\hat{\varphi}_i(w), \hat{\varphi}_i(w')$ ($1 \leq i \leq 3$). Let $h = j_0(w) - j_0(w')$. Then we need to calculate the limit of

$$h^{-1}((A\zeta + B/C\zeta + D) - \zeta) = h^{-1}(C\zeta + D)^{-1}(-C\zeta^2 + (A - D)\zeta + B)$$

as $h \to 0$. As the reader can verify, these polynomials are of forms $A = A_0 + A_1$ (and similar for B, C, D) such that A_0 etc. are polynomials in $\hat{\varphi}_i(w)$ and A_1 etc. are of a symmetrical form so that

$$\lim_{w' \to w} h^{-1} A_1 = A_2$$

where A_2 is a polynomial in $\hat{\varphi}_i(w), \hat{\varphi}_i'(w)$ (and similar for B, C, D). The polynomials A_2 etc. are quite complicated but the values A_0 etc. come out to

$$A_0 = D_0 = (\hat{\varphi}_1(w) - \hat{\varphi}_2(w))(\hat{\varphi}_2(w) - \hat{\varphi}_3(w))(\hat{\varphi}_3(w) - \hat{\varphi}_1(w))$$

and $B_0 = C_0 = 0$. When we substitute these identities into equation (14), we find that the limit L_2 exists (since it is the absolute value of a limit we can calculate), and that

$$L_2(\zeta, w) = |D_0^{-1}(-C_2\zeta^2 + (A_2 - D_2)\zeta + B_2)|.$$

We can now very easily obtain a bound for this value for all ζ, w from the fact that there are bounds on $\hat{\varphi}_i(w), \hat{\varphi}_i'(w)$ (as these are analytic functions on a compact domain in $\mathbb{C}$) and $|\zeta|$, and the fact that $|D_0|$ is bounded away from 0 because we are assuming that $\sigma(\varphi_i(w), \varphi_j(w)) \geq r$ for all $1 \leq i < j \leq 3$ and $w \in K$.

Proof of Theorem 12. For each $w \in U$, define a Möbius transformation g_w by the conditions

$$g_w(0) = \varphi_1(w) \qquad g_w(1) = \varphi_2(w) \qquad g_w(\infty) = \varphi_3(w) \tag{15}$$

For each $f \in \mathcal{F}$, define a new function G_f by

$$G_f(w) = g_w^{-1}(f(w)) \tag{16}$$

The reader can verify that G_f is analytic and that G_f omits the values $0, 1, \infty$ on U. Therefore by Theorem 9, $\{G_f : f \in \mathcal{F}\}$ is normal. Now suppose that $K \Subset U$. By Proposition 14 there exists $r > 0$ such that

$$\sigma(\varphi_i(z), \varphi_j(z)) \geq r \qquad (z \in K, 1 \leq i < j \leq 3)$$

If we take the spherical derivative of G_f we find

$${G_f}^\sharp(w) = f^\sharp(w)(g_w^{-1})^\sharp(f(w)) + S(f, w) \tag{17}$$

where

$$S(f, w) = \lim_{w' \to w} \frac{\sigma(g_w^{-1}(f(w)), g_{w'}^{-1}(f(w)))}{\sigma(w, w')}$$

By Lemma 4,

$$r^3/\pi \leq (g_w^{-1})^\sharp(z) \leq \pi/r^3 \qquad (z \in K). \tag{18}$$

We wish to show that there is a bound B on $S(f, w)$ which does not depend on $f \in \mathcal{F}$ or $w \in K$. In fact we will find a bound on

$$T(z, w) = \lim_{w' \to w} \frac{\sigma(g_w^{-1}(z), g_{w'}^{-1}(z))}{\sigma(w, w')} \qquad (w \in K, z \in \mathbb{C}_\infty)$$

Applying Lemm 3 to g_w, we find that it is sufficient to find a bound on

$$U(z,w) = \lim_{w'\to w} \frac{\sigma((g_{w'} \circ g_w^{-1})(z), z)}{\sigma(w,w')} \qquad (w \in K, z \in \mathbb{C}_\infty)$$

Let $H_{w,w'} = g_{w'} \circ g_w^{-1}$. Then $H_{w,w'}$ is a Möbius transformation mapping each $\varphi_i(w)$ to $\varphi_i(w')$. Thus Lemma 16 gives a bound on U. If we look again at equation (17) we see that L is bounded and $(g_w^{-1})^\sharp(f(w))$ is bounded away from 0. Therefore since $G_f^\sharp$ is bounded on K by normality, $f^\sharp$ must also be bounded on K. So $\mathcal{F}$ is normal.

Proof of Theorem 13. The proof follows that of Theorem 12. This time we know that $\mathcal{F}$ is abnormal so that $\left\{f^\sharp \,:\, f \in \mathcal{F}\right\}$ is unbounded on some compact $K \Subset U$. By the same argument as in the proof of Theorem 12, $\{G_f^\sharp \,:\, f \in \mathcal{F}\}$ must also be unbounded on K. (Here we use boundedness of $S(f,w)$ from the proof of Theorem 12 and the fact that $(g_w^{-1})^\sharp(f(w))$ is bounded *above* by equation (18).) Therefore $\{G_f \,:\, f \in \mathcal{F}\}$ is abnormal and so by Theorem 10 there is $f \in \mathcal{F}$ and $z \in U$ such that $G_f(z) \in \{0,1,\infty\}$. But then $f(z) \in \{\varphi_1(z), \varphi_2(z), \varphi_3(z)\}$, as required.

Theorem 17. *Let $\mathcal{F}$ be a class of functions which is abnormal on an open set U. Let $m > 0$, and suppose that for each $f \in \mathcal{F}$ and $i \in \{1,2,3\}$ there is $a_{i,f}$ such that $\sigma(a_{1,f}, a_{2,f}, a_{3,f}) \geq m$ for all $f \in \mathcal{F}$. Then there is an $f \in \mathcal{F}$, $i \in \{1,2,3\}$ and $z \in U$ such that f(z) $= a_{i,f}$.*

Proof. For each $f \in \mathcal{F}$ let h_f be the unique Möbius transformation taking $(a_{1,f}, a_{2,f}, a_{3,f})$ to $(0,1,\infty)$, and let $G_f = h_f \circ f$. Then by Lemma 5, $\{G_f \,:\, f \in \mathcal{F}\}$ is also abnormal on U and so by Theorem 10 there is $z \in U$ and $f \in \mathcal{F}$ such that $G_f(z) \in \{0,1,\infty\}$, whence $f(z) \in \{a_{i,f}\}$.

