Journal of Universal Computer Science, vol. 10, no. 9 (2004), 1212-1238
submitted: 14/11/04, accepted: 28/5/04, appeared: 28/9/04

The Tiling of the Hyperbolic 4D Space by the 120-cell is Combinatoric

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Abstract: The splitting method was defined by the author in [Margenstern 2002*a*, Margenstern 2002*d*]. It is at the basis of the notion of combinatoric tilings. As a consequence of this notion, there is a recurrence sequence which allows us to compute the number of tiles which are at a fixed distance from a given tile. A polynomial is attached to the sequence as well as a language which can be used for implementing cellular automata on the tiling.

The goal of this paper is to prove that the tiling of hyperbolic 4D space is combinatoric. We give here the corresponding polynomial and, as the first consequence, the language of the splitting is not regular, as it is the case in the tiling of hyperbolic 3D space by rectangular dodecahedra which is also combinatoric.
[1]

Key Words: cellular automata, hyperbolic plane
Category: F.1.1, F.1.3

1 Introduction

Starting from a certain time, several papers appeared which, more or less, claimed that they belong to a combinatoric approach to hyperbolic geometry.

This is also the goal of this paper which belongs to a rather long sequence of papers by the same author, alone or with co-authors: [Chelghoum et al. 2003, 2004, Grigorieff et al. 2002, 2004, Herrmann et al. 2000, 2002, 2003, Iwamoto et al., 2002, 2003, 2004, Margenstern 2000*a*,*b*, 2002*a*-*d*, 2003*a*-*g*, Margenstern et al. 1999, 2000, 2001, 2002*a*-*c*, 2003*a*-*c*]. All these papers also belong to the same line of works. Although the term of *combinatoric tilings* appears only on recent papers of the author, all papers above quoted are within the scope of this method. While most of the papers deal with tilings in the hyperbolic plane, a few of them deal with tilings in the hyperbolic 3D space, see [Margenstern 2002*a*, Margenstern et al. 2002*b*, Margenstern et al. 2002*c*, Margenstern et al. 2003*c*].

[1] The present paper is an extended abstract of a full technical report published both by the LITA, the University of Metz, France and by the CDMTCS, the University of Auckland, New-Zealand, see [Margenstern 2003*a*].

The goal of this paper is to prove that the tiling of $I\!H^4$ which is built from the 120-cell by recursive reflections in its hyper-faces and of the images in their hyper-faces is combinatoric.

In principle, this property gives us tools in order to implement cellular automata in the $4D$ hyperbolic space. Due to the length of the proof that the tiling is combinatoric, in this paper we do not give implementation details. We just remind the guidelines of [Margenstern et al. 2003*b*] which are still valid in this context. A fully detailed implementation as well as the important question of implementation complexity require additional details which are not given in this paper. Some of these properties are in [Margenstern 2003*e*], but this paper also does not deal with this issue. A forthcoming paper will present be devoted to such an implementation. This issue may be of interest for physics as, most certainly, this is the case for $3D$ space, see [Margenstern et al. 2003*c*], but also for other domains requiring $4D$ spaces.

Although the question is not fully addressed in this paper, the motivation of the series of papers leading to it and including it **is** the implementation of cellular automata in hyperbolic spaces. As such, due to the algorithmic considerations of its point of view, the paper belongs to computer science.

In the second section, we recall the splitting method and we recall the definition of a combinatoric tiling. In the third section we briefly recall the notions of hyperbolic geometric which are needed in order to understand the paper. We also indicate how to deal with the hyperbolic $4D$ space. In the fourth section, as a first step of the proof, we construct the splitting of the space which leads to the tiling by following the traditional stages of construction of the 120-cell. In the fifth section, we deal with the polynomial and the language of the splitting. We also prove that this language is not regular, which gives us an additional information.

2 The splitting method

2.1 The geometric side

The method is based on the notion of a **basis of splitting** which was introduced in [Margenstern 2002*d*]. Such a basis consists of two finite families of closed, simply connected sets of the considered geometric space $\mathcal{X}$. The sets are also supposed to be with a non-empty interior. The sets of the first family, say $S_0, \ldots, S_k$ are called **regions** and they are assumed to be unbounded. The sets of the second family, say, $P_0, \ldots, P_h$ are called the **generating tiles** and they are bounded. Moreover, these two families of sets have the following properties:

(i) $\mathcal{X}$ splits into finitely many copies of S_0,

(ii) any S_i splits into one copy of some P_ℓ, the **leading tile** of S_i, and finitely many copies of S_j's, where **copy** means an **isometric image**, and where, in condition (ii), the copies may be of different S_j's, S_i being possibly included.

As usual, it is assumed that the interiors of the copies of the $\mathcal{P}_\ell$'s and the copies of the S_j's which are involved are pairwise disjoint.

From a basis of splitting of $\mathcal{X}$, if any, we define a tree A which is associated with the basis by induction. The root of the tree is the leading tile of S_0. The sons of the root are the leading tiles of the copies of the S_j's which enter the splitting of S_0 by condition (ii). Then, we apply this argument recursively on the leading tiles of the regions already obtained, applying condition (ii) to the copies of the S_j's.

This recursive process shows that A is an infinite tree with finite branching. We call A the **spanning tree of the splitting**, where the splitting refers to the basis of splitting $S_0, \ldots, S_k$, $P_0, \ldots, P_h$.

Definition – *Say that a tiling of X is* **combinatoric** *if it has a basis of splitting and if the spanning tree of the splitting yields exactly the restriction of the tiling to S_0, where S_0 is the head of the basis.*

An illustrative example is given by the splitting of $I\!H^2$, the hyperbolic plane, which generates the pentagrid, tiling $\{5, 4\}$ in Schläfli notations, for instance, see [Coxeter 1963, Sommerville 1958], *i.e.* five sides and an interior angle of $\frac{\pi}{2}$. See figure 2, in section 3.

In this paper, the tiling which we consider has a single generating tile, *i.e.* $h = 0$.

2.2 The algebraic side

The splitting method has also algebraic consequences. From [Margenstern 2002a], we know that when a tiling is combinatoric, there is a polynomial which is attached to the spanning tree of the splitting.

More precisely, we have the following result:

Lemma 1 – ([Margenstern 2002a]) *Let $\mathcal{T}$ be a combinatoric tiling, and denote a basis of splitting for $\mathcal{T}$ by $S_0, \ldots, S_k$ with $P_0, \ldots, P_h$ as its generating tiles. Let $\mathcal{A}$ be the spanning tree of the splitting. Let M be the square matrix with coefficients m_{ij} such that m_{ij} is the number of copies of S_{j-1} which enter the splitting of S_{i-1} in condition (ii) of the definition of a basis of splitting. Then the number of nodes of $\mathcal{A}$ of the n^{th} generation are given by the sum of the coefficients of the first row of M^n. More generally, the number of nodes of the n^{th} generation in the tree which is constructed as $\mathcal{A}$ but which is rooted in a node being associated to S_i is the sum of the coefficients of the $i+1^{\text{th}}$ row of M^n.*

This matrix is called the **matrix of the splitting** and we call **polynomial of the splitting** the characteristic polynomial of this matrix, being possibly divided by the greatest power of X which it contains as a factor. Denote the polynomial by P. From P, we easily infer a recurrence equation which allows us to compute very easily the number u_n of nodes of the level n in $\mathcal{A}$. This gives us also the number of nodes of each kind at this level by the coefficients of M^n on the first row: we use the same equation with different initial values. Sequence $\{u_n\}_{n \in \mathbb{N}}$ is called the **recurrent sequence of the splitting**.

First, as in [Margenstern 2000a, 2002a], number the nodes of $\mathcal{A}$ level by level, starting from the root and, on each level, from the left to the right. Second, consider the recurrent sequence of the splitting, $\{u_n\}_{n \geq 1}$: it is generated by the polynomial of the splitting. As we shall see, it turns out that the polynomial has a greatest real root β and that $\beta > 1$. Sequence $\{u_n\}_{n \geq 1}$ is increasing. Now, it is possible to represent any positive number n in the form $n = \sum\limits_{i=0}^{k} a_i.u_i$, where $a_i \in \{0..b\}$, where $b = \lfloor \beta \rfloor$, see [Fraenkel 1985, Hollander 1998], for instance. String $a_k \dots a_0$ is called a representation of n. In general, the representation is not unique and it is made unique by an additional condition: we take the representation which is maximal with respect to the lexicographic order on the words on $\{0..b\}$. The set of these representations is called the **language of the splitting**.

