Journal of Universal Computer Science, vol. 9, no. 5 (2003), 398-422
submitted: 22/5/02, accepted: 26/7/02, appeared: 28/5/03 © J.UCS

Fibonacci Type Coding for the Regular Rectangular Tilings of the Hyperbolic Plane

Maurice Margenstern
(Laboratoire d'Informatique Théorique et Appliquée
EA 3097, UFR MIM, Université de Metz
margens@lita.sciences.univ-metz.fr)
Gencho Skordev
(CeVis, Fachbereich Mathematik
Universität Bremen
skordev@cevis.uni-bremen.de)

Abstract: The study of cellular automata (CA) on tilings of hyperbolic plane was initiated in [6]. Appropriate tools were developed which allow us to produce linear algorithms to implement cellular automata on the tiling of the hyperbolic plane with the regular rectangular pentagons, [8, 10]. In this paper we modify and improve these tools, generalise the algorithms and develop them for tilings of the hyperbolic plane with regular rectangular s-gons for $s \geq 5$. For this purpose a combinatorial structure of these tilings is studied.

Key Words: cellular automata, tiltings, hyperbolic plane
Category: F.1.1, F.1.3

1 Introduction

There are some recent developments in a combinatorial approach to the hyperbolic geometry. As an example, we can quote [1, 5, 14]. The idea is to grasps features of the hyperbolic geometry by appropriate algorithms and use them to perform some constructions or to prove that particular constructions are not possible.

The present paper belongs to this general trend. It is a generalization and a simplification of [6, 7, 8, 9, 10].

In [6] a particular cellular automaton (CA) on the pentagrid $\mathcal{T}_5$ (the tiling of the hyperbolic plane by regular rectangular pentagons) was defined. This CA solved the SAT problem in a polynomial time. The key point in the construction and the implementation of this CA was the combinatorial structure of the tiling $\mathcal{T}_5$, revealed by a splitting procedure, proposed in [6]. It was described by an appropriate spanning tree T_5 of the adjacency graph of the tiling $\mathcal{T}_5$. With the help of this tree a special coding of the tiles of $\mathcal{T}_5$ was introduced, see [10]. The coding was given by the Fibonacci representation of the numeration labels of the nodes of the tree T_5. It turns out that this coding is appropriate for the implementation of CA on $\mathcal{T}_5$, [2].

As in [6], we work for simplicity with the tiling $\mathcal{T}_s^{sw}$ of the south-west corner of the hyperbolic plane, induced by the tiling $\mathcal{T}_s$ of the hyperbolic plane by regular rectangular s-gons for $s \geq 5$.

First we describe the combinatorial structure of the tiling $\mathcal{T}_s^{sw}$ by an appropriate spanning tree T_s of the adjacency graph of the tiling $\mathcal{T}_s^{sw}$. Then we introduce a Fibonacci type coding of the tiles of $\mathcal{T}_s^{sw}$. In the case $s = 5$ this coding is different from the Fibonacci coding, being introduced in [10]. At the end we present several algorithms, in particular we give an algorithm for the codes of the tiles adjacent to a given tile. This algorithm uses as input only the code of the given tile.

The results of this paper were announced in [11].

2 The tiling $\mathcal{T}_s$ of the hyperbolic plane

We use the conformal model of the hyperbolic plane H^2 in the unit disc B^2 (also called Poincaré disc model), see [15], Ch. I, p. 50. In this model the hyperbolic plane is the open disc $B^2 = \{(x, y) \in R^2 : x^2 + y^2 < 1\}$. The boundary of this disc is called the absolute of H^2. The straight lines (h-lines) of H^2 are the intersections with B^2 of circles or Euclidean straight lines, orthogonal to the absolute. In particular, all diameters of the unit disc B^2 are h-lines. For a given h-line, the circle or the Euclidean line containing it, is called its support (or supporting circle/line). The angle between two intersecting h-lines is the Euclidean angle between their supports.

The group of the isometries of the hyperbolic plane H^2 (h-isometries) is generated by the reflections (h-reflections) in the h-lines. They are defined as follows. Let k be a h-line supported by the Euclidean straight line $\tilde{k}$. The h-reflection in k is the Euclidean reflection in $\tilde{k}$, restricted on H^2. Let k be a h-line supported by the circle $\tilde{k}$. The h-reflection in k is the inversion in $\tilde{k}$, restricted on H^2, see [15] , Ch. 3 for details.

For every natural number $s \geq 5$ there is a unique (up to h-isometry) tiling of the type $\{s, 4\}$ of H^2 by regular s-gons with all angles equal to $\pi/2$ (rectangular s-gons), see [15], Ch. II, Table 6, p. 217. The Schläfli symbol $\{s, 4\}$ is self explanatory: it denotes a tiling of H^2 by regular s-gons with four tiles at every vertex.

We shall consider a particular tiling $\mathcal{T}_s$ of the type $\{s, 4\}$ constructed as follows. Consider the vertical and the horizontal diameters d_1, and d_2 of the unit disc B^2. They intersect at the center M_0 of the disc and divide the disc in four quarters. We shall construct the restriction $\mathcal{T}_s^{sw}$ of the tiling $\mathcal{T}_s$ in the south-west quarter of B^2. Then the whole tiling $\mathcal{T}_s$ is generated by the images of $\mathcal{T}_s^{sw}$ w.r.t. the h-reflections in the diameters d_1 and d_2.

The tiling $\mathcal{T}_s^{sw}$ is defined by induction. Consider the regular s-gon $\mathcal{P}^0$ in the south-west quarter of the disc B^2 with a vertex M_0 and two sides, supported by the diameters d_1 and d_2, respectively. We say that $\mathcal{P}^0$ is the tile of the zero-th generation of $\mathcal{T}_s^{sw}$. The sides of $\mathcal{P}^0$, not supported by d_1 or d_2, are called free sides of the zero-th generation.

Assume that the tiles of the k-th generation of the tiling $\mathcal{T}_s^{sw}$ are constructed and that their free sides of the k-th generation are defined. The tiles of the $(k+1)$-th generation are the images of the tiles of the k-th generation w.r.t. the h-reflections in the free sides of the k-th generation. The free sides of the $(k+1)$-th generations are the sides of the tiles of the $(k+1)$-th generation, which are not sides of tiles of the k-th generation and are not supported by the diameters d_1 or d_2. In such a way the tiling $\mathcal{T}_s^{sw}$ is constructed. For $s = 5$ and $s = 6$ see Fig. 1.