Theorem 18. *Suppose that $\mathcal{F}$ is a class of functions on an open set $U \subset \mathbb{C}_\infty$, and that $\mathcal{F}$ is abnormal at $z_0 \in U$. Suppose also that K_1, K_2, K_3 are compact sets in $\mathbb{C}_\infty$ such that $\sigma(K_1, K_2, K_3) > 0$. Then there exist $i \in \{1,2,3\}$ and $f \in \mathcal{F}$ such that $K_i \subset f(U)$.*

Proof. By countable choice we can assume that $\mathcal{F}$ is countable: just choose $f_{m,n} \in \mathcal{F}$ such that $f_{m,n}^\sharp$ exceeds m on $\overline{B}_\sigma(z_0, 2^{-n})$, and then replace $\mathcal{F}$ by $\{f_{m,n} \,:\, m,n \in \mathbb{Z}^+\}$. Choose $r > 0$ such that $B_\sigma(z_0, r) \subset U$. Each $f \in \mathcal{F}$ maps the compact $\overline{B}_\sigma(z_0, r/2)$ to a compact set J_f and the open $B_\sigma(z_0, r)$ to an open U_f such that $J_f \Subset U_f$, by Theorem 1. So we can choose $r(f) > 0$ such that $(J_f)_{r(f)} \subset U_f$. For each $i \in \{1,2,3\}$, consider

$$\mu_i(f) = \sup\left\{\sigma(z, J_f) \,:\, z \in K_i\right\}.$$

Either $\mu_i(f) > 0$ or $\mu_i(f) < r(f)$. In the former case choose $a_{i,f} \in K_i$ so that $\sigma(a_{i,f}, J_f) > 0$. Otherwise choose $a_{i,f} \in K_i$ such that $\sigma(a_{i,f}, J_f) < r(f)$. (This requires countable choice, which is why we made sure $\mathcal{F}$ is countable.) By Theorem 17 above, since $\mathcal{F}$ is abnormal on $B(z_0, r/2)$, there are $f \in \mathcal{F}$, $i \in \{1, 2, 3\}$ and $w \in B_\sigma(z_0, r/2)$ such that $f(w) = a_{i,f}$. Then $\sigma(a_{i,f}, J_f) \not> 0$, so we must have $\mu_i(f) < r(f)$. Therefore $K_i \subset (J_f)_{r(f)} \subset U_f = f(B_\sigma(z_0, r)) \subset f(U)$.

5 Julia sets and Fatou sets

Let R be a rational map on $\mathbb{C}_\infty$. Consider the family of iterations of this function: $\mathcal{F} = \{R^n : n \in \mathbb{Z}^+\}$. The Julia set of R is defined to be the set of points $z \in \mathbb{C}_\infty$ at which $\mathcal{F}$ is abnormal. Similarly the Fatou set of R is defined to be the set of points $z \in \mathbb{C}_\infty$ at which $\mathcal{F}$ is normal. These sets are denoted by $J(R)$ and $F(R)$ respectively. It follows immediately from the definitions of local normality and abnormality that $J(R)$ is closed and $F(R)$ is open and that $J(R) \cap F(R) = \emptyset$.

Given a map R we say that a set X is **forward invariant** under R if $R(X) \subset X$ and **backward invariant** under R if $R^{-1}(X) \subset X$. We also say that X is **completely invariant** under R if it is both forward and backward invariant (that is, if $z \in X \Leftrightarrow R(z) \in X$).

Theorem 19. *Let R be a rational map on $\mathbb{C}_\infty$. Then $J(R)$ and $F(R)$ are completely invariant under R.*

Proof. The proof is left as an exercise for the reader: apply Theorem 8 several times.

Theorem 20. *Let R be a rational map on $\mathbb{C}_\infty$ of degree ≥ 2, and let $p \in \mathbb{Z}^+$. Then $J(R) = J(R^p)$ and $F(R) = F(R^p)$.*

Proof. Omitted. Apply Theorem 8 in each case.

Lemma 21. *Let R be a rational map on $\mathbb{C}_\infty$ of degree $d \geq 2$. Let $\lambda > 0$. Then there are $\lambda_1, \lambda_2 \in (0, \lambda)$ such that for all $z_0 \in \mathbb{C}_\infty$ one of the following conditions holds:*

1. *given any enumeration $z_1, \ldots, z_d$ of the predecessors of z_0 under R, there is $k \in \{1, 2, \ldots, d\}$ such that $\sigma(z_0, z_i) \geq \lambda_1$ for all $i \in \{1, 2, \ldots, d\}$ s.t. $i \neq k$*
2. *$R(B_\sigma(z_0, \lambda_2)) \subset B_\sigma(z_0, \lambda_2)$.*

Proof. Let δ be a modulus of continuity for R on $\mathbb{C}_\infty$. Let $r = \delta(1/2)/4$. Let $K = 3^{-\frac{1}{2}}$. Suppose that $\lambda > 0$ is given and define

$$s := \min\left\{\lambda, r, \tfrac{1}{4}(K^{-1}\, r^2)\right\}$$

Let Δ be the function defined in equation (4). Let $\lambda_1 = \frac{1}{2}\Delta(0,s)$. Let $\lambda_2 = \Delta(0,s)$. Note that $\lambda_1, \lambda_2 < s \leq \lambda$.

Now fix $z_0 \in \mathbb{C}_\infty$. Without loss of generality we can assume that $z_0 = 0$, because we can conjugate R by a Möbius isometry which takes z_0 to 0. Let $z_1, \ldots, z_d$ be the R-predecessors of $z_0 = 0$. For each $1 \leq i \leq d$, either $\sigma(0, z_i) \geq \lambda_1$ or $\sigma(0, z_i) \leq \Delta(0,s)$. If the former holds for all but one i then we are done, so we may assume that the latter holds for two values of i, which we can take to be 1 and 2. It follows that $|z_1| \leq s$ and $|z_2| \leq s$. Now let $\hat{R} = j_0 \circ R \circ i_0$, defined on $B(0,s)$. We prove for all $z \in B(0,s)$ that $R(z) \in B(0,s)$. Then when we change back to the $\mathbb{C}_\infty$-metric, we will immediately get $R(B_\sigma(z_0, \lambda_2)) \subset B_\sigma(z_0, \lambda_2)$, as desired.

Given any $w \in \overline{B}(0, 3r)$ we have $\sigma(0,w) \leq |w| \leq 3r$ so $\sigma(w, z_1) \leq 4r = \delta(\frac{1}{2})$ so $\sigma(\hat{R}(w), 0) \leq \frac{1}{2}$, which in turn means that $|\hat{R}(w)| \leq 3^{-\frac{1}{2}} = K$. So in particular, $\hat{R}$ is an analytic function on $\overline{B}(0,3r)$, where it can be represented as

$$\hat{R}(z) = g(z)(z - z_1)(z - z_2) \qquad (z \in \overline{B}(0, 3r))$$

for some g analytic on $\overline{B}(0,3r)$. Also, $\|\hat{R}\|_{\overline{B}(0,3r)} \leq K$. Now suppose that $w \in S(0, 2r)$. we have $|z_i| \leq s \leq r$ $(i = 1, 2)$, so $|w - z_i| \geq r$, so $|g(w)| \leq Kr^{-d}$. By the maximum principle (Theorem 5.5.2, [Bishop and Bridges 1985]), it follows that

$$\|g\|_{\overline{B}(0,2r)} \leq Kr^{-2}.$$

Now, suppose that z is any point in $B(0,s)$. Then $z \in \overline{B}(0, 2r)$ and so

$$\begin{aligned}|\hat{R}(z)| &= |g(z)| \cdot |z - z_1| \cdot |z - z_2| \\ &< K\, r^{-2}\, (2s)^2 \\ &\leq s.\end{aligned}$$

Which completes the proof.