Although this notion was introduced in order to study tilings of hyperbolic spaces, it may apply also in the euclidean case or other situations of unbounded manifolds. The notion was successfully applied to various tilings of hyperbolic spaces, mainly $\mathbb{H}^2$ and $\mathbb{H}^3$, the hyperbolic $3D$ space, see [Margenstern 2000a, 2002c,a, Margenster et al. 2002b,c, 2003c, Grigorieff et al. 2002, 2004].

The method also applies to $\mathbb{H}^4$:

Theorem 1 – *The tiling of $\mathbb{H}^4$ which is built on the* 120*-cell by tessellation is combinatoric. The polynomial of the splitting is:*

$$P_{\mathbb{H}^4}(X) = X^4 - 116X^3 + 366X^2 - 116X + 1.$$

The language of the splitting which is associated to it is not regular.

We turn now to the proof of the theorem.

3 General tools

3.1 About hyperbolic geometry

We refer the reader to [Meschkowski 1964] for introductory material on hyperbolic geometry. In order to make the paper as self-contained as possible and also to spare space for this paper, we shall use Poincaré's disk model of the hyperbolic plane, see figure 1, below, and its natural generalisations to higher dimensions.

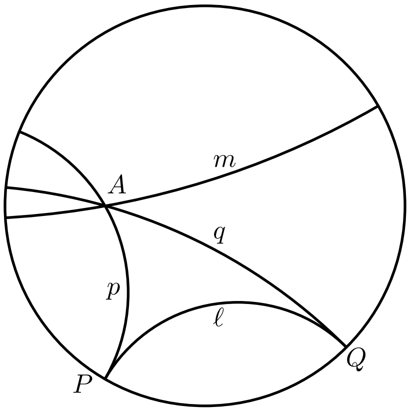

Figure 1. The Poincaré model of $I\!H^2$

The open unit disk U of the euclidean plane constitutes the points of the hyperbolic plane, $I\!H^2$. The border of U, ∂U is called the set of *points at infinity*. Lines are the trace in U of its diameters or the trace in U of circles which are orthogonal to ∂U. The model has a very remarkable property, which it shares with the half -plane model: hyperbolic angles between lines are the euclidean angles between the corresponding circles.

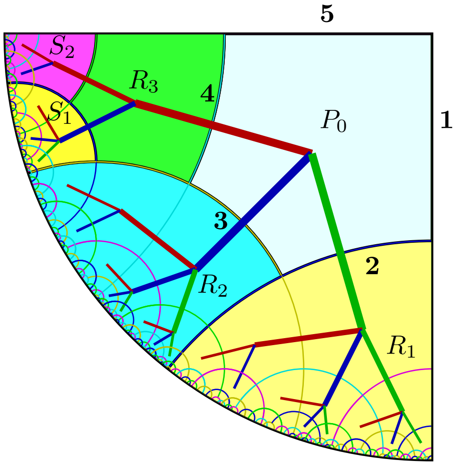

Figure 2. The splitting of $\mathcal{Q}$ which is associated to the pentagrid. Notice the construction of the spanning tree.

In order that the reader could be more familiarised to hyperbolic geometry and in order to illustrate the splitting method by a concrete example, we give in figure 2 the application of the method to the pentagrid which is the tiling obtained by the tessellation of the rectangular regular pentagon.

In the situation which is illustrated by figure 2, notice that we have two regions in the basis of the splitting: S_0 is $\mathcal{Q}$ and S_1 is the region denoted by R_3 in the figure.

The model is easily generalised to higher dimension: in [Margenstern et al. 2002*b*,*c*, 2003*c*], it is done for $I\!H^3$, where the hyperbolic $3D$ space is the interior of the unit ball of $I\!R^3$. The planes are the trace in the ball of spheres which are orthogonal to the unit sphere, diametral planes being considered as a limit case. Lines are intersections of planes.

As a model of the hyperbolic $4D$ space, we take the interior of the unit ball of $I\!R^4$. Hyper-planes are the trace in the ball of hyper-spheres which are orthogonal to the unit hyper-sphere, diametral hyper-planes being a limit case. Planes are intersections of hyper-planes and lines are intersections of planes.

3.2 The fourth dimension

It is a *lieu commun* to say that it is very difficult to see objects in four dimensions. However a lot of works are devoted to this subject, see [Coxeter 1963, Sommerville 1958], for instance. For an introduction to the 120-cell and for related papers, we refer the reader to [Stillwell 2001].

In our study, we shall use the **dimensional analogy** between the $2D$, $3D$ and $4D$ spaces, already pointed out by Coxeter, see [Coxeter 1963] where the limitations of the method are also discussed. Here, our analogy is grounded on the fact that, in all cases, the topology of a hyperbolic space is **locally** the same as the topology of the euclidean space in which a model of the hyperbolic space is realised.

3.2.1 The incidence relations

Consequently, in order to find the incidence relations between objects of different dimensions in $I\!H^2$, $I\!H^3$ and $I\!H^4$, it is enough to look at the same relations for the analogous objects of the euclidean spaces, respectively $I\!E^2$, $I\!E^3$ and $I\!E^4$.

In $I\!E^4$, there are sixteen hyper-cubes around the point $(0,0,0,0)$. They are defined by the following formula: $\prod_{i=0}^{3}(-1)^{\epsilon_i}.[0,1]$, where $\epsilon_i \in \{0,1\}$, and with the convention that $(-1).[0,1] = [-1,0]$ and, of course, $1.[0,1] = [0,1]$. We can extend this formula by introducing an exponent to the interval which occurs in it. The formula becomes now $\prod_{i=0}^{3}(-1)^{\epsilon_i}.[0,1]^{\alpha_i}$, with $(-1)^{\epsilon_i}.[0,1]^0 = \{0\}$, again by convention. This allows us to obtain all elements of dimension 1, 2 and 3 which pass through point $(0,0,0,0)$. Thus, we obtain the information for $I\!H^4$ and for the 120-cell see below in tables 1.

We invite the reader to apply the same arguments to $I\!H^3$ and to $I\!H^2$. He/she will find the other information given by tables 1.

These tables contain also the characteristic given by the Schläfli notation of the tiling with the 120-cell: $\{5,3,3,4\}$. This means that the $2D$-faces are pentagons, that any edge is shared by three of the $3D$-faces dodecahedra and that any vertex is shared by four of the $3D$-faces.

	0	1
1	4	
2	4	2

	0	1	2
1	6		
2	12	4	
3	8	4	2

	0	1	2	3
1	8			
2	12	6		
3	32	12	4	
4	16	8	4	2

	0
1	2

	0	1
1	3	
2	3	2

	0	1	2
1	4		
2	6	3	
3	4	3	2

Tables 1. The tables of incidence in $I\!H^2$, $I\!H^3$ and $I\!H^4$ and then in the pentagon, the dodecahedron and the 120-cell.

3.2.2 Schlegel diagrams

In this paper, we shall make use of another tool, a very old one, which is also used in [Sommerville 1958]: **Schlegel diagrams**.

The Schlegel diagram of a figure F in a $3D$ space, either euclidean, elliptic or hyperbolic, is the central projection of F on a plane, starting from a particularly chosen point of the space. For more details on these diagrams, see for instance, [Sommerville 1958, Epstein et al. 1992, Hilbert et al. 1990]. Such a representation takes place in the euclidean plane but, as already indicated, it can be used for the study of $3D$ spaces, whatever the geometry is.

The same tool can be used for $4D$ space, either euclidean, elliptic or hyperbolic, as it is done in [Sommerville 1958]. Indeed, we make use of $3D$ projections from which we take the Schlegel diagram.

Recall that in $I\!H^4$, as in the other hyperbolic spaces, whatever the dimension is, there are no similitude. Consequently, all 120-cells of $I\!H^4$ are isometric. And so, in this sense, there is a single 120-cell from the point of view of Schlegel diagrams. A global view of the 120-cell does not allow us to grasp it from an algorithmic point of view. On the opposite, local views are worth of interest, later we call them **maps**. They will allow us to prove that the tiling which is based on the tessellation of the 120-cell is combinatoric.