Let u_k be the number of the tiles of the k-th generation of the tiling $\mathcal{T}_s^{sw}$. The sequence $\mathcal{U}_k = (u_k)_{k \geq 0}$ is not a geometrical progression, since some tiles of the $(k+1)$-th generation are obtained as images of two different tiles of the k-th generation. This is the reason why there is a combinatorial structure of the tiling $\mathcal{T}_s^{sw}$. We shall describe it in the next section.

Since the area of the hyperbolic disc of radius r depends exponentially on r, we expect that u_k depends exponentially on k.

3 Combinatorial structure of the tiling $\mathcal{T}_s^{sw}$

The combinatorial structure of the tiling $\mathcal{T}_s^{sw}$ is revealed by a splitting procedure. We shall define first the splitting of special regions of the hyperbolic plane: the s-corners and the s-strips. Then we apply this construction by induction, starting with the given s-corner, the south-west quarter of H^2: it is split in a sequence of s-corners and s-strips. The combinatorial description of this sequence of regions is given by the tree T_s, which is also a spanning tree of the adjacency graph of the tiling $\mathcal{T}_s^{sw}$. Moreover, the nodes of the k-th generation of T_s correspond to the tiles of the k-th generation of the tiling $\mathcal{T}_s^{sw}$.

At the end, using the tree T_s, we shall prove that the sequence $\mathcal{U}_s = (u_k)_{k \geq 0}$ is a solution of a linear recurrence equation of a second order with constant coefficients. This gives an explicit formula for the number u_k of the tiles of the k-th generation of $\mathcal{T}_s^{sw}$.

3.1 s-corners and s-strips in the hyperbolic plane

Let l, m be two orthogonal h-arrows (half h-lines) with common point M in the hyperbolic plane. Assume that the orientation from l to m coincides with the positive orientation of H^2. We call the right angle $\angle(l, m)$ *a corner*, and denote it by $C = (l, m, M)$ (we consider the right angle as a region). An example is the corner $C^0 = (l_0, l_{s-1}, M_0)$, where l_{s-1} is the half diameter supported by the vertical diameter d_1, and pointing to the south, and l_0 is the half diameter, supported by the horizontal diameter d_2, and pointing to the west (l_0 and l_{s-1} are hyperbolic arrows (h-arrows) in the hyperbolic plane H^2 with common point M_0, see Fig. 2).

For every corner $C = (l, m, M)$ there is a unique h-isometry, preserving the orientation of the hyperbolic plane, which maps the corner $C^0 = (l_0, l_{s-1}, M_0)$ on C, in such a way that the images of l_0 and of l_{s-1} are l, and m, respectively.

An *s-corner* is a pair $(C, \mathcal{P})$, where $C = (l, m, M)$ is a corner, and $\mathcal{P}$ is the regular rectangular s-gon with a vertex M and two sides, supported by the h-arrows l and m, respectively. The s-gon $\mathcal{P}$ is called the leading s-gon of the s-corner $(C, \mathcal{P})$. The corner $C^0 = (l_0, l_{s-1}, M_0)$ determines the s-corner $(C^0, \mathcal{P}^0)$, where $\mathcal{P}^0$ is the tile of the zero-th generation of the tiling $\mathcal{T}_s^{sw}$, see Fig. 2.

A strip $S = (p, PQ, q)$ is a region in the hyperbolic plane H^2, bounded by the h-arrow p, the segment PQ and the h-arrow q, where

- p is an h-arrow with initial point P. The angle between p and PQ is $\pi/2$, and the orientation from p to PQ coincides with the positive orientation of the hyperbolic plane;
- q is a h-arrow with initial point Q. The angle between QP and q is $\pi/2$, and the orientation from QP to q coincides with the positive orientation of the hyperbolic plane;
- the hyperbolic length of the segment PQ is equal to the hyperbolic length of a side of a regular rectangular s-gon (such a length is unique in the hyperbolic plane).

An example is the strip $S^0 = (l_1, M_1M_0, l_{s-1})$, where the segment M_1M_0 is supported by the h-arrow l_0, see Fig. 2.

For every strip $S = (p, PQ, q)$ there is a unique h-isometry, preserving the orientation of H^2, which maps $S^0 = (l_1, M_1M_0, l_{s-1})$ on S, in such a way that the image of the h-arrow l_1 is p, the image of the segment M_1M_0 is PQ, and the image of the h-arrow l_{s-1} is q.

With every strip $S = (p, PQ, q)$ we associate the s-gon $\mathcal{P}$ with side PQ and two other sides, supported by p and by q, respectively. We call $\mathcal{P}$ the leading s-gon of the strip S and the pair $(S, \mathcal{P})$ - *s-strip*. The leading s-gon $\mathcal{P}^0$ of the strip $S^0 = (l_1, M_1M_0, l_{s-1})$ is the tile of the zero-th generation of the tiling $\mathcal{T}_s^{sw}$.

3.2 The splitting procedure

The splitting procedure was introduced in [6] for 5-corners and 5-strips in the hyperbolic plane. Here we shall define it in general. There are several possibilities for this as this was already noticed for the case $s = 5$ in [10]. We choose a particular one, which is the most appropriate for our purposes. For the following definitions consult Fig. 2.

Splitting of s-corners

Since all s-corners are h-isometric to the s-corner $(C^0, \mathcal{P}^0)$, it is enough to define the splitting only for that one.

First we introduce some notations, see Fig. 2. Remind that one of the sides of the leading s-gon $\mathcal{P}^0$ of the s-corner $(C^0, \mathcal{P}^0)$ is the segment M_0M_1, supported by the h-arrow l_0. We denote this side by $\mathbf{0}$. A second side of $\mathcal{P}^0$ is supported by the h-arrow l_{s-1}. We denote this side by $(\mathbf{s-1})$. The other sides of the s-gon $\mathcal{P}^0$ are denoted by $\mathbf{1}, \mathbf{2}, \ldots, (\mathbf{s-2})$ in the positive orientation. Let M_i be the intersection point of the sides $(\mathbf{i-1})$ and $\mathbf{i}$, for $1 \leq i \leq s-1$.

Moreover we denote by:

- l_i the h-arrow supporting the side $\mathbf{i}$ and with the initial point M_i, for $1 \leq i \leq s-4$;
- l_{s-2} the h-arrow supporting the side $(\mathbf{s-2})$ and with the initial point M_{s-1};
- $\tilde{l}_i$ the h-arrow supported by the h-arrow l_i and with the initial point M_{i+1}, for $0 \leq i \leq s-4$;
- $\tilde{l}_{s-1}$ the h-arrow supported by l_{s-1} and with the initial point M_{s-1};
- $\tilde{l}_{s-2}$ the h-arrow supported by l_{s-2} and with the initial point M_{s-2}.