Given a rational map R, define an R**-chain** to be a sequence $(z_1, \ldots, z_n)$ of points such that $R(z_i) = z_{i+1}$ $(1 \leq i \leq n-1)$.

Lemma 22. *Let R be a rational map of degree $d \geq 2$. Let $\lambda > 0$ and $n \in \mathbb{Z}^+$. Then there exist $\lambda_1 \in (0, \lambda)$ such that for every $z_0 \in \mathbb{C}_\infty$ one of the following conditions holds:*

1. *There is an R-chain $(z_{-n}, \ldots, z_{-1}, z_0)$ such that $\sigma(z_i, z_j) \geq \lambda_1$ for all i, j such that $-n \leq i < j \leq 0$.*
2. *There is $k \in \mathbb{Z}^+$ such that $k \leq n$ and a neighbourhood U of z_0 such that $U \subset B_\sigma(z_0, \lambda)$ and $R^k(U) \subset U$.*

Proof. We proceed by induction on n. The $n = 1$ case follows immediately from Lemma 21. Suppose that our result holds for some $n \in \mathbb{Z}^+$. Let R be a rational map of degree ≥ 2 and let $\lambda > 0$. Apply the induction hypothesis to the value λ to obtain a constant $\lambda_1' \in (0, \lambda)$ such that for every $z_0 \in \mathbb{C}_\infty$, either case (1) or case (2) holds for the value n. Now apply Lemma 21 to the function R^{n+1} and the value λ. Let $\lambda_1'', \lambda_2'' \in (0, \lambda)$ be the values thereby derived. Let

$$\lambda_1 = \min\{\lambda_1', \delta(\lambda_1'/2), \lambda_1''\},$$

where δ is the modulus of continuity of R on $\mathbb{C}_\infty$. We wish to prove that this constant λ_1 satisfies the $n+1$ case of the present lemma. So fix $z_0 \in \mathbb{C}_\infty$. By the choice of λ_1' above (induction hypothesis applied to λ) we obtain that either case (1) or case (2) of the present lemma holds with regard to z_0. If case (2) holds, then we are done, because the value k will do just as well for $n + 1$ as for n. So we may suppose that case (1) holds. Let $(z_{-n}, \dots, z_0)$ be the resulting R-chain. Thus $\sigma(z_i, z_j) \geq \lambda_1'$ whenever $-n \leq i < j \leq 0$. By choice of constants λ_1'' and λ_2'', either case (1) or case (2) of lemma 21 holds with regard to z_0 (and the function R^{n+1}). If case (2) holds, then we are done: let $k = n + 1$ and $U = B_\sigma(z_0, \lambda_2'')$ to get case (2) of the present lemma. So suppose that case (1) of Lemma 21 holds. The R-predecessors of z_{-n} (of which there are $d \geq 2$) form a subsequence of the R^{n+1}-predecessors of z_0, so we can choose an R-prececessor z_{-n-1} of z_{-n} so that $\sigma(z_{-n-1}, z_0) \geq \lambda_1''$. It remains to prove that $\sigma(z_i, z_j) \geq \lambda_1$ whenever $-n-1 \leq i < j \leq 0$. If $i = -n-1$ and $j = 0$ then result follows from choice of z_{-n-1}. If $i > -n-1$ then result follows from the induction hypothesis (since $\lambda_1 \leq \lambda_1'$). So we may suppose that $i = -n-1$ and $j < 0$. Therefore the induction hypothesis gives $\sigma(z_{i+1}, z_{j+1}) \geq \lambda_1'$. But then $\sigma(z_i, z_j) \geq \delta(\lambda_1'/2) \geq \lambda_1$, so our proof is complete.

Lemma 23. *Let R be a rational map of degree ≥ 2, and suppose that W is an open set which meets $J(R)$. Then there exists $n \in \mathbb{Z}^+$ such that*

$$J(R) \subset R^n(W) \cup R^{n+1}(W) \cup R^{n+2}(W) \tag{19}$$

Proof. Suppose that $w_0 \in J(R) \cap W$ and that $B_\sigma(w_0, 2r) \subset W$. Apply lemma 22 to the rational function R with $\lambda = 1/2$ and $n = 2$. Let λ_1 be the constant so obtained. The set $\{R^n : n \in \mathbb{Z}^+\}$ is an abnormal family on $\overline{B}_\sigma(w_0, r)$ and so there is some $z \in \overline{B}_\sigma(w_0, r)$ and $n \in \mathbb{Z}^+$ such that $(R^n)^\sharp(z) > k_0(r, \lambda_1)$, where k_0 is the function constructed in Theorem 11. We prove that this n satisfies equation (19).

Let $z_0 \in J(R)$. By choice of λ_1, either case (1) or case (2) of Lemma 22 holds. But case (2) is not possible: for suppose there is a neighbourhood U of z_0 such that $U \subset B_\sigma(z_0, 1/2)$ and $R^k(U) \subset U$ for some $k \in \{1, 2\}$. Then the family $\{R^{km} : m \in \mathbb{Z}^+\}$ is normal on U by Theorem 9 (since we can choose three distinct points lying outside $B_\sigma(z_0, 1/2)$, and since $R^{km}(U) \subset U$ for all $k \in \mathbb{Z}^+$), and therefore $z_0 \in F(R^k) = F(R)$, contradicting our initial assumption that $z_0 \in J(R)$. So we must have case (1): there is an R-chain (z_{-2}, z_{-1}, z_0) such that $\sigma(z_i, z_j) \geq \lambda_1$ whenever $-2 \leq i < j \leq 0$. By Theorem 11, there is $\zeta \in B_\sigma(z, r)$ such that $R^n(\zeta) = z_i$ for some $i \in \{-2, -1, 0\}$. Thus

$$z_0 \in R^{n-i}(B_\sigma(w_0, 2r)) \subset R^{n-i}(W).$$

Since z_0 was a general element of $J(R)$, we have shown that $J(R) \subset R^n(W) \cup R^{n+1}(W) \cup R^{n+2}(W)$, as desired.

Theorem 24. *Let R be a rational map of degree ≥ 2. Then every point in $J(R)$ is a limit point of $J(R)$.*

Proof. Fix $z_0 \in J(R)$. Given any open neighbourhood W of z_0, we know from Lemma 23 that there exists $n \in \mathbb{Z}^+$ such that $J(R) \subset R^n(W) \cup R^{n+1}(W) \cup R^{n+2}(W)$. Apply lemma 22 to the function R and the constants $\lambda = 1/2$ and $n = 3$. For the point z_0, case (2) cannot hold: for suppose that U is a neighbourhood of z_0 such that $U \subset B_\sigma(z_0, 1/2)$ and $R^k(U) \subset U$ for some $k \in \{1, 2, 3\}$. Then clearly $R^{km}(U) \subset U$ for all $m \in \mathbb{Z}^+$, and so by Theorem 9 (since we can choose three distinct points outside $U \subset B_\sigma(z_0, 1/2)$, we must have $z_0 \in F(R^k) = F(R)$, contradicting our assumption that $z_0 \in J(R)$.