Now, we fix some notations and conventions for an intensive use of this tool. To that purpose, look at figure 3 which represents the hyper-face of the 0^{th} generation.

The dodecahedron is projected on a face F from a point A which has an orthogonal projection on F and which is outside the dodecahedron. We look on the diagram from the reflection of A in F.

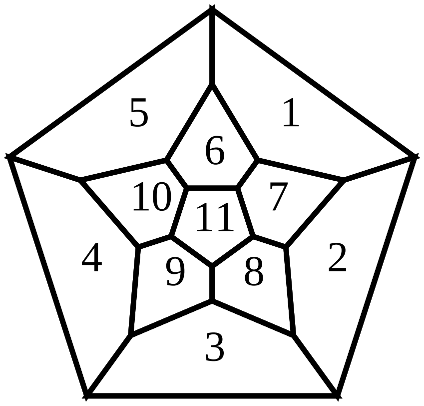

Figure 3. The 0^{th} *generation*

The figure represents a true Schlegel diagram of a dodecahedron. We numbered the sides of the dodecahedron from 0 up to 11. Face 0 does not appear directly: it is the 'hidden face' which is bordered by faces 1 up to 5.

Thanks to this numbering, we identify all the elements of the dodecahedron by the numbers of the faces which share the element. Accordingly, edges are identified by two numbers and vertices are identified by three numbers.

We shall also classify the faces in three **rings**. The **outer** ring contains faces from 1 up to 5. The **inner** ring contains faces 6 up to 10. The last ring contains face 11.

In many places, we shall call face 3 the **feet** of the dodecahedron, face 0, its **back**, and face 11 its **top**.

3.2.3 Inside, outside and orthogonal completion

Consider a part M of $I\!H^4$ and let $\mathcal{H}$ be a hyper-plane. The complement of $\mathcal{H}$ in $I\!H^4$ consists of two parts H_i and H_o which do not intersect and which are open **half-spaces** of $I\!H^4$. We put $\overline{H}_i = H_i \cup \mathcal{H}$, and $\overline{H}_o = H_o \cup \mathcal{H}$. If $\overline{H}_i \supset M$, respectively $\overline{H}_o \supset M$, we say that $\overline{H}_i$, respectively $\overline{H}_o$, is the **inside** of $\mathcal{H}$ with respect to M. The complement in $I\!H^4$ of the inside is the **outside**.

In the rest of the paper, the rôle of M will be played by the 120-cell, and the rôle of $\mathcal{H}$ will be played by the hyper-plane generated by a hyper-face of the

120-cell.

We shall also have to define these notions for regions R of $I\!H^4$. We shall consider regions which are the intersection of a finite sequence $\mathcal{H}_1, \ldots, \mathcal{H}_m$ of half-spaces such that $\partial\mathcal{H}_i$ and $\partial\mathcal{H}_{i+1}$ are orthogonal for $i \in \{1..m{-}1\}$, where $\partial\mathcal{H}_i$ is the hyper-plane which is the border of $\mathcal{H}_i$. We also assume that for $i \in \{2..m{-}1\}$, $\mathcal{H}_{i+1} \supseteq \bigcap\limits_{j=1}^{i} \mathcal{H}_j$. In this case, the rôle of M is played by the intersection $\mathcal{R}$ of the $\mathcal{H}_i$'s. The **outside** of $\mathcal{R}$ is the union of the outsides of the $\partial\mathcal{H}_i$'s. Notice that for $i \in \{1..m\}$, the inside of $\partial\mathcal{H}_i$ is the same, whether we take M as $\bigcap\limits_{j=1}^{i-1} \mathcal{H}_j$ or as $\bigcap\limits_{j=1}^{m} \mathcal{H}_j$.

In particular, we shall apply this definitions to the basic region which we shall study: the **hyper-corner**. It is defined by four dodecahedra which are pairwise orthogonal, which have pairwise a common face and which share together a single vertex.

Also we notice that the reflection in a hyper-plane P exchanges the two half-spaces which it determines in the complement of P. And so, a reflection in a hyper-face exchanges its inside and its outside. Next, consider two hyper-planes P_1 and P_2 such that their insides and outsides are defined with respect to the same set M. If P_1 and P_2 are perpendicular, the reflection in P_i transforms points which lie in the inside, respectively the outside, of P_j into points which lie also in the inside, respectively the outside, of P_j.

In the traditional construction of the 120-cell, as it is explained in [Sommerville 1958], we often 'place' dodecahedra on some of the free faces of the currently constructed solid. What *to place* means in this context has to be explained.

Now, consider a region, as described above, and assume that $\mathcal{H}_i$ is generated by a rectangular dodecahedron Δ_i with $\Delta_i \subset \mathcal{R}$. Consider another rectangular dodecahedron $\mathcal{D}$ which shares a face F with Δ_m. We say that $\mathcal{D}$ is obtained by **orthogonal completion** with respect to $\mathcal{R}$ if $\mathcal{D}$ is orthogonal to Δ_m and if $\mathcal{D}$ is in $\mathcal{R}$. Notice that there are exactly two dodecahedra $\mathcal{D}'$ and $\mathcal{D}''$ which share F and which are orthogonal to Δ_m. The condition to be in the inside of $\mathcal{R}$ fixes the choice between them. When the reference to $\mathcal{R}$ will be superfluous, we shall say simply **orthogonal completion**.

4 The splitting

The traditional construction of the 120-cell can be seen as a process in which each step consists in the construction of a hyper-plane $\mathcal{F}_i$ in such a way that when the process is completed, we obtain a sequence of hyper-planes $\mathcal{F}_1, \ldots, \mathcal{F}_{120}$ such that $\mathcal{F}_i$ is orthogonal to $\mathcal{F}_{i+1}$ for $i \in \{0..119\}$ and such that for a suitable choice of the half-spaces determined by the $\mathcal{F}_i$'s, the intersection of these half-spaces is exactly the 120-cell. The traditional construction splits the whole process in nine stages.

In this paper, we shall perform the splitting at the same time as we construct the hyper-faces of the 120-cell. The traditional construction proceeds in a symmetrical way and mainly aims at counting the elements of the 120-cell: its vertices, its edges, its faces and its dodecahedra. We have to visit again this construction in order to identify each dodecahedron and to fix its respective positions with its neighbouring ones. To make this point clear, we start with a new visit to the $3D$ case.

4.1 In lower dimensions

The right hand part of figure 4 illustrates the splitting of an octant in $I\!H^3$ as it is indicated in [Margenstern et al. 2002*b*,*c*, 2003*c*]. The left hand part of the figure illustrates another splitting which is based on the construction of the dodecahedron by generations, following the guidelines of the construction of the 120-cell. Indeed, the left hand splitting yields three octants and four half-octants (eight-faced regions), while the right hand one yields five octants and no half-octant.

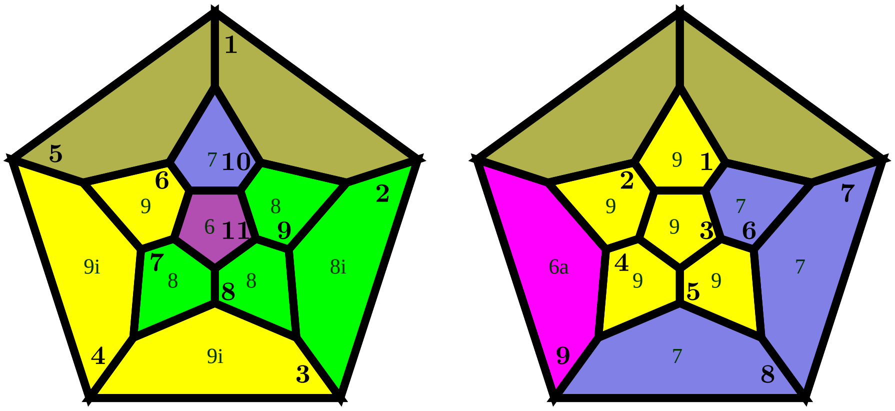

Figure 4. In the left hand side: splitting the octant in $I\!H^3$, generation after generation. The first generation is indicated by i, the second corresponds to the inner ring and the third generation contains a single element: the top of the initial dodecahedron. In bold symbols, we give the numbers of the faces.
In the right hand side: the splitting of the former papers.