Then we have the corners $C_{i+1} = (\tilde{l}_i, l_{i+1}, M_{i+1}),\ 0 \leq i \leq s-5$, and $C_{s-2} = (l_{s-2}, \tilde{l}_{s-1}, M_{s-1})$ and the strip $S_{s-3} = (\tilde{l}_{s-4}, M_{s-3}M_{s-2}, \tilde{l}_{s-2})$. Denoting their leading s-gons by $\mathcal{P}_i$, we have the s-corners $(C_i, \mathcal{P}_i)$, for $1 \leq i \leq s-2,\ i \neq s-3$, and the s-strip $(S_{s-3}, \mathcal{P}_{s-3})$.

Then we define *the splitting of the s-corner* $(C^0, \mathcal{P}^0)$ as the representation

$$\overline{C^0 \setminus \mathcal{P}^0} = C_1 \cup C_2 \cup \ldots \cup C_{s-4} \cup S_{s-3} \cup C_{s-2}$$

of the closure of the complement of $\mathcal{P}^0$ in the corner C^0 as an union of the $(s-3)$ corners, and the s-strip defined above.

We call $C_1, C_2, \ldots, C_{s-4}, S_{s-3}, C_{s-2}$ *descendants of the s-corner* $(C^0, \mathcal{P}^0)$. Later on we consider them in this order.

The particularity of this definition is the choice of the position of the s-strip between the corners: it is the penultimate.

Splitting of s-strips

Since all strips are h-isometric to the strip $(S^0, \mathcal{P}^0)$, it is enough to define the splitting of this strip. We shall use the notations introduced by the definition of the splitting of the s-corner $(C^0, \mathcal{P}^0)$, see Fig. 2.

The splitting of the s-strip $(S^0, \mathcal{P}^0)$ is defined as the representation

$$\overline{S^0 \setminus \mathcal{P}^0} = C_2 \cup \ldots \cup C_{s-4} \cup S_{s-3} \cup C_{s-2}$$

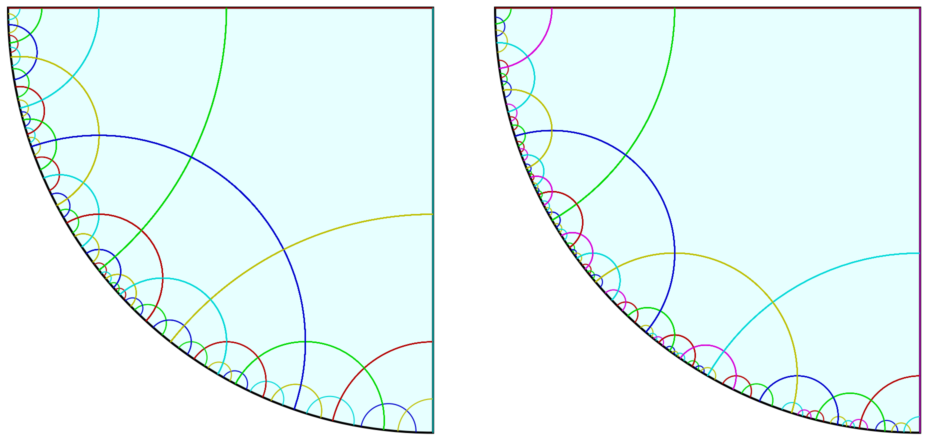

Figure 1 The tiling for $s = 5$ (left), and for $s = 6$ (right)

$\square$

of the closure of the complement of its leading tile $\mathcal{P}^0$ in the s-strip S^0 as a union of $(s-4)$ corners and one strip. We call $C_2, \ldots, C_{s-4}, S_{s-3}, C_{s-2}$ *descendants of the s-strip* $(S^0, \mathcal{P}^0)$. Later on we consider them in this order.

The particularity of our definition is again the choice of the position of the s-strip between the corners: it is also the penultimate.

3.3 Recurrent splitting of the s-corner

Now we shall apply the splitting procedure by induction, starting with the initial s-corner $(C^0, \mathcal{P}^0)$. We call it the corner of the zero-th generation. The descendants of $(C^0, \mathcal{P}^0)$ (w.r.t. the splitting procedure)

$$(C_1, \mathcal{P}_1), (C_2, \mathcal{P}_2), \ldots, (C_{s-4}, \mathcal{P}_{s-4}), (S_{s-3}, \mathcal{P}_{s-3}), (C_{s-2}, \mathcal{P}_{s-2}),$$

given in this order, are called regions (s-corners or s-strip) of the first generation.

Observe that the leading s-gons $\mathcal{P}_1, \ldots, \mathcal{P}_{s-2}$ of the regions of the first generation are the tiles of the first generation of the tiling $\mathcal{T}_s^{sw}$.

Now we proceed by induction. Assume that the regions (s-corners or s-strips) of the k-th generation are defined, and that their leading s-gons are all the tiles of the k-th generation of the tiling $\mathcal{T}_s^{sw}$. Applying the splitting procedure to the s-corners and s-strips of the k-th generation (in the given order) we obtain the regions (s-corners and s-strips) of the $(k+1)$-th generation. We consider them in the order which is inherited from the order of the regions of the k-th generation and the order of descendants of the corresponding s-corners and s-strips.

Obviously, the leading s-gons of the regions of the $(k+1)$-th generation are all the tiles of $(k+1)$-th generation of the tiling $\mathcal{T}_s^{sw}$.