Therefore we must have case (1) of Lemma 22: there is an R-chain $(z_{-3}, \ldots, z_0)$ such that $z_i \neq z_j$ whenever $-3 \leq i < j \leq 0$. These four points all lie in $J(R)$ by theorem 19 and so by the pigeonhole principle there must be i, j, k with $-3 \leq i < j \leq 0$ and $0 \leq k \leq 2$ such that $z_i, z_j \in R^{n+k}(W)$. Therefore there are $w_1, w_2 \in W$ such that $R^{n+k}(w_1) = z_i$ and $R^{n+k}(w_2) = z_j$. Since R^{n+k} is continuous, $w_1 \neq w_2$. By Theorem 19, $w_1, w_2 \in J(R)$. Either $z_0 \neq w_1$ or $z_0 \neq w_2$. Either way, there is a point in $J(R) \cap W$ other than z_0. Since W was an arbitrary neighbourhood of z_0, we have shown that z_0 is not isolated in $J(R)$.

We need a very small lemma before proving the next theorem.

Lemma 25. *Let $n \in \mathbb{Z}^+$ and let $S_n = \{1, 2, \ldots, n\}$. Suppose that $f : S_n \to S_n$. Then there is $x \in S_n$ and $N \in \mathbb{Z}^+$ such that $f^N(x) = x$.*

Proof. Apply the pigeonhole principle to $\{1, f(1), \ldots, f^n(1)\}$.

Theorem 26. *Let R be a rational map of degree ≥ 2 and suppose that W is an open set which meets $J(R)$. Then there is an $n \in \mathbb{Z}^+$ such that $J(R) \subset R^n(W)$.*

Proof. Suppose that W is an open set meeting $J(R)$ at w. We know from theorem 24 that w is a limit point of $J(R)$ and so we can choose three distinct points $w_1, w_2, w_3 \in W \cap J(R)$. Choose a positive $r \leq \sigma(w_1, w_2, w_3)$ such that

$B_\sigma(w_i, r) \subset W$. Then the compact sets $K_i := \overline{B}_\sigma(w_i, r/3) \Subset W$ have chordal distance $\geq r/3$ from one another and so we can apply Theorem 18 to each of the sets K_i to obtain $k(i) \in \{1, 2, 3\}$ and $n(i) \in \mathbb{Z}^+$ such that $K_{k(i)} \subset R^{n(i)}(K_i)$. By Lemma 25 there is $l \in \{1, 2, 3\}$ and $N \in \mathbb{Z}^+$ such that $K_l \subset R^N(K_l)$. We can then prove by induction that $R^{mN}(K_l) \subset R^{(m+1)N}(K_l)$ for all $m \in \mathbb{N}$. Since $w_l \in J(R) = J(R^N)$, by the previous theorem there is $n \in \mathbb{Z}^+$ such that

$$J(R) = J(R^N) \subset R^{Nn}(K_l) \cup R^{N(n+1)}(K_l) \cup R^{N(n+2)}(K_l) = R^{N(n+2)}(K_l).$$

and therefore $J(R) \subset R^{N(n+2)}(W)$.

6 Critical points

Let R be a rational map on $\mathbb{C}_\infty$. We say that $z \in \mathbb{C}_\infty$ is a **critical point** of R if $R^\sharp(z) = 0$. We outline some important facts about critical points which we will need later when we come to construct repelling cycles.

Theorem 27. *Counting multiplicities, there are $2d - 2$ critical points of any rational map R, where d is the degree of R.*

Proof. We omit the proof, as it is sufficiently like the classical version: see the proof of Theorem 2.7.1, [Beardon 1991].

Proposition 28. *Let f be an analytic function on an open set $U \subset \mathbb{C}_\infty$ and $B_\sigma(w_0, r_0) \subset f(U)$. Suppose that $z_0 \in U$ is such that $f(z_0) = w_0$. Suppose also that for any compact $K \Subset B_\sigma(w_0, r_0)$ there exists a compact $L \Subset U$ such that $f^{-1}(K) \subset L$. Suppose finally that $f^\sharp$ is nonzero on U. Then there is a unique analytic function $g : B(w_0, r_0) \to U$ such that $g(w_0) = z_0$ and $f \circ g = \mathrm{id}_{B_\sigma(w_0, r_0)}$.*

Proof. The proof is omitted for reasons of space. This proposition is a generalisation of the local inverse function theorem on $\mathbb{C}$, which is proved much as in the classical case. See e.g. [Gamelin 2001], p. 234.

Theorem 29. *Let R be a rational map on $\mathbb{C}_\infty$ of degree ≥ 2. Let z_0 be an attracting fixed point of R. Then there is a critical point c of R such that $R^n(c) \to z_0$ as $n \to \infty$.*

Proof. Choose $r > 0$ and $\theta \in (0, 1)$ such that

$$z \in B_\sigma(z_0, r) \Rightarrow \sigma(R(z), z_0) \leq \theta \cdot \sigma(z, z_0).$$

(This can be done by considering a Taylor expansion of f local to z_0, considered as an analytic $\mathbb{C}$-function.) Let c be a critical point of R. Apply Lemma 22 to R with $\lambda = r/2$ and $n = 2$ to obtain $\lambda_1 \in (0, \lambda)$. So either there is an R-chain

(z_1, z_2, c) such that $\sigma(z_1, z_2, c) \geq \lambda_1$ or there is a neighbourhood U of c such that $U \subset B_\sigma(c, r/2)$ and $R^k(U) \subset U$ for some $k \in \{1, 2\}$. In the latter case choose three distinct points $a_1, a_2, a_3 \in U$ and let $\lambda_1 = \sigma(a_1, a_2, a_3)$. (So we define the constant λ_1 differently according to which of these two cases hold.) Let k_0 be the function given in Theorem 11 and choose $n \in \mathbb{Z}^+$ such that $R^\sharp(z_0)^n < (1 + k_0(r/2, \lambda_1))^{-1}$. We shall prove that either

(i) there is an analytic inverse S to R^n definable on $B_\sigma(z_0, r/2)$, or

(ii) there is a critical point c_0 of R^n such that $R^n(c_0) \in B_\sigma(z_0, r)$.

Suppose that (i) holds. Then $S^\sharp(z_0) = (R^\sharp(z_0))^{-n} > k_0(r/2, \lambda_1)$ by choice of n, and so by Theorem 11 there is a point $z \in B_\sigma(z_0, r/2)$ such that $S(z)$ is one of the three distinct points z_1, z_2, c or one of the three distinct points a_1, a_2, a_3, depending on how λ_1 was defined above. In the former case, $R^{n-j}(c) \in B_\sigma(z_0, r)$ for some $j \in \{0, 1, 2\}$ and so clearly $(R^m(c))_{m \in \mathbb{Z}^+}$ converges to z_0 by choice of r. In the latter case, $R^n(a_l) = z$. But $a_l \in U$ and $R(U) \subset U$ so $z \in U \subset B_\sigma(c, r/2)$ and therefore $c \in B_\sigma(z_0, r)$ so that, again, $(R^m(c))_{m \in \mathbb{Z}^+}$ converges to z_0.