We obtain the other regions for a basis of the splitting by shadowing successively one face of the outer ring on the figure of the left hand part of figure 4. The reader may easily check that we obtain the matrix of the splitting which is indicated by table 2.

A simple computation which we checked with *Maple*8 shows us that the polynomial of the new splitting is again polynomial $P_{\mathbb{H}^3}$ given in [Margenstern et al. 2002*c*, 2003*c*]:

$$P_{\mathbb{H}^3}(X) = X^3 - 9X^2 + 9X - 1.$$

	9	8	7	6
3	3	4	1	1
2	2	4	1	1
1	1	4	1	1
0	1	3	1	1

Table 2. The matrix of the splitting for the other splitting of the octant in $\mathbb{H}^3$.

The same construction can be performed for the rectangular pentagonal grid of $\mathbb{H}^2$, we leave this to the reader.

Starting from this point, the term *dodecahedron* stands for *rectangular regular* dodecahedron.

4.2 The first and the second generation.

4.2.1 The first generation.

The first region S_0 of the basis of the splitting is a sixteenth of $\mathbb{H}^4$, *i.e.* a **hyper-corner** which we already defined as the intersection of the half-spaces which are determined by four pairwise perpendicular hyper-planes.

We fix a 120-cell $\mathcal{M}_0$ at the extremal point O of S_0, which belongs to all the hyper-planes which generated S_0. We notice that this hyper-corner can be represented by the four dodecahedra of 120-cell $\mathcal{M}_0$ which meet at O. It is not difficult to see that we can reconstruct the hyper-corner from $\mathcal{M}_0$, once we fix one of its vertices: from incidence table 1, we know that precisely four dodecahedra meet in the vertex and that they are pairwise perpendicular.

Let $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ be these four dodecahedra. Their respective positions at their common vertex, one with respect to each other, are symmetric. Call $\mathcal{D}_0$ the **bottom** of $\mathcal{M}_0$, and $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ its **walls**. The terms of *bottom* and of *walls* could be exchanged. They are used in order to break the symmetry.

Next, we characterise the image of a shift of $\mathcal{M}_0$ along one of the edges being issued from O.

Hyper-corner Lemma – *Let four dodecahedra $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ share a common vertex O and be pairwise perpendicular, sharing pairwise a common face. Consider another vertex B, sharing an edge of these dodecahedra with O.*

Consider the three dodecahedra sharing OB, for instance, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. Consider one of these dodecahedra, say $\mathcal{D}_1$. Let F and G be the two faces of $\mathcal{D}_1$ sharing OB. Then the image under the shift along OB of the 120-cell $\mathcal{M}$ constructed on $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ is the reflection of $\mathcal{M}$ in the image of $\mathcal{D}_0$ under the same shift. In particular, we have that the image of $\mathcal{D}_1$ under the shift is its reflection in H, where H is the third face of dodecahedron $\mathcal{D}_1$ which contains B. As a consequence, this reflection is in the same hyper-plane as $\mathcal{D}_1$.

The result of the lemma is illustrated by figure 5, below. Face F of the lemma is determined by OB and OC in the figure. Similarly, face G is determined by OB and OD, and face H by BM and BN.

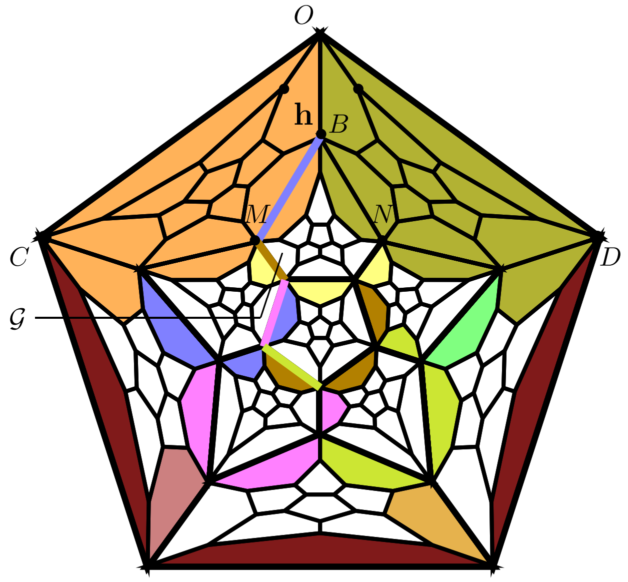

Figure 5. Splitting the basic hyper-corner: first generation
Notice the various shifts being indicated by a coloured broken line.

Traditionally, one of the fourth dodecahedra which we introduced is called the initial one and it defines the 0^{th} generation. Here it is $\mathcal{D}_1$.

The first generation is obtained by the orthogonal completion of all the faces of $\mathcal{D}_1$. Of course, the completion is relative to $\mathcal{M}_0$, but we may also see it, at this stage, as being relative to the initial hyper-corner.

From the above lemma and coming back to definitions, we see that the dodecahedra which are put on the faces of $\mathcal{D}_1$ by the orthogonal completion can be obtained by successive shifts along the edges of $\mathcal{D}_1$.

From now on, we number the dodecahedra as it is indicated in the right hand part of figure 4 and they will be denoted by $\mathcal{E}_i$, with $i \in \{1..9\}$ for the first generation.

Now, consider the splitting. Let us have a closer look at $\mathcal{E}_1$. As shift $\mathbf{h}$ along OB is in the same $3D$ space as $\mathcal{D}_2$, we see that this hyper-plane is a wall for $\mathbf{h}(\mathcal{M}_0) = \mathcal{M}_1$. Similarly, $\mathcal{D}_2$ is also a wall for $\mathcal{M}_1$. Two other walls are determined by $\mathbf{h}(\mathcal{D}_0)$ which is $\mathcal{E}_1$ and by $\mathbf{h}(\mathcal{D}_1)$ which determines the same hyper-plane as $\mathcal{D}_1$ because OB belongs to this hyper-plane. An analogous shift which is indicated in figure 5 allows us to obtain $\mathcal{M}_2$ from $\mathcal{M}_1$.

Now, consider face $\mathcal{G}$ which is indicated in figure 5. It is identified with the opposite face to it in face 10 of $\mathcal{D}_1$ thanks to the incidence table. Four dodecahedra share this face. We already know two of them: $\mathcal{E}_1$ and $\mathcal{E}_2$. The others are the reflections of them in the face, this can be checked on the hyper-cube, using the argument of sub-section 3.2.1. Now, $\mathcal{E}_1$ defines a wall for $\mathcal{E}_2$, and it is easy to see that the 120-cell obtained from $\mathcal{M}_1$ by the reflection in $\mathcal{E}_2$ is the same as the one which is obtained from $\mathcal{M}_2$ by the reflection in $\mathcal{E}_1$. Accordingly, one reflection must be prohibited while the other one must be enabled.

We indicate this choice by shadowing the image of $\mathcal{G}$ which lies in the representation of $\mathcal{E}_2$ as it is indicated in figure 5. This is a shadowing which looks like to the one which the author introduced in [Margenstern et al. 2002*b*], but which is not obtained in the same way. Here we take advantage of the identification by shadowing one representation of the face and by un-shadowing the other one.

What we discussed about face $\mathcal{G}$ and about the walls of $\mathcal{M}_1$ induces two simple rules:

Shading Rules:

- *If a face of a new generation is adjacent to a shaded face of the previous generation, it must be shaded.*
- *When two identified faces appear at the same generation, one of them must be shaded and the other one must remain un-shaded.*

In these rules, generations are understood in the meaning given to this word during the construction of the 120-cell.

The starting point in the application of these rules is that the walls and the bottom of the initial 120-cell are considered as shaded: recall that the bottom is also a wall in this regard.

As a result, the first generation gives us three new regions which can be represented, respectively, by the bottom of the considered 120-cell. In the first case, it is a dodecahedron with four shaded faces, in the second case, it is a dodecahedron with five shaded faces and in the third case, it is a dodecahedron with six shaded faces. Notice that this corresponds to the barred edges of the bottom in the $3D$ situation. The contribution of the first generation to the different types of regions is given in table 3.