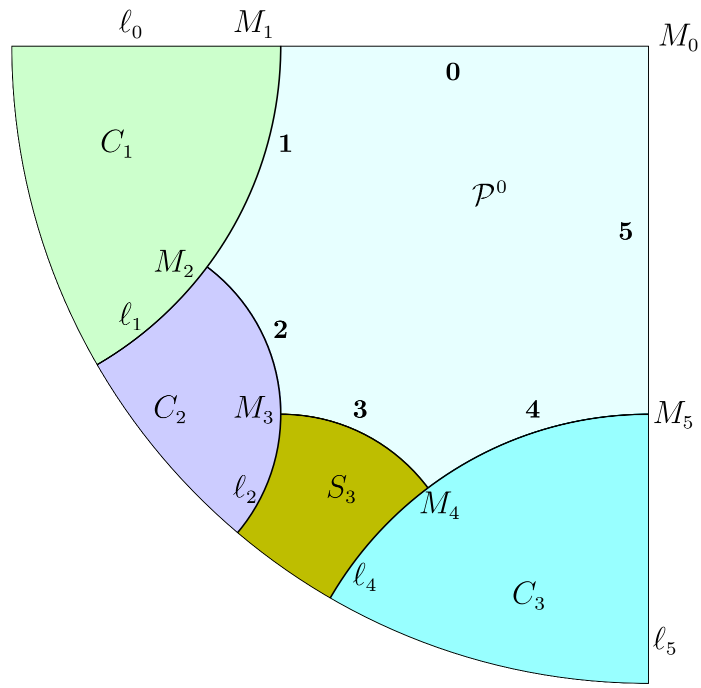

Figure 2 The splitting of an s-corner with $s = 6$

3.4 The spanning tree T_s

The recurrent splitting procedure of the s-corner $(C^0, \mathcal{P}^0)$ is described by a tree T_s. The nodes of this tree correspond to the s-corners and s-strips of the hierarchical splitting of $(C^0, \mathcal{P}^0)$. Accordingly, the generation is preserved. Let v be a node, corresponding to the region R (s-corner or s-strip). The descendants of v correspond to the descendants of R w.r.t. the splitting of R. We transfer the order of the sets of regions of the k-th generation to the set of nodes of the k-th generation of the tree T_s.

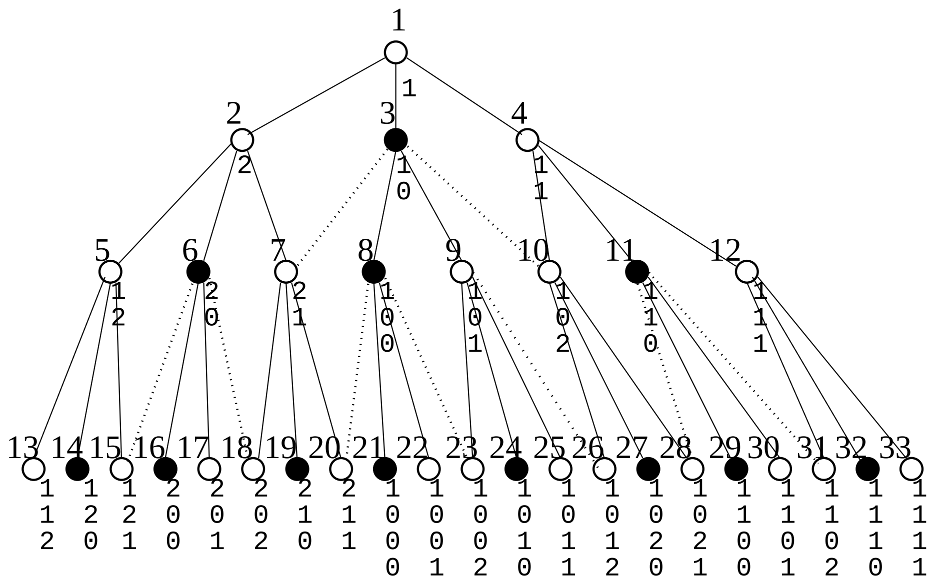

Figure 3 The tree associated to the splitting of Figure 2 for $s = 5$

$\square$

We think of this tree T_s as being embedded in the Euclidean plane. On the top of the tree there is the root: it corresponds to the s-corner $(C^0, \mathcal{P}^0)$ of the zero-th generation. On the k-th line below, there are the nodes of the k-th generation. They are displayed in the order from left to right which is inherited from the order of the k-th generation w.r.t. the splitting, see Fig. 3, 4.

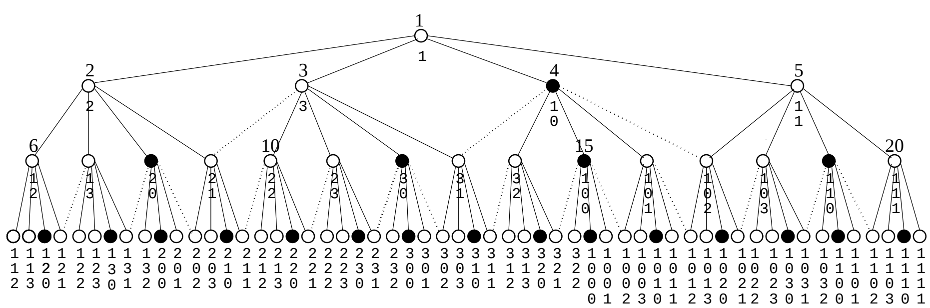

Figure 4 The tree associated to the splitting of Figure 2 for $s = 6$

$\square$

Observe also that the nodes of the tree T_s are in one-to-one correspondence, preserving the generation, with the tiles of the tiling $\mathcal{T}_s^{sw}$. Therefore the tree T_s is *a spanning tree of the adjacency graph of the tiling* $\mathcal{T}_s^{sw}$.

3.5 The status of the nodes and the substitution, generating the tree T_s

There are two type of regions in the splitting procedure: s-corners and s-strips, and therefore there are two types of nodes of the tree T_s. To distinguish them we introduce *the status* $\sigma(v)$ *of the node* $v \in T_s$. We say that the status $\sigma(v)$ is equal to one if the node corresponds to an s-corner, and that it is equal to zero, if the node corresponds to an s-strip. Symbolically, in the figures below, the nodes of status one are represented by small white discs, and the nodes of status zero are represented by small black discs.

The tree T_s is fully described by its root, which is of status one, and by a set of *rewriting rules*, we call it also *substitution*, μ given symbolically on Fig. 5.

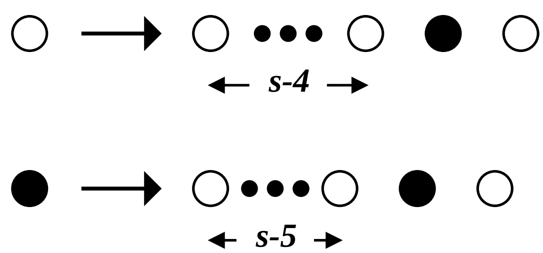

Figure 5 The substitution μ, which generates the tree T_s

The formal definition of the substitution $\mu : \{0,1\} \longrightarrow \{0,1\}^*$ which we shall use in the last section, is

$$\mu(0) = 1^{s-5}01$$
$$\mu(1) = 1^{s-4}01,$$

where $\{0,1\}^*$ is the monoid of all words written by 0 and 1, and 1^a01 is a short notation for the string beginning with a symbols 1 followed by 0 and then by 1.