Now suppose that (ii) holds. Let C be an enumeration of the $2d - 2$ critical points of R. Since

$$0 = (R^n)^\sharp(c_0) = R^\sharp(c_0) \cdot R^\sharp(R(c_0)) \cdot \cdots \cdot R^\sharp(R^{n-1}(c_0))$$

We can see that the distance from c_0 to the finite set

$$C \cup R^{-1}(C) \cup \ldots \cup (R^{n-1})^{-1}(C)$$

must be zero. (For if it was greater than 0, we would have $(R^n)^\sharp(c_0) > 0$.) So since c_0 lies in the open set $B_\sigma(z_0, r)$, there is a point $w \in R^{-q}(C) \cap B_\sigma(z_0, r)$ for some $q < n$ so that once again we obtain a critical point $c = R^q(w)$ of R such that $(R^m(c))_{m \in \mathbb{Z}^+}$ converges to z_0. It remains to prove our claim that either (i) or (ii) must hold. Let $c_1, \ldots, c_N$ be an enumeration of the critical points of R^n (where $N = 2d^n - 2$). For each i, either $\sigma(R^n(c_i), z_0) > r/2$ or $\sigma(R^n(c_i), z_0) < r$. If the latter holds for some i then let $c_0 = c_i$ and we have (ii). So we may suppose that the former holds for all $i \in \{1, \ldots, N\}$. Then for any $z \in (R^n)^{-1}(B_\sigma(z_0, r/2))$ and any c_i, we have $R^n(z) \neq R^n(c_i)$ and so $z \neq c_i$. Thus $(R^n)^\sharp$ is nonzero on $(R^n)^{-1}(B_\sigma(z_0, r/2))$. So we can apply Proposition 28 to obtain the desired inverse to R^n on $B(z_0, r/2)$. (Note that the hypothesis of Proposition 28 concerning compact sets is automatically satisfied because the domain of R^n is all of $\mathbb{C}_\infty$.) Thus we have satisfied either (i) or (ii) above, thereby completing the proof.

Corollary 30. *Let R be a rational map on $\mathbb{C}_\infty$ of degree ≥ 2. Let $(z_1, \ldots, z_m)$ be an attracting m-cycle of R. Then there is a critical point c of R such that $R^n(c) \to \{z_1, \ldots, z_m\}$ as $n \to \infty$.*

Proof. z_1 is a fixed point of R^m and so by the above there is a critical point c' of R^m such that $R^{mn}(c') \to z_1$ as $n \to \infty$. We can choose a neighbourhood N of c so that all points in N similarly converge to z_1. Since $(R^m)^\sharp(c') = 0$ we have $R^\sharp(c') \cdots R^\sharp(R^{m-1}(c')) = 0$ and so there must be a critical point c of R lying in one of the open sets $N, \ldots, R^{m-1}(N)$. This point will necessarily converge to the cycle $\{z_1, \ldots, z_m\}$ as desired.

7 Finding repelling cycles

We are now ready to prove the main result of this paper, that any rational function R of degree $d \geq 2$ always has repelling cycles, which necessarily lie in $J(R)$. We shall state this formally at the end of this section along with some other results of the same form. We break down the proof of this theorem into a series of claims.

Let R be a rational function of degree $d \geq 2$. Then R can be expressed in lowest terms as P/Q where P and Q are polynomials with coefficients in $\mathbb{C}$. Our aim is to find a repelling cycle of R, and so it does not matter if we commute R with a Möbius isometry. Therefore we shall assume that $R(0) \neq 0, \infty$ and that $R(\infty) \neq 0, \infty$. From these assumptions it follows that that P and Q are monic of degree d.

For any $t \in \mathbb{C}$ define

$$R_t(z) = \frac{(1-t)P(z) + t}{(1-t)Q(z) + tz^d} = \frac{P_t(z)}{Q_t(z)}.$$

Claim 1 *There exists a finite set of points $\{t_1, \ldots, t_N\} \in \overline{B}(0,2) \sim \{0,1\}$ such that for any $t \in \overline{B}(0,2) \sim \{t_1, \ldots, t_N\}$, the zeros of P_t are pairwise distinct from the zeros of Q_t.*

Proof. Let $z_1, \ldots, z_{2d} \in C$ be the roots of the polynomial $P(z)z^d - Q(z)$. Consider the rational function S defined by

$$S(z) = \frac{Q(z)}{Q(z) - z^d}$$

Note that this is a rational function of degree d expressed in lowest terms: Since $Q(0) \neq 0$, the zeros of the numerator are distinct from the zeros of the denominator. So we can consider S as a rational function on $\mathbb{C}_\infty$. Let $t_i = S(i_0(z_i))$ for $1 \leq i \leq 2d$. Then $t_i \in \mathbb{C}_\infty$. Suppose that $t \in \mathbb{C}$ and $i_0(t) \neq t_i$ for all $1 \leq i \leq 2d$. We wish to prove that for any $z \in \mathbb{C}$, either

$$(1-t)P(z) + t \neq 0 \qquad \text{or} \qquad (1-t)Q(z) + tz^d \neq 0.$$

So fix $z \in \mathbb{C}$. For any $1 \leq i \leq 2d$, either $z \neq z_i$ or $Q(z)/(Q(z) - z^d) \neq t$. We consider these two cases in turn.

Suppose that $z \neq z_i$ for all $1 \leq i \leq 2d$. Then $P(z)z^d \neq Q(z)$. Either $t \neq 1$ or $t \neq 0$. If $t \neq 1$ then $(1-t)P(z)z^d \neq (1-t)Q(z)$ and so either $(1-t)P(z)z^d + tz^d \neq 0$ or $(1-t)Q(z) + tz^d \neq 0$. In the former case it follows that $(1-t)P(z) + t \neq 0$. If $t \neq 0$ then

$$tP(z)z^d + (1-t)Q(z)P(z) \neq tQ(z) + (1-t)P(z)Q(z)$$

so either $tP(z)z^d + (1-t)Q(z)P(z) \neq 0$ or $tQ(z) + (1-t)Q(z)P(z) \neq 0$. It then follows that either $tz^d + (1-t)Q(z) \neq 0$ (cancel P in first equation) or $(1-t)P + t \neq 0$ (cancel Q in second equation). This deals with the case where $z \neq z_i$ for all $1 \leq i \leq 2d$. So we consider the second case, where

$$\frac{Q(z)}{Q(z) - z^d} \neq t \tag{20}$$

We know that the polynomials $Q(z)$ and $Q(z) - z^d$ have no common zeros so either $Q(z) \neq 0$ or $Q(z) - z^d \neq 0$ for the particular z we have fixed. In the latter case we can multiply equation 20 by $Q(z) - z^d$ to get $tz^d + (1-t)Q(z) \neq 0$. So suppose that $Q(z) \neq 0$. Either $|Q(z) - z^d| > 0$ (a case we have just dealt with) or $|Q(z) - z^d| < (1 + |t|)^{-1}|Q(z)|$. In the latter case $|t(Q(z) - z^d)| < |Q(z)|$ and so $tz^d + (1-t)Q(z) \neq 0$.