At last, remark the **analogy** of situation between the $4D$ case and the $3D$- and $2D$- cases. We already stressed it and the reader can check again the parallelism in the rôles played by the vertices, the lines and the faces in, respectively, $I\!H^2$, $I\!H^3$ and $I\!H^4$.

4.2.2 The second generation and the third generation.

Now, we turn to the second generation in the construction of the 120-cell which consists of the dodecahedra which are obtained by the orthogonal completion of the faces which are in the inner rings of the dodecahedra of the first generation.

Starting from this generation, we shall first establish a **map** of the identifications between faces of the dodecahedra of the considered generation. Then, taking advantage of these identifications, we shall proceed to the splitting for this stage of the construction.

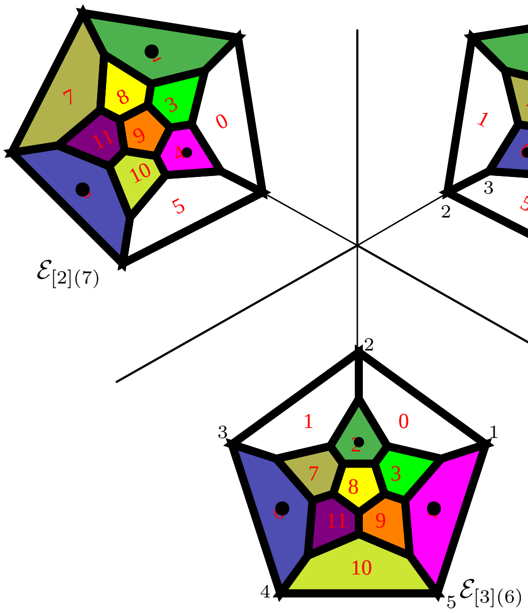

Figure 6. The 2^{nd} *generation in the Schlegel diagram:*
the colours represent corresponding faces, a face marked with a dot occurs in another dodecahedron of the second generation corresponding faces have the same number.

Above, figure 6 illustrates the map of the second generation. A close examination of the closest vertices of the first generation around a fixed vertex of $\mathcal{D}_1$, see figure 5, shows that the three dodecahedra of the second generation which are put on inner faces around a same vertex of $\mathcal{D}_1$ represent the same dodecahedron. They are identified but we have to see them as *different views* of the dodecahedron.

In figure 6 we use an auxiliary numbering of the faces of the dodecahedra of the second generation which are identified in order to establish an exact correspondence between the identified elements.

These identification allow us to obtain the splitting of the second generation. As a starting point, notice that $\mathcal{E}_{10}$ is obtained from $\mathcal{E}_1$ as this dodecahedron was obtained from $\mathcal{D}_1$. Notice also that several views of the same dodecahedron gives us additional information on the shadowing: what must be shaded in a view must also be shaded in the others.

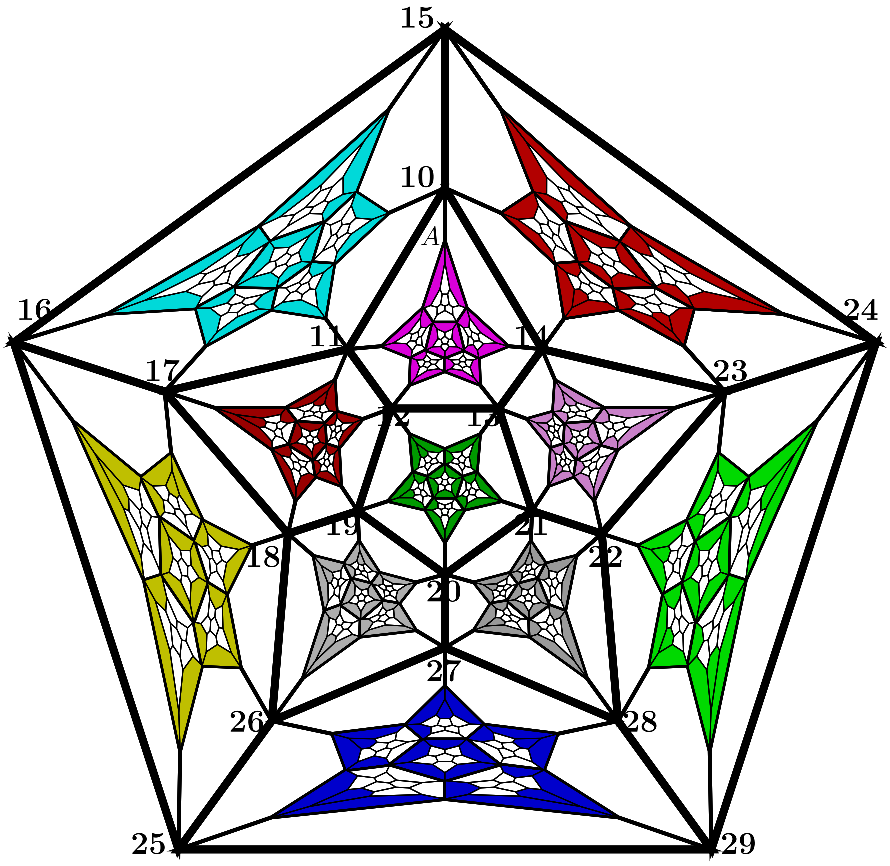

Figure 7. Splitting the basic hyper-corner: second and third generation

Figure 7 should be considered dynamically, putting and shadowing the dodecahedra one after another from $\mathcal{E}_{10}$ up to $\mathcal{E}_{29}$. The figure shows us a new type of region with a bottom which has also six shaded faces but which is different from the six-shaded faced dodecahedron of the first generation. In table 3, the types of regions are numbered by the non-shaded faces. The 6-faced type of the first generation is called $6a$ and the 6-faced type given by the second one is called $6b$. The contribution of the second generation is given in table 3.

The third generation is obtained by the orthogonal completion of the tops of the dodecahedra of the first generation, see figure 7. The third generation provides us with 6-faced regions of the type $6a$, see table 3.

4.3 The fourth and fifth generations.

The fourth generation is obtained by the orthogonal completion of the inner faces of the dodecahedra of the third generation. From the identifications, we obtain that the dodecahedra of the fourth generation are identified by pairs: the pairs belong to dodecahedra which are put on faces of dodecahedra of the third generation which are 'symmetric' in the sense of the Schlegel diagram, with respect to an edge of $\mathcal{D}_1$, see figure 9. Also, from figure 6, we know that all the faces of the dodecahedra of the second generation are now covered.

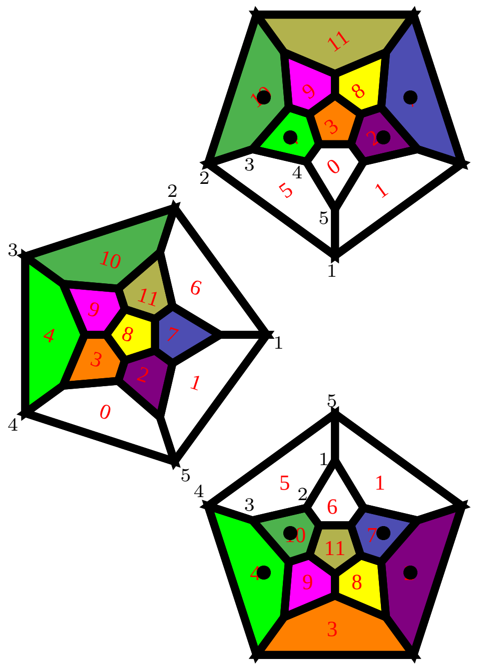

Figure 8. The 4^{th} *generation in the Schlegel diagram:*
the colours represent corresponding faces, a dotted face occurs in another dodecahedron of the fourth generation
identified faces are numbered the same

As in the case of the second generation, we use an auxiliary numbering in order to establish the map of figure 8 for the exact correspondence between the faces of the identified dodecahedra.

Applying again the same arguments as we did in the previous generations, we now proceed to the splitting which is displayed in figure 9.

From the map, we see that the regions of the fourth generation have a bottom with at least four shaded faces. Two new types of regions are given by the fourth generation: 5-faced and 4-faced regions. The total contribution of the generation is provided us by table 3.