With the string $v_0 \ldots v_k$ of consecutive nodes of the same generation in T_s we associate the word $\sigma(v_0)\ldots\sigma(v_k) \in \{0,1\}^*$. Then the string of the consecutive descendants of the nodes $v_0, \ldots, v_k$ is associated with the word $\mu(\sigma(v_0)\ldots\sigma(v_k))$ and we have $\mu(\sigma(v_0)\ldots\sigma(v_k)) = \mu(\sigma(v_0))\ldots\mu(\sigma(v_k))$.

The tree T_s describes a combinatorial structure of the tiling $\mathcal{T}_s^{sw}$. Its tiles are of two types: tiles of status one, and tiles of status zero. Tiles $T(v)$ of status one correspond to nodes v of T_s of status $\sigma(v) = 1$, and tiles $T(v)$ of status zero correspond to nodes v of status $\sigma(v) = 0$.

4 The number of the tiles of the k-th generation of $\mathcal{T}_s^{sw}$

Here we shall use the combinatorial structure of the the tiling $\mathcal{T}_s^{sw}$, which we described in the previous section, to determine the number u_k of the tiles of the

k-th generation of $\mathcal{T}_s^{sw}$. We shall obtain also the number $\tilde{u}_k$ of the tiles of the k-th generation of the tiling $\mathcal{T}_s$.

Proposition 1

Let u_k, $k \geq 0$, be the number of the tiles of the k-th generation of the tiling $\mathcal{T}_s^{sw}$. Then the sequence $\mathcal{U}_k = (u_k)_{k \geq 0}$ satisfies the recurrent equation : $u_{k+2} = (s-2)u_{k+1} - u_k$ with initial terms $u_0 = 1$ and $u_1 = s-2$. Therefore, we have

$$u_k = \frac{1}{\sqrt{s(s-4)}}(\lambda_s^{k+1} - \lambda_s^{-k-1}),$$

where $\lambda_s > 1$ and λ_s^{-1} are the roots of the equation

$$\lambda^2 - (s-2)\lambda + 1 = 0.$$

Proof

Let v be a node v of status $\sigma(v) = 1$ of the tree T_s. The number of its descendants is $(s-2)$. For a node v of status $\sigma(v) = 0$ it is $(s-3)$. Denote by f_k the number of all nodes v of the k-th generation of status $\sigma(v) = 1$, and by d_k the number of all nodes v of the k-th generation of status $\sigma(v) = 0$. Then $u_k = f_k + d_k$, for $k \geq 0$, and $f_0 = 1, d_0 = 0$. From the substitution μ it follows that:

$$f_{k+1} = (s-3)f_k + (s-4)d_k,$$

$$d_{k+1} = f_k + d_k.$$

Therefore, $d_{k+1} = u_k$ for $k \geq 0$ and

$$u_{k+2} = (s-2)u_{k+1} - u_k, \ k \geq 0,$$

i.e., the sequence $\mathcal{U}_k = (u_k)_{k \geq 0}$ is a solution of the recurrent equation of second order

$$x_{k+2} - (s-2)x_{k+1} + x_k = 0, k \geq 0, \tag{1}$$

with the initial conditions $x_0 = 1, x_1 = s-2$.

Denote by λ_s and λ_s^{-1} the roots of the characteristic equation $\lambda^2 - (s-2)\lambda + 1 = 0$ of the equation (1), and let $\lambda_s > 1$. Then all solutions of (1) are linear combinations of the fundamental solutions $(\lambda_s^k)_{k \geq 0}$ and $(\lambda_s^{-k})_{k \geq 0}$. Then, using the initial values $u_0 = 1, u_1 = s-2$, we obtain the assertion.

□

Remark

For $s = 5$ the number u_k is the $(2k+1)$-th Fibonacci number, see [6].

5 The number of the tiles of the k-th generation of the tiling $\mathcal{T}_s$

The tiling $\mathcal{T}_s$ and its tiles of the k-th generations are defined by induction as in the case of the tiling $\mathcal{T}_s^{sw}$. The s-gon $\mathcal{P}^0$ is called the s-gon of the zero-th generation of $\mathcal{T}_s$, and all its sides are called free sides of the zero-th generation. The images of the s-gon $\mathcal{P}^0$ w.r.t. h-reflections in the free sides of zero-th generations are called tiles of the first generation of $\mathcal{T}_s$. The sides of the tiles of the first generation, which are not sides of the tile of the zero-th generation, are called free sides of the first generation. Further, by induction, as in the case of the construction of the tiling $\mathcal{T}_s^{sw}$, we define the tiles and free sides of the k-th generation of $\mathcal{T}_s$. Eventually, we obtain the complete tiling $\mathcal{T}_s$.

Corollary 1

Let u_k be the number of the tiles of the k-th generation of the tiling $\mathcal{T}_s^{sw}$. Let $\tilde{u}_k$ be the number of the tiles of the k-th generation of the tiling $\mathcal{T}_s$. Then it follows that

$$\tilde{u}_{k+1} = s u_k, \;\; k \geq 0 \text{ and } \tilde{u}_0 = 1.$$

Therefore, we have

$$\tilde{u}_{k+1} = \frac{s}{\sqrt{s(s-4)}}(\lambda_s^{k+1} - \lambda_s^{-k-1}), \; k \geq 0 \text{ and } \tilde{u}_0 = 1,$$

where $\lambda_s > 1$ and λ_s^{-1} are the roots of the equation

$$\lambda^2 - (s-2)\lambda + 1 = 0.$$

Proof

The tiling $\mathcal{T}_s$ consists of the tiling $\mathcal{T}_s^{sw}$ and its images, w.r.t. the h-reflections in the vertical and the horizontal diameters d_1 and d_2 of the unit disc B^2. Therefore

$$\tilde{u}_{k+2} = u_{k+2} + 2u_{k+1} + u_k, \; k \geq 0,$$

and $\tilde{u}_0 = 1, \; \tilde{u}_1 = s$.

Then the both assertions follow, since the sequence $(u_k)_{k\geq 0}$ satisfies the recurrent equation (1).

$\square$

Remarks

1. The sequence $(\tilde{u}_{k\geq 0})$ satisfies the recurrent equation (1).

2. As expected $\tilde{u}_k$ (and u_k) depend exponentially on k. Their growths is given by the dominant root λ_s of the characteristic equation of the recurrent relation (1).

6 Fibonacci type coding of the nodes of the tree T_s

Denote the set of the nodes of the tree T_s by V_s. We defined before the status function $\sigma : V_s \longrightarrow \{0,1\}$. Here we shall define the numeration function $\nu : V_s \longrightarrow \{1,2,\ldots\}$. Then, following the approach of [8, 10], we use the numeration function for the definition of a Fibonacci type coding of the nodes of T_s. The coding which we introduce here, is different, for the case $s = 5$, from the coding, proposed in [8, 10]. The new coding is better adapted to the problem we consider, and this is the reason why the algorithms which are given later, are simpler.