Now fix $i \leq 2d$. We prove that $t_i \neq 0, 1$. First we prove that $Q(z_i) \neq 0$. P and Q are in lowest terms so either $Q(z_i) \neq 0$ or $P(z_i) \neq 0$. But we also know that either $Q(z_i) \neq 0$ or $z_i \neq 0$ (because $R(0) \neq \infty$). Since $Q(z_i) = P(z_i)z_i^d$, we obtain in any case that $Q(z_i) \neq 0$. Now, if $Q(z_i) \neq 0$ then $t_i \neq 0$. But also $z_i \neq 0$ (because $P(z_i)z_i^d = Q(z_i) \neq 0$) and so $Q(z_i) \neq Q(z_i) - z_i^d$, whence $t_i \neq 1$.

If we eliminate those t_i sufficiently close to ∞ and map the rest back to $\mathbb{C}$ via j_0, then we obtain the desired sequence, and the claim is proved.

Now fix a sequence of distinct odd integers $p_1, \ldots, p_n$. such that $(p_i, p_j) = 1$ for all $1 \leq i < j \leq n$. We shall hereafter additionally demand of R that $R^{p_i}(\infty) \neq \infty (1 \leq i \leq n)$. Observe that this is a safe assumption to make, because we can conjugate R by a Möbius isometry which ensures this as well as our previous assumptions on R. Let $p = p_i$ for some fixed but arbitrary $1 \leq i \leq n$. Consider $R_t(z)$ as a rational function in $\mathbb{C}[t](z)$. (That is, consider t as an indeterminate, construct the polynomial ring $\mathbb{C}[t]$ and consider P_t and Q_t as coprime polynomials in $\mathbb{C}[t][z]$.) As such, the function R_t can be iterated and the resulting rational function expressed as a quotient of coprime polynomials:

$$R_t^p(z) = \frac{P_{t,p}(z)}{Q_{t,p}(z)}.$$

(Note that the subscript t has now become purely formal, although one can substitute values of t so long as we then revert to considering rational functions in $\mathbb{C}(z)$.)

Consider the polynomial $T_{t,p} \in \mathbb{C}[t][z]$ given by

$$T_{t,p}(z) = P_{t,p}(z) - zQ_{t,p}(z).$$

When we substitute $t = 0$ we know (from the assumption that $R^p(\infty) \neq \infty$) that $Q_{0,p}$ has a nonzero leading coefficient in $\mathbb{C}$. So the leading coefficient of $Q_{t,p}$ must also be nonzero in $\mathbb{C}[t]$. Therefore $T_{t,p}$ has degree $d^p + 1$. We are interested in the roots of $T_{t,p}$ because they represent p-cycles. However we are looking for proper p-cycles (for reasons that will become clear later on), and in order to make this distinction we need to divide out the fixed points of R. Therefore we observe the following:

Claim 2 *$T_{t,1}$ divides $T_{t,p}$ over the ring $\mathbb{C}[t][z]$.*

Proof. Straightforward algebra. Left to the reader.

So let $U_{t,p} = T_{t,p}/T_{t,1} \in \mathbb{C}[t][p]$. We wish to consider the roots of $U_{t,p}$. The problem is that the field $\mathbb{C}[t](z)$ is not discrete (see [Mines et al. 1988]) as is needed in order to construct an extension in which $U_{t,p}$ *has* roots. Although this problem can be circumvented by further abstraction (namely, by considering the coefficients of P and Q as further indeterminates and working over $\mathbb{Z}$, later replacing the coefficients with values in $\mathbb{C}$ once the appropriate field extension has been obtained), we need not concern ourselves with these details because we do not need to work with the roots individually, only with symmetric polynomial expressions over these roots. For the time being, assume that $z_1, \dots, z_N$ are new indeterminates intended to stand for the roots of T, where $N = d^p - d$. We can take the formal derivative of R_t in $\mathbb{C}[t](z)$:

$$R_t'(z) = \frac{P_t'(z)Q_t(z) - P_t(z)Q_t'(z)}{(Q_t(z))^2}$$

Observe that by the chain law for derivatives:

$$(R_t^p)'(z) = \prod_{i=0}^{p-1} R_t'(R^i(z))$$

(where $R^0(z) = z$). Define $V \in \mathbb{C}[t](z_1, \dots, z_N)$ by

$$V(z_1, \dots, z_N) = \prod_{i=1}^{N} (R_t^p)'(z_i).$$

Suppose that V is written as a quotient of polynomials $A, B \in \mathbb{C}[t][z_1, \dots, z_N]$. We do not need to assume that $(A, B) = 1$ and so we can assume that A and B are symmetric in the z_i (meaning any permutation of $\{z_1, \dots, z_N\}$ leaves A, B unchanged) and can therefore be represented as polynomials over $\sigma_1, \dots, \sigma_N$

(with coeffs. in $\mathbb{C}[t]$) where $\sigma_1 = \sum_i z_i, \sigma_2 = \sum_{i<j} z_i z_j, \ldots \sigma_N = z_1 z_2 \cdots z_N$, according to the fundamental theorem on symmetric polynomials (constructively presented in [Mines et al. 1988], Ch. II, Theorem 8.1).

By multiplying out $T(z) = T_N(z - z_1) \cdots (z - z_N)$ we see that the σ_i values can be resolved into elements of $\mathbb{C}[t]$, and therefore that A and B can also be represented as elements of T. We therefore have a rational expression

$$\Lambda_p(t) = A(t)/B(t)$$

. Where A and B are polynomials in t. Note that neither A nor B will necessarily be of determinate degree, nor do we necessarily know if $(A, B) = 1$ as elements of $\mathbb{C}[t]$.

We need to prove that B is nonzero. To this end we compute the value of $B(0) = B(z_1, \ldots, z_N)(0)$. Since $R^p(z_i) = z_i$ and $R^p(\infty) \neq \infty$ we must have $R(z_i) \neq \infty$ and therefore $Q(z_i) \neq 0$. But $B(0)$ is just a product of terms $Q(z_i)^2$ and so must be nonzero. We can perform a similar calculation for the rational function $R_1 : z \mapsto z^{-d}$. This function also satisfies $R_1^p(\infty) \neq \infty$ (because we have stipulated that $p = p_i$ is odd), and so $B(1) \neq 0$ as well.

Since B is nonzero, by Lemma 6, we can determine a finite set $\{b_1, \ldots, b_m\} \subset \mathbb{C}_\infty$ of zeros of B. We have further shown that $0, 1 \neq b_i$ $(1 \leq i \leq m)$. If we combine this finite set of points with the finite set given by Claim 1 we find that we have proved the following:

Claim 3 *There is a finite set $\{c_1, \ldots, c_M\}$ of points in $\mathbb{C}{\sim}\{0,1\}$ and an analytic function Λ_p on the open set $L_p := B(0,2){\sim}\{c_1, \ldots, c_M\}$ such that for any $t \in L_p$, R_t is a rational function of degree d and $|\Lambda_p(t)|$ is equal to the product $\prod_i (R_t^p)^\sharp(z_i)$ taken over any enumeration of the roots of $U_{t,p} \in \mathbb{C}[z]$.*

A couple of observations are perhaps necassary to complete the proof of this claim: first note that wherever $B(t) \neq 0$, it follows that $Q_t(z_i)^2 \neq 0$ for all i and therefore that $R_t^p(\infty) \neq \infty$, so that we can compute the fixed points of R_t^p for this specific t and find them to be equal to the zeros of the polynomial T with this value of t substituted in. Secondly, we are able to switch to the spherical derivative in the above claim because if $R^p(z) = z$ then

$$R^\sharp(z) \cdots R^\sharp(R^{p-1}(z)) = |R'(z)| \cdots |R'(R^{p-1}(z))|.$$

(Just use equation (2) repeatedly and cancel terms.)