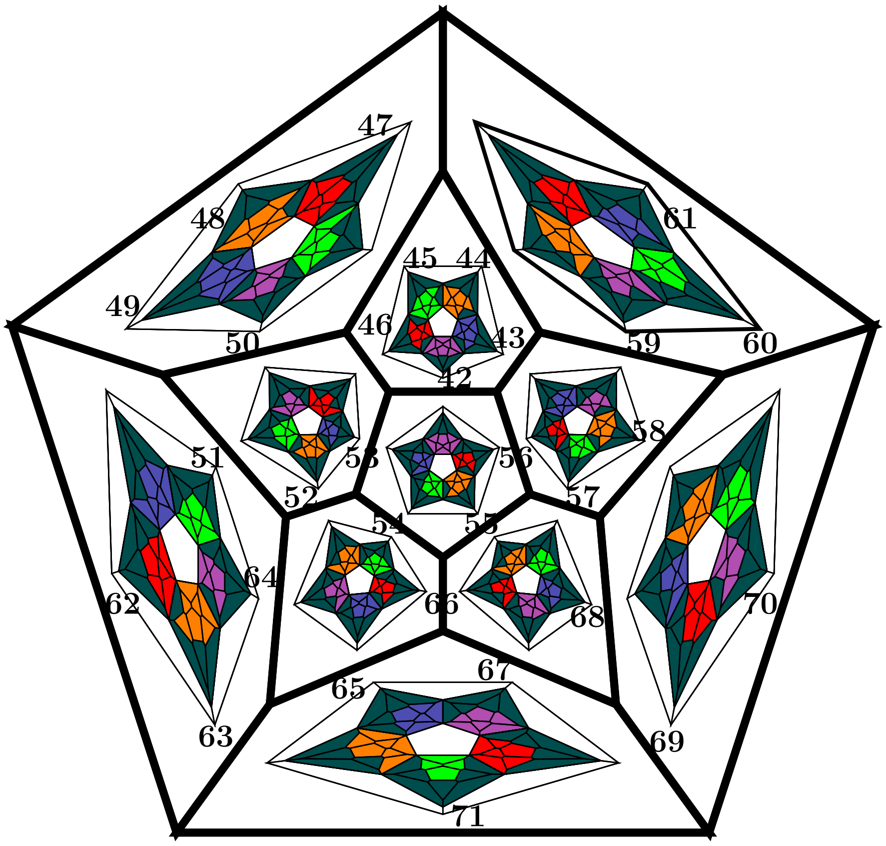

Figure 9. Splitting the basic hyper-corner: the walls for the fourth generation

The fifth generation gives us the same types of regions as the third generation and it also contributes with the same number of copies of the regions as the third generation.

4.4 The sixth and seventh generations.

4.4.1 The sixth generation

This time, we perform the orthogonal completion of the inner faces of the dodecahedra of the fifth generation. From the identified elements of the previous generation, we get that here again, the dodecahedra are identified by groups of three.

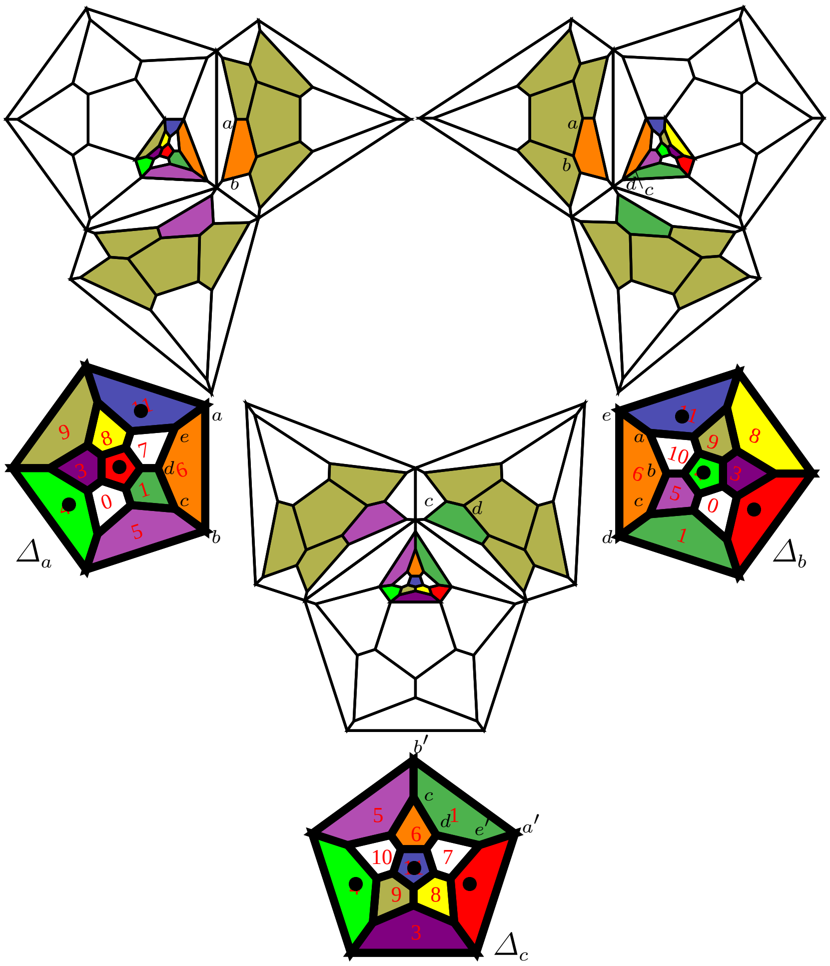

Figure 10. The 6^{th} *generation in the Schlegel diagram: superposition of two different zooms: a face marked with a dot occurs in another dodecahedron of the sixth generation; identified faces are numbered the same*

Figure 10 represents two views of such a group of three identified dodecahedra. One view shows the places of the dodecahedra of the sixth generation with respect to the fifth and the fourth generations. A zoom on the new identified dodecahedra is presented in the same figure in order to show the correspondence between their faces.

From figure 10, we obtain that the regions which appear in the splitting for this generation are based on a bottom with at most six un-shaded faces. A new type of region is yielded by the sixth generation: it has a 3-faced bottom. See the contribution of the generation on table 3.

Next, figure 11 shows us the splitting. As for figure 9, we have to see the figure dynamically, putting the dodecahedra one after another, the three views together and, at the same time, shadowing them one after another, taking into account the three views. Notice that here also, the dodecahedra which are built on the back of $\mathcal{D}_1$ are not represented, but they are taken into account.

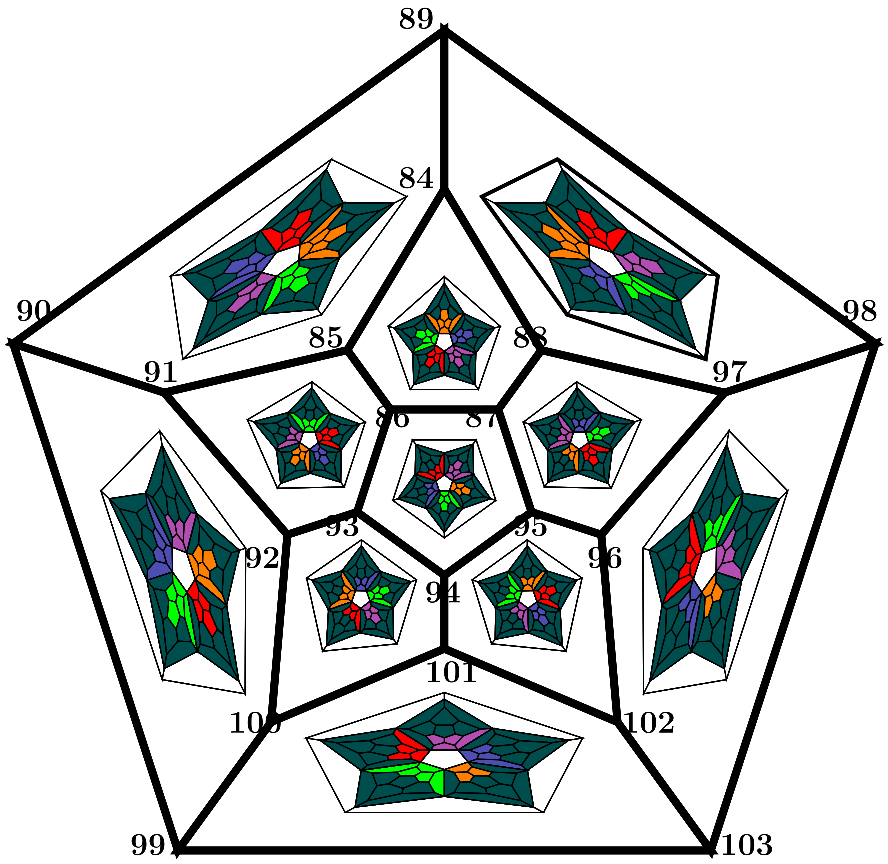

Figure 11. Splitting the basic hyper-corner: the walls being defined by the sixth generation

4.4.2 The seventh generation

For the seventh generation, we perform the orthogonal completion of the tops of the dodecahedra of the fifth generation.