6.1 The numeration function on T_s

For the definition of the numeration function $\nu : V_s \longrightarrow \{1,2,\ldots\}$ we enumerate the nodes of the tree T_s, starting from the root: we label it by 1 and then continue the enumeration for every next generation of the nodes from left to right, i.e., in the order, which we introduced before. In this way we define the function $\nu : V_s \longrightarrow \{1,2,\ldots\}$. We call its value $\nu(v)$ the numeration label of the node v.

Since the nodes of the tree T_s are in one-to-one correspondence with the tiles of the tiling $\mathcal{T}_s^{sw}$, *par abus de langage*, we say also that $\nu(v)$ is the numeration label of the tile $T(v)$, corresponding to the node v.

Now we use the greedy numeration system with a basis $\mathcal{U}_s = (u_n)_{n\geq 0}$ of the natural numbers. We remind shortly only what we need. For more information see [3].

As we have

$$\sup_n \frac{u_{n+1}}{u_n} = \lambda_s = \frac{1}{2}(s - 2 + \sqrt{s^2 - 4s}),$$

and as the integer part $[\lambda_s]$ of the real number λ_s is $(s-3)$, the appropriate alphabet A_s for this greedy representation is $A_s = \{0, 1, \ldots, s-3\}$, see [3].

Then every natural number m is represented as

$$m = \alpha_k u_k + \cdots + \alpha_0 u_0,\ \alpha_0, \ldots, \alpha_k \in A_s,\ \alpha_k \neq 0. \tag{2}$$

We associate with the natural number m the word $\alpha_k \ldots \alpha_0 \in A_s^*$, where A_s^* is the monoid of all concatenations (words) of the elements of the alphabet A_s. We say that the word $\alpha_k \ldots \alpha_0$ represents the natural number m, w.r.t. the basis sequence $\mathcal{U}_s = (u_n)_{n\geq 0}$, and shall write $\overline{\alpha_k \ldots \alpha_0}$ for $\alpha_k u_k + \cdots + \alpha_0 u_0$.

Remark

Representations with a few leading zeros are also used and we use them later on.

The representation (2) is not unique, i.e., several words (elements of A_s^*) correspond to the natural number m. We consider the so called *normalised greedy representation* $U(m)$ *of* m. This representation is the maximal word, representing

m, w.r.t. the lexicographic order in A_s^*, induced by the natural order in A_s. The normalised greedy representation is obtained by the greedy algorithm, [3].

We define *the Fibonacci type code*, or simple *code*, of the node $v \in V_s$ (and the corresponding tile $T(v) \in \mathcal{T}_s^{sw}$) by

$$c(v) = U(\nu(v)),\ v \in T_s,$$

i.e., by the normalised greedy representation of the numeration label $\nu(v)$ of the node v, w.r.t. the basis sequence $\mathcal{U}_s = (u_n)_{n\geq 0}$.

The first generations of the tree T_s for $s = 5$ and $s = 6$ are represented on the Fig. 3 and 4. The numeration labels and the codes of the nodes are given on these figures. The code is given as a column below the corresponding node. It has to be read from the top to the bottom. The nodes of status one are represented by small white discs, and the nodes of status zero are represented by small black discs. The doted lines are for a further use.

Remark

The normalised greedy representation of the natural numbers, w.r.t. the Fibonacci sequence $\mathcal{F} = (F_n)_{n\geq 0}$, was used for the coding of the tiles of the pentagrid $\mathcal{T}_s^{sw}$ in [10]. In this case the sequence $\mathcal{U}_5 = (u_n)_{n\geq 0}$, used above as a basis, is the sequence of the odd Fibonacci numbers.

6.2 The code and the status of a node

Here we observe that the status of a node v is given by the rightmost letter of its code. This is important for the algorithms, presented in the next sections.

Proposition 2

Let v be a node of the tree T_s with code $c(v) = \alpha_k \dots \alpha_0$. Then the node v has a status $\sigma(v) = 0$ if and only if the last letter α_0 of its code is zero, i.e.,

$$\sigma(v) = \begin{cases} 0 & \alpha_0 = 0 \\ 1 & \alpha_0 \neq 0, \end{cases}$$

and so, the penultimate descendant of the node v has a code $\alpha_k \dots \alpha_0 0$.

Proof

It follows from the technical lemmas in the last section.

□

Remark

The notion of *the continuator of a given node* of the tree T_s, w.r.t. the greedy representation used for the coding, was defined in [10]. In our case the continuator of the node v with the code $\alpha_k \dots \alpha_0$ is the node with the code $\alpha_k \dots \alpha_0 0$, i.e., it coincides with the descendant of v of status zero. In general, using as a basis for the greedy representation sequences different from the our sequence $\mathcal{U}_s = (u_n)_{n\geq 0}$, this may be not the case.

7 An algorithm for the code of the predecessor and the descendants of a given node

Here we consider the following question: given a node $v \in T_s$ with a code $c(v) = \alpha_k \dots \alpha_0$, find the codes of the predecessor and descendants of v. For both cases, the tool is given by Proposition 2.

7.1 Algorithm for the code of the predecessor of a given node

Let v_1 be the predecessor of the node v. There are two cases:

– the node v has status $\sigma(v) = 0$. Then its code $c(v) = \alpha_k \dots \alpha_1 0$. From Proposition 2, we get that the code of v_1 is $c(v_1) = \alpha_k \dots \alpha_1$;

– the node v has status $\sigma(v) = 1$, and its code is $c(v) = \alpha_k \dots \alpha_1 1$. This implies that the node v is the last descendant of the node v_1. Then the previous descendant v' of v_1 has code $c(v') = \alpha_k \dots \alpha_1 0$, and by Proposition 2 the code of v_1 is $c(v_1) = \alpha_k \dots \alpha_1$;

– the code of v is $c(v) = \alpha_k \dots \alpha_1 \alpha_0$, with $\alpha_0 \geq 2$. Then the node v precedes in T_s the continuator v' of the node v_1. Let the code of the node v' be $c(v') = \gamma_k \dots \gamma_1 0$. Then the code of the node v_1 is $c(v_1) = \gamma_k \dots \gamma_1$.

  The word $\gamma_k \dots \gamma_1 0$ is the normalised greedy representation of the natural number $\overline{\alpha_k \dots \alpha_0} + s - 2 - \alpha_0$. We find the code $\gamma_k \dots \gamma_1 0$ of the node v_1 by an addition of the number $s-2-\alpha_0$ to $\overline{\alpha_k \dots \alpha_1 \alpha_0}$ in the greedy numeration system with the basis $\mathcal{U}_s = (u_n)_{n\geq 0}$.