We wish to compute $\lambda_p(1)$. The fixed points of R_1^p are the roots of $z^{d^p+1} - 1 = 0$, with the roots of $z^{d+1} - 1$ removed. Also

$$(R_1^p)'(z) = -d^p z^{-d^p - 1}$$

so we obtain

$$\lambda_p(1) = (-d^p)^{d^p - d} = d^{p(d^p - d)}.$$

We can now construct repelling cycles for R. The point z belongs to a repelling n-cycle if and only if

$$R^n(z) = z \qquad \text{and} \qquad (R^n)^\sharp(z) > 1.$$

If $|\lambda_p(0)| > 1$ then there is some z belonging to a repelling p-cycle. On the other hand if $|\lambda_p(0)| < 1$ then we can use Theorem 29 to find a critical point c which converges to a p-cycle. There are only so many critical points, so we can only have finitely many attracting p-cycles. We shall nudge the value of t away from zero so as to ensure that either $\lambda_p(t) > 0$ or $\lambda_p(t) < 0$. But the problem is that we may be counting a fixed point of R as both a p_i-cycle and a p_j-cycle for different values p_i and p_j. However:

Claim 4 *If for some $t \in L_p$ and $z_0 \in \mathbb{C}$ we have $U_{t,p}(z_0) = 0$ and $(R_t^p)'(z_0) \neq 1$, then $R_t(i_0(z_0)) \neq i_0(z_0)$.*

Proof. If $U_{t,p}(z_0) = 0$ then $P_{t,p}(z_0) - z_0 Q_{t,p}(z_0) = 0$ and so $Q_{t,p}(z_0) \neq 0$. Therefore $R_t^p(z_0) = z_0$. Let G be R_t^p considered as a map on a dense open subset of $\mathbb{C}$. Choose a suitable small neighbourhood N of z_0. in which $f : w \mapsto G(w) - w$ is analytic. Then $f(z_0) = 0$ and $f'(z_0) \neq 0$. In this case we can choose $\delta > 0$ such that $f(z) = (z - z_0)g(z - z_0)$ on $B(z_0, \delta) \subset N$ for some analytic g on $B(0, \delta)$. Since $0 \neq f'(z_0) = g(0)$, we can further assume, by adjusting δ if necessary, that g is bounded away from 0 on $B(0, \delta)$. If $R_t(i_0(w_0)) = i_0(w_0)$ for some $w_0 \in B(z_0, \delta)$ then w_0 is a zero of $T_{t,1}$. Since z_0 is a zero of $U_{t,p}$ we have that (w_0, z_0) is a *pair* of zeros of $T_{t,p}$ (in the sense that $(z - w_0)(z - z_0) \mid T_{t,p}$). Therefore (z_0, w_0) is a pair of zeros of f contradicting the separation of f into $(z - z_0)g(z - z_0)$ above. Thus the fixed points of R_t map back via j_0 to points a positive distance from z_0 (otherwise we can obtain the above contradiction), from which it follows that $R_t(i_0(z_0)) \neq i_0(z_0)$.

Lemma 31. *Let $f_1, \ldots, f_n$ be nonzero analytic functions on $\overline{B}(0, r)$ with $f_i(0) = 0$ for $1 \leq i \leq n$. Let $M > 0$ be a constant. Then there is $r_1 \in (0, r]$ and integers $m(i) \in \mathbb{Z}^+$ $(1 \leq i \leq n)$ such that if we represent f_i by*

$$f_i(z) = a_{i,1} z + \cdots a_{i,m(i)} z^{m(i)} + g_i(z) z^{m(i)+1} \tag{21}$$

(where g_i is analytic on $B(0, r)$) then for all $z \in S(0, r_1)$ and $1 \leq i \leq n$ we have $a_{i,m(i)} \neq 0$ and

$$|f_i(z) - a_{i,m(i)} z^{m(i)}| \leq M |a_{i,m(i)} z^{m(i)}|.$$

Proof. Since f_i is nonzero, we can choose $m(i), a_{i,j}, g_i$ so that equation 21 is satisfied, with $a_{i,m(i)} \neq 0$ $(1 \leq i \leq n)$. Define $G_i = \|g_i\|_{\overline{B}(0,r)}$. Let

$$s = \min\left\{\tfrac{1}{2} M |a_{i,m(i)}| (1 + G_i)^{-1} \;:\; 1 \leq i \leq n\right\}$$

and let $r_1 = \min\{s, r, 1\}$. Then for all i, $M|a_{i,m(i)}| - G_i r_1 > 0$ and so either

$$\max\{|a_{i,1}|, \ldots, |a_{i,m(i)-1}|\} \leq r_1^n (M|a_{i,m(i)}| - G_i r_1) \qquad (1 \leq i \leq n) \qquad (22)$$

(with the left-hand side equal to 0 when $m(i) = 1$) or there is some $1 \leq j \leq n$ and $m'(j) < m(j)$ such that $a_{j,m'(j)} \neq 0$. In the latter case we can replace $m(j)$ with $m'(j)$ and repeat the above procedure. This can happen only a certain number of times (at most $\sum_i (m(i) - 1)$) and so in the end we are sure to arrive at the case given by equation 22. In this case we have for all $z \in S(0, r_1)$ and $1 \leq i \leq n$:

$$\sum_{j=1}^{m(i)-1} |a_{i,j}| + G_i r_1^{n+1} \leq M|a_{i,m(i)}| r_1^n.$$

From which we get

$$|f_i(z) - a_{i,m(i)} z^{m(i)}| \leq M|a_i, m(i) z^{m(i)}| \qquad (1 \leq i \leq n)$$

as desired.

We are now in a position to construct a repelling cycle for R. Choose finitely many pairwise coprime odd positive integers $p_1, \ldots, p_n$, where $n = 10d - 6$. For each $1 \leq i \leq n$ define Λ_{p_i} as above. For a fixed but general $1 \leq i \leq n$ let $p = p_i$. We have calculated that $|\Lambda_p(1)| > 1$ so either $|\Lambda_p(0)| > 1$ or Λ_p is non-constant. In the former case we can produce a fixed point z of R^p such that $(R^p)^\sharp(z) > 1$, as desired. Also, either $|\Lambda_p(0)| > 0$ or $|\Lambda_p(0)| < 1$ If the latter case occurs for $2d - 1$ of the indices i (say, the set $I_1 \subset \{1, \ldots, n\}$) then we can obtain zeros z_i of U_{0,p_i} such that $R^{p_i}(z_i) = z_i$ and $(R^{p_i})^\sharp(z_i) < 1$. By Lemma 31 above, $R(z_i) \neq z_i$. Since p_i and p_j are coprime for distinct i, j, it follows that the cycles $C_i = (z_i, \ldots, R^{p_i - 1} z_i)$ are distinct (in that if $a \in C_i$, $b \in C_j$ for $i \neq j \in I_1$ then $a \neq b$). But by Corollary 30 there must be a critical point c_i which converges to the p_i-cycle of z_i under R. Thus the c_i must be distinct from one another. But there are only $2d - 2$ critical points of R by Theorem 27 and so we obtain a contradiction. Thus for at least $(10d - 6) - (2d - 2) = 8d - 4$ of the indices i, we must have $\Lambda_{p_i}(0) \neq 0$. Let I be the set of such indices and for each $i \in I$ define