Figure 12 gives us the map of the situation. Notice that in this map, we represent three dodecahedra of the seventh generation which are around the same dodecahedron $\mathcal{D}$ of the sixth. However, in order to establish all the identifications, the three views of $\mathcal{D}$ are represented in the figure: they constitute the 'inner ring' of dodecahedra. From the map, we see that three faces of the seventh generation are identified and that each one has two views on the map. And so, face 8 of $\mathcal{E}_{105}$ is face 9 of $\mathcal{E}_{104}$, face 8 of $\mathcal{E}_{104}$ is face 9 of $\mathcal{E}_{103}$ and face 8 of $\mathcal{E}_{103}$ is face 9 of $\mathcal{E}_{105}$. This shows us that the tops of $\mathcal{E}_{103}$, $\mathcal{E}_{104}$ and $\mathcal{E}_{103}$ belong to the same dodecahedron: the single dodecahedron of the eight generation.

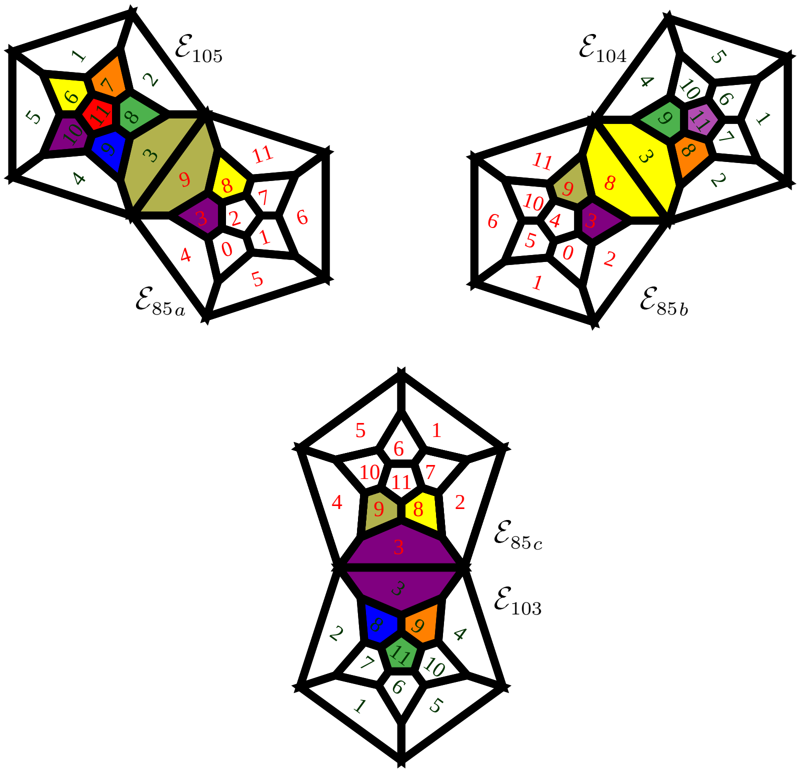

Figure 12. The 7^{th} *generation in the Schlegel diagram: superposition of two different zooms: a face marked with a dot occurs in another dodecahedron of the seventh generation; corresponding elements are numbered the same in the sixth generation only*

We can now proceed to the splitting as in the previous generations, see figure 13. From the map of figure 12, we know that the regions of the seventh generation have a bottom which has at most six un-shaded faces. The seventh generation introduces two new types of region: one with a 2-faced bottom and the other one with a 3-faced one, see table 3.

We conclude this section with the splitting by noticing that the eight generation contributes with a single region which has a 0-faced bottom.

5 Matrix, polynomial and language

We gather the information being yielded by the previous sections in table 3.

This information concerns the splitting of S_0 only. However, the same table allows us to obtain the splitting of the other regions.

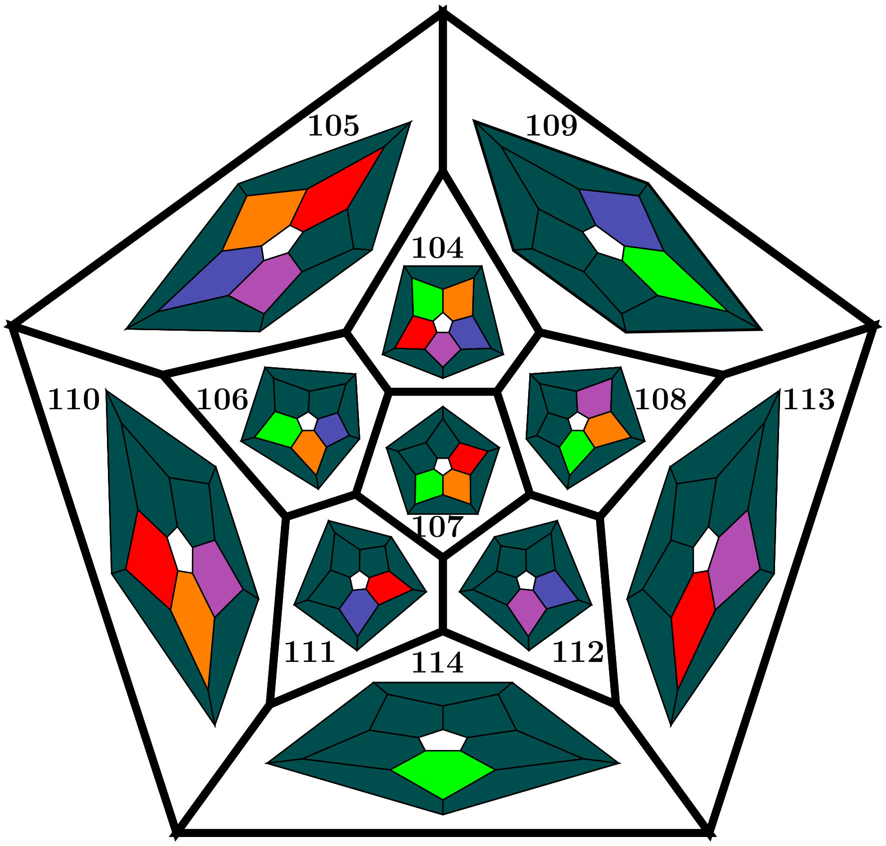

Figure 13. Splitting the basic hyper-corner: the walls being defined by the seventh generation

	9	8	7	$6a$	$6b$	5	4	3	2	1	0
I	5	0	3	1	0	0	0	0	0	0	0
II	1	9	9	0	1	0	0	0	0	0	0
III	0	0	0	12	0	0	0	0	0	0	0
IV	0	1	9	9	1	9	1	0	0	0	0
V	0	0	0	12	0	0	0	0	0	0	0
VI	0	0	0	0	1	9	9	1	0	0	0
VII	0	0	0	1	0	1	4	4	1	1	0
VIII	0	0	0	0	0	0	0	0	0	0	1
9	6	10	21	35	3	19	14	5	1	1	1

Table 3. The matrix of the splitting: counting the contribution of each generation for the splitting of a hyper-corner.

Indeed, the difference between the regions concerns the first generation only. As the other generations are inside the intersection of the half-spaces being determined by the hyper-faces of the first generation, they also contribute to the splitting of the other regions, and with the same number of S_i's.