7.2 Algorithm for the codes of the descendants of a given node

As before $v \in T_s$, and its code $c(v) = \alpha_k \dots \alpha_0$. To find the codes of the descendants of v we consider two cases:

– the status $\sigma(v)$ of v is one. Then the node v has $(s-2)$ descendants. The penultimate of them is the continuator. Its code is $\alpha_k \dots \alpha_0 0$. Then the code of the last descendant of v is $\alpha_k \dots \alpha_0 1$. The codes of the first $(s-4)$ descendants of v are obtained by substracting (in the greedy numeration system) $1, 2, \dots, (s-4)$ from the code $\alpha_k \dots \alpha_0 0$;

– the status $\sigma(v)$ of v is zero. Then v has $(s-3)$ descendants. The penultimate of them v' has status zero and code $\alpha_k \dots \alpha_0 0$. The last descendant has code $\alpha_k \dots \alpha_0 1$. The codes of the first $(s-5)$ descendants of v are obtained by substracting (in the greedy numeration system) $1, 2, \dots, (s-5)$ from the code $\alpha_k \dots \alpha_0 0$.

Remark

For a given node v there is a unique path in T_s, connecting it with the root of the tree T_s. As a corollary from the previous rules we obtain an algorithm which gives the codes of the consecutive nodes of this path, starting with the code of the node v.

8 Algorithm for the codes of the tiles adjacent to a given tile in T_s^{sw}

Let $v \in T_s$ has a code $c(v) = \alpha_k \dots \alpha_0$. The node v corresponds to the tile $T(v) \in T_s^{sw}$. As we mentioned before, we say that $\alpha_k \dots \alpha_0 \in A_s^*$ is the code of the tile $T(v)$.

Here we address the following question: how to find the codes of the tiles $T(w)$, adjacent to $T(v)$ in T_s^{sw}, from the code of v. Among these tiles $T(w)$ are the tiles, corresponding to the predecessor and the descendants of the node v. We already know how to find their codes. We shall find the codes of the rest of the tiles $T(w)$, adjacent to $T(v)$ in $\mathcal{T}_s^{sw}$.

On the Fig. 3, 4 the node v is connected by a doted lines with these nodes w.

We have three types of tiles in the tiling T_s^{sw}: the tile of generation zero, the boundary tiles, and the inner tiles. We consider them separately:

– *the tile $T(v)$ of generation zero* corresponds to the root v of the tree T_s. It has $(s-2)$ neighbour tiles in T_s^{sw}. The root is a node of a status one, and has $(s-2)$ descendants in T_s. Therefore we know the codes of all tiles adjacent to $T(v)$ in T_s^{sw};

– *the boundary tiles* are the tiles with one side, supported by the vertical or the horizontal diameters d_1, d_2 of the unit disc B^2. Every boundary tile $T(v)$, has $(s-1)$ adjacent tiles in T_s^{sw}. The node v, corresponding to $T(v)$ has status one, and is connected with $(s-1)$ nodes in the tree T_s. Therefore we know the codes of all tiles adjacent to a given boundary tile;

– *inner tiles* are the tiles of T_s^{sw}, which are not boundary tiles, and are different from the root. We consider two cases:

- *inner tiles $T(v)$ of a status $\sigma(v) = 0$.*

  The node v, corresponding to such a tile has $(s-3)$ descendants and one predecessor in the tree T_s. We know how to obtain their codes. Two more nodes w_1 and w_2, not connected with v in T_s, correspond to tiles $T(w_1)$ and $T(w_2)$, adjacent to $T(v)$ in T_s^{sw}. We shall find their codes.

  Let v be a node of the k-th generation. Denote by v_1 the node with the numeration label $\nu(v_1) = \nu(v) - 1$, and by v_2 the node with the numeration label $\nu(v_2) = \nu(v)+1$. They are nodes of the k-th generation. Then the node w_1, corresponding to the tile $T(w_1)$, is the last descendant of the node v_1, and the node w_2, corresponding to the tile $T(w_2)$, is the first descendant of the node v_2.

  To find the codes of the nodes w_1 and w_2 we need the codes of the nodes v_1 and v_2, respectively: the code of v_1 is obtained by substracting one in the greedy numeration system from the code of the node v, and the code of v_2 is obtained by adding one to the code of the node v.

  Let $c(v_1) = \gamma_k \ldots \gamma_0$. Then the code $c(w_1)$ of w_1 is $\gamma_k \ldots \gamma_0 1$.

  Let $c(v_2) = \delta_k \ldots \delta_0$. Then the code $c(w_2)$ of w_2 is obtained by substracting $(s-4)$ from the code $\delta_k \ldots \delta_0 0$ in the greedy numeration system.

- *inner tiles $T(v)$ of a status $\sigma(v) = 1$:*

  ∗ *the predecessor v' of v is of a status zero, and v is the last descendant of v'*

  The node v is connected with $(s-1)$ nodes in the tree T_s. There is one more node w, not connected with v in T_s, with corresponding tile $T(w)$, adjacent to $T(v)$ in the tiling $\mathcal{T}_s^{sw}$. We shall find its code.

  Consider the node v_1 with the numeration label $\nu(v_1) = \nu(v) + 1$. Then the node w is the first descendant of the node v_1. We know its code from the previous section.

  ∗ *the predecessor v' of v has status one, and v is the last descendant of v'.*

  The node v is connected with $(s-1)$ tiles in the tree T_s. There is one more node w, not connected with v in T_s, with corresponding tile $T(w)$, adjacent to $T(v)$ in the tiling $\mathcal{T}_s^{sw}$. We shall find its code.

  Consider the node v_1 with the numeration label $\nu(v_1) = \nu(v') + 1$. Then the node w coincides with v_1. Its code is obtained by adding one to the code of the node v' in the greedy numeration system.

  ∗ *the predecessor v' of v has status one, and does not belong to the previous two cases*

The node v is connected with $(s-1)$ tiles in the tree T_s. There is one more node w, not connected with v in T_s, with corresponding tile $T(w)$, adjacent to $T(v)$ in the tiling $\mathcal{T}_s^{sw}$. We shall find its code.

Consider the node v_1 with the numeration label $\nu(v_1) = \nu(v) - 1$. Then the node w, corresponding to the tile $T(w)$, is the last descendant of the node v_1. The code of v_1 is obtained by substracting one from the code of the node v in the greedy numeration system. Let $c(v_1) = \alpha_k \ldots \alpha_0$. Then $c(w) = \alpha_k \ldots \alpha_0 1$.