$$f_i(z) = \frac{\Lambda_{p_i}(0)}{\Lambda_{p_i}(z)} - 1.$$

Then each f_i is a nonconstant analytic function on a neighbourhood of 0 with $f_i(0) = 0$. Thus we can apply Lemma 31 to produce $r_1 > 0$ and $m(i) \in \mathbb{Z}^+$ $(i \in I)$ such that for all $z \in S(0, r_1)$ and $i \in I$,

$$f_i(z) = a_i z^{m(i)} + h_i(z)$$

where $a_i \neq 0$ and

$$|h_i(z)| \leq \tfrac{1}{2} |a_i| r_1^{m(i)}.$$

We will also choose r_1 small enough so that

$$\overline{B}(0, r_1) \Subset \mathrm{dom}(f_i) \qquad (i \in I).$$

Now let us define the quadrant $Q \subset S(0,1)$ by

$$Q = \{\zeta \in S(0,1) \,:\, \arg \zeta \in [-\pi/4, \pi/4]\}$$

and for each $i \in I$ let

$$Q_i = \left\{\zeta \in S(0,1) \,:\, a_i |a_i|^{-1} \zeta^{m(i)} \in Q\right\}$$

Let χ_i be the characteristic function of the set Q_i considered as an integrable function on $S(0,1)$ (equipped with the obvious measure so that $\mu(S(0,1)) = 2\pi$). Then clearly χ_i is integrable and $\int \chi_i = \pi/2$. Let ψ be the step function on $S(0,1)$ defined by $\psi = \sum_{i \in I} \chi_i$. Then clearly $\int \psi = (8d-4)\pi/2$ and so there must be a point $\zeta_0 \in S(0,1)$ such that ζ_0 lies in the domain of each χ_i and $\psi(\zeta_0) > (8d-4)/4 - 1$. So there must be a finite subset $I_0 \subset I$ of cardinality $(8d-4)/4 = 2d-1$ such that $\zeta_0 \in Q_i$ for all $i \in I_0$.

Let $\zeta = \zeta_0 r_1$. The reader can verify that for all $i \in I_0$,

$$\mathrm{Re}\, f_i(\zeta) > 0$$

and therefore that

$$|\Lambda_{p_i}(\zeta)| < |\Lambda_{p_i}(0)| \qquad (i \in I_0).$$

Thus we have for each $i \in I_0$, either $|\Lambda_{p_i}(\zeta)| < 1$ or $|\Lambda_{p_i}(0)| > 1$. If the former case holds for all $i \in I_0$ then the rational function R_ζ (which is of degree d) has $2d-1$ distinct proper cycles which are all attracting, and so by Theorem 30 there is a critical point converging to each cycle: a contradiction, since there are only $2d-2$ critical points of R_ζ by Theorem 27. Thus there must be at least one $i \in I_0$ such that $|\Lambda_{p_i}(0)| > 1$ and from this we immediately obtain a proper repelling p_i-cycle of R. We state what we have proved.

Theorem 32. *Let R be a rational function of degree $d \geq 2$. Let $p_1, \ldots, p_N$ be pairwise coprime odd integers, where $N = 10d-6$. Then there is $1 \leq i \leq N$ such that there exists a repelling p_i-cycle of R.*

Note that with a little extra care, we can easily reduce the value of N to $6d-4$.

Lemma 33. *Let R be a rational map of degree $d \geq 2$. Let $z \in J(R)$ and let U be a neighbourhood of z. Then for any $n \in \mathbb{Z}^+$ there is an $m \in \mathbb{Z}^+$ and a point $w \in U$ such that $R^{mn+2}(w) = w$.*

Proof. Let $z \in J(R)$, and let U be an open neighbourhood of z. Since $J(R)$ contains no isolated points, we can assume that z is not one of the finitely many critical *values* of R^2 (i.e. the points $R^2(c)$ where c is a critical point of R^2). Thus there are at least four distinct points in $R^{-2}(z)$ (since $\deg(R^2) = d^2 \geq 4$) and so we can choose three of them z_1, z_2, z_3 distinct from z. We can also choose $r > 0$ such that there are analytic local inverses S_1, S_2, S_3 to R on $\overline{B}_\sigma(z,r)$ which map z to z_1, z_2, z_3 respectively, and which map $\overline{B}_\sigma(z,r)$ to a compact neighbourhood W_i of z_i. We can also assume that the chordal distance $\sigma(Z_i, W_j)$ is greater than 0 for $1 \leq i < j \leq 3$, so that there is a constant $k > 0$ such that for any $\zeta \in B_\sigma(z,r)$ and any $1 \leq i < j \leq 3$, $\sigma(S_i(\zeta), S_j(\zeta)) \geq k$. We know that $z \in J(R) = J(R^n)$ so that $\{R^{mn} : m \in \mathbb{Z}^+\}$ is abnormal at z. Thus by Theorem 13 there is some $w \in B_\sigma(z,r)$, $m \in \mathbb{Z}^+$ and $i \in \{1,2,3\}$ such that $S_i(w) = R^{mn}(w)$, so that $R^{mn+2}(w) = w$. Since r can be chosen arbitrarily small, we have completed the proof.

Theorem 34. *Let R be a rational function of degree ≥ 2. Then $J(R)$ is contained in the closure of the set of periodic points of R.*

Proof. Immediate from Lemma 33.

With a lot more work, we can in fact prove the following, whose proof is omitted for reasons of space. (One refines the proof technique of Theorem 32 so as to construct a repelling cycle which meets an arbitrary neighbourhood of a point in $J(R)$.)

Theorem 35. *Let R be a rational function of degree ≥ 2. Then $J(R)$ is equal to the closure of the set of repelling periodic points of R.*

8 Concluding Remarks

An obvious next step in the development of this theory, is to prove that the sets $J(R)$ are compact. This is hard to do, and there are even reasons (to do with the high fractal dimension of some Julia sets) to suspect that it is not possible to prove compactness for general $J(R)$, or at least that such a result would have substantial implications even for the classical theory. Compactness for a large class of Julia sets can be proved however, and I hope to make this the subject of a future paper. Another question worth further consideration is how one would develop this theory computationally so as to produce feasible algorithms for e.g. approximating Julia sets. The major obstacle (there are of course many others) to this is equation (11) which emerges from the constructive proof of Schottky's theorem: the constant in this equation is so large that it excludes any of the above theorems from practical computational use. In light of this it would be interesting to know if any results which reduce this constant can be obtained. (For example, one might restrict consideration to Picard functions f on $B(0,1)$ which are polynomials of degree n, and find a reasonable polynomial bound in terms of n on $|f(0)|$.)

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