From this remark and from the information of the previous sections, we have 11 regions the splitting of which is given by the following table which constitutes the matrix of the splitting. The regions are determined by the number of shaded/un-shaded faces of their bottom, taking into account that there are two non-isometric regions with a 6-faced bottom, as we already noticed: they are denoted by $6a$ and $6b$ in tables 3 and 4. We refer the reader to [Margenstern 2003a] for a proof of this difference between the regions $6a$ and $6b$.

9	6	10	21	35	3	19	14	5	1	1	1
8	5	10	21	35	3	19	14	5	1	1	1
7	4	10	21	35	3	19	14	5	1	1	1
$6a$	3	11	20	35	3	19	14	5	1	1	1
$6b$	2	12	20	35	3	19	14	5	1	1	1
5	2	11	20	35	3	19	14	5	1	1	1
4	2	10	20	35	3	19	14	5	1	1	1
3	1	11	19	35	3	19	14	5	1	1	1
2	1	10	19	35	3	19	14	5	1	1	1
1	1	10	18	35	3	19	14	5	1	1	1
0	1	10	18	34	3	19	14	5	1	1	1

Table 4. The matrix of the splitting.

The characteristic polynomial of the matrix of the splitting, as being computed by $Maple8$, is the following:

$$P_0(X) = X^{11} - 116X^{10} + 366X^9 - 116X^8 + X^7.$$

Accordingly, the polynomial of the splitting is:

$$P(X) = X^4 - 116X^3 + 366X^2 - 116X + 1.$$

We notice that P is a reciprocal polynomial. From this, it is not difficult to obtain an algebraic expression for its roots by using an auxiliary equation of the second degree. This gives us:

$$\begin{aligned}
X_1 &= 29 + 5\sqrt{30} + \sqrt{1590 + 290\sqrt{30}},\\
X_2 &= 29 + 5\sqrt{30} - \sqrt{1590 + 290\sqrt{30}},\\
X_3 &= 29 - 5\sqrt{30} + \sqrt{1590 - 290\sqrt{30}},\\
X_4 &= 29 - 5\sqrt{30} - \sqrt{1590 - 290\sqrt{30}}.
\end{aligned}$$

whose approximate values are:

$$\begin{array}{ll}
X_1 = 112.7633976 & X_3 = 2.880593478\\
X_2 = 0.00886813 & X_4 = 0.347150762
\end{array}$$

This computation of the roots shows us that the greatest root of the polynomial is not a Pisot number. Also, clearly, as $P(X)$ is irreducible, it is not divided by a polynomial of the form $X^k + X^{k-1} + \ldots + 1$ and so, as it is proved by theorem 8.1 of [Hollander 1998], the language of the splitting is not regular.

Now, if $u(0) = 1$, and if we assume $u(-1) = 0$, $u(-2) = 0$ and $u(-3) = 0$, we get, successively:

1	2	3	4	5
116	13 090	1 476 100	166 450 115	18 769 479 064

which are the number of regions of, respectively, the first, the second, the third, the fourth and the fifth level in the spanning tree of the splitting.

As it is indicated in [Margenstern 2003a], there are other ways to split the regions with the same basis of splitting which also yields the same polynomial for the splitting. From this property, it is not difficult to prove the following consequence, see [Margenstern 2003a]:

Theorem 2 − *There are continuously many spanning trees for the tiling being defined by the tessellation of the* 120*-cell in* $I\!H^4$*, all of them being defined by the same basis of splitting. All of them also yield the same polynomial for the splitting.*

We find here for $I\!H^4$ a property which we proved in $I\!H^2$, with the pentagrid, see [Margenstern 2000a].

As a concluding remark, I indicate an alternative proof of the correctness of the splitting. Considering the sixteen hyper-corners which constitute the whole $4D$ hyperbolic space, it is not difficult to see that the number w_n of new 120-cells which is obtained at generation n, by reflection on the hyper-faces of the 120-cells of generation $n-1$, is given by the same induction sequence with appropriate initial values. In [Margenstern 2003*e*], I indicate a way to obtain this sequence as follows. Fix a 120-cell $\mathcal{M}_0$ of the tiling which will be considered as the origin and to which we associate the identity on $I\!H^4$. We consider the reflections in the hyper-faces of $\mathcal{M}_0$ as **generators** of a group Γ of endomorphisms of $I\!H^4$ which leaves the tiling invariant. For any tile $\mathcal{M}$, we consider the **words** on the alphabet of the generators of Γ which corresponds to products of these generators which allow to transform $\mathcal{M}_0$ into $\mathcal{M}$. This can be also seen as a **path** in the tiling which allows to go from $\mathcal{M}_0$ to $\mathcal{M}$ by moving from a tile to a neighbouring one. Define the distance of $\mathcal{M}$, denoted by $d(\mathcal{M})$, as the shortest length of a word which determines a path from $\mathcal{M}_0$ to $\mathcal{M}$. In [Margenstern 2003*e*], I call **pattern of distances**, the set of values $d(\mathcal{M}_V)-d(\mathcal{M})$ when V runs over the neighbours of $\mathcal{M}$ in the tiling. It is not difficult to see that there are finitely many patterns of distances when M runs other all the tiling. In [Margenstern 2003*e*], I prove that these patterns are characterised by their numbers of $+1$ and -1 and that there are four of them. I also define an incidence matrix A for these patterns. Taking into account the multiplicity which is induced by considering all the reflections, I obtain a new matrix B with coefficients those of A **weighted** by multiplicities. The characteristic polynomial of B is exactly P, and so I obtain the same sequence $\{w_n\}_{n\in I\!N}$.

6 Conclusion

As a conclusion for this extended abstract, I would draw the attention of the reader on the fact that a lot of properties of the tiling of $I\!H^4$ by the 120-cell have to be found.

The example of the language of the splitting is an important one. We know that the language is not regular. It would be interesting to know whether it is context-free or not. On the few data which I have at the present moment, see an indication about them in [Margenstern 2003*a*], it seems to me that we may conjecture that the language is also not context-free.

Another direction is the implementation of cellular automata which was done in the case of $I\!H^3$ in [Margenstern et al. 2002*b*,*c*, 2003*c*]. It seems that the same technique could be applied here and that the same complexity result would be obtained. Indeed, as it is indicated in [Margenstern et al. 2002*b*,*c*, 2003*c*], the language of the splitting allows us to define a system of **coordinates** for the tiles of the tiling. Considering the spanning tree, we number the tiles in a hyper-corner

S_0 starting from the root, level by level and, on each level, from the left to the right. The coordinate of a tile is defined as the maximal greedy representation of its number in the basis defined by sequence $\{u(n)\}_{n\in\mathbb{N}}$. In order to define coordinates for all the tiles of the tiling, we take sixteen hyper-corners at a vertex of the tiling which we take as an origin and we associate two numbers to a tile: the number of its hyper-corner and the number of the tile in the spanning tree associated to this hyper-corner. Using local maps as in [Margenstern et al. 2003*c*] and also [Margenstern 2003*e*], which indicate the relative positions of the tiles using the numbering, we can extract the coordinates of the neighbours of a tile τ from the coordinate of τ in its hyper-corner. Such an alorithm is the basis of any implementation of cellular automata on a tiling and from what we just said, we see that such an algorithm is feasible for the grid which we consider. Its time and space complexity should be the same as in the $3D$ case, *i.e.* cubic and quadratic, respectively, see [Margenstern et al. 2003*c*].

From the point of view of the combinatoricity of tilings, it would be interesting to know whether the property also holds for the other tessellations of $\mathbb{H}^4$: there are still two of them, taking into account that the tiling of the 600-cell is the dual of this one.

At last, we may conclude this section by indicating a kind of 'natural' conclusion: there are no tessellation of the hyperbolic space $\mathbb{H}^n$ for $n \geq 5$, see for instance [Sommerville 1958] for a proof of this property. There are purely combinatoric properties which bar this possibility. Maybe this is also connected with the properties of the language of the splitting?

Acknowledgements

I am much in debt to Gencho Skordev and to Vanya Stoyanova who, a couple of years ago, draw my attention on Schlegel diagrams, repeating at that time the constructions of [Sommerville 1958] for the 120-cell. This allowed me to define the shadowing technique for the $3D$ hyperbolic case which, together with the splitting method, is the key of [Margenstern et al. 2002*b*,*c*, 2003*c*].

I am also very much in debt to Cris Calude for his interest in this work.

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