Remark

It is clear that there are uncountable many ways to perform the hierarchical splitting procedure of the s-corner: on every step we have a choice of the position of the s-strip. It was shown in [10] in the case $s = 5$ that there are also uncountable many ways, which posses nice properties. There is one possibility among these, which is better than the others, as far as it provides us with a very simple linear algorithm for the path from a node to the root of the tree T_5, and for the codes of the tiles adjacent to a given tile in T_5^{sw}. These algorithms are different from the algorithms presented here.

9 The language associated to the tiling $\mathcal{T}_s^{sw}$

For the notions, used in this section, see [4].

We consider the language $L_s \subset A_s^*$ of all normalised greedy representations of the natural numbers, w.r.t. the basis sequence $\mathcal{U}_s = (u_k)_{k \geq 0}$. The language L_s consists of all codes of the nodes of the tree T_s. We say that L_s *is the language associated to the tiling* $\mathcal{T}_s^{sw}$.

Proposition 3

The language L_s, associated to the tiling $\mathcal{T}_s^{sw}$, is a regular language.

Proof

The assertion follows from [3].

A direct simple argument is the following description of the language L_s. It consist of all words $\alpha_k \ldots \alpha_0 \in A_s^*$, which satisfy:

- $\alpha_k \neq 0$;
- the words $(s-3)(s-4)^l(s-3)$ are not subwords of $\alpha_k \ldots \alpha_0$, for any $l \geq 0$.

□

Remarks

1. An appropriate splitting construction could be applied to the tiling $\mathcal{T}_e^{ne}$ of the north-east quarter of the Euclidean plane by unit squares. In this case

we also have two types of regions: corners, and strips. A corner is isometric by an Euclidean isometry to the north-east quarter, and the strip is isometric to the product of the arrow $[0, +\infty)$ and the unit segment. The tree T_e, describing the recurrent splitting procedure of the corner, has nodes of two types: of status one, corresponding to the corners, and nodes of status zero, corresponding to the strips. The root of T_e is a node of status one. The substitution $\mu_e : \{0, 1\} \longrightarrow \{0, 1\}^*$, generating the tree T_e, is given by: $\mu_e(0) = 0$, and $\mu_e(1) = 01$. Here we used the same notation as for the definition of the substitution μ, which generates the tree T_s, associated with the tiling $\mathcal{T}_s^{sw}$.

The number of the nodes of the k-generation of the tiling $\mathcal{T}_e^{ne}$ is $(k+1)$. The sequence $\mathcal{U}_e = (k+1)_{k\geq 0}$ is not appropriate as a basis sequence for a greedy representation of the natural numbers.

2. In the work in progress we consider the tiling of the 3-dimensional hyperbolic space by regular dodecahedra with dihedral right angles and faces regular rectangular 5-gons. There is a splitting procedure in this case also, but the situation is more complicated and some important differences with the 2-dimensional case appear, see [12, 13].

10 Technical lemmas

Here we give the technical lemmas about the tree T_s, and the coding of its vertices, which are used in the proofs of the Propositions 2 and in the algorithms which we presented before. By reading these lemmas it is useful to consult Fig. 6.

From the definitions of the tree T_s, and the substitution, $\mu : \{0, 1\} \longrightarrow \{0, 1\}^*$, generating it, we have:

Lemma 1
Let $\mu^k : \{0, 1\}^* \longrightarrow \{0, 1\}^*$ be the k-th iteration of the substitution μ. Then

$$\mu^k(1) = \sigma(v_1) \ldots \sigma(v_{u_k}),$$

where $v_1, \ldots, v_{u_k}$ are the nodes of the k-th generation of the tree T_s, and $\sigma : V_s \longrightarrow \{0, 1\}$ is the status function.

$\square$

For what follows, we need some notations.

Let the code of the node $v \in V_s$ be $c(v) = \alpha_l \ldots \alpha_0 \in A_s^*$. By $T_s(v) = T_s(\alpha_l \ldots \alpha_0)$ we denote the subtree of T_s with root v, e.g., $T_s = T_s(1)$.

By $T_s(v)_k = T_s(\alpha_l \ldots \alpha_0)_k$ denote the k^{th} level of the tree $T_s(v)$, i.e., the string of all consecutive nodes of the k-th generation of this tree, in the order which is inherited from the tree T_s.

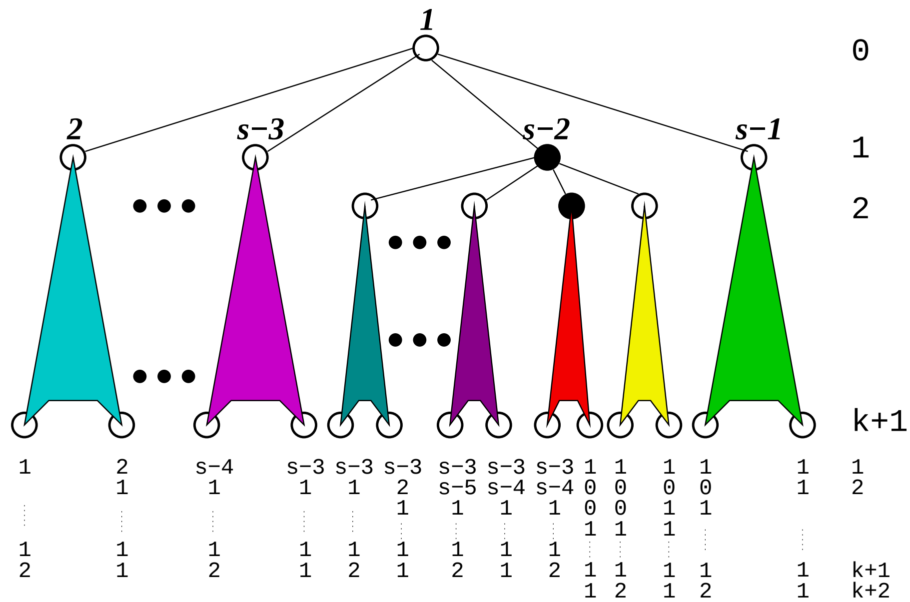

Figure 6 The decomposition of $\mathcal{T}_s(1)_{k+1}$ for the proof of Lemma 3.

In this figure, numbers of the rightmost column indicate the level of the tree and then, the length of the codes of the nodes that lie on the $k{+}1^{th}$ level. Vertical words are these codes of level $k{+}1$. We indicate, for each subtree, the code of the leftmost element and the code of the rightmost one.

$\square$

For a given string $v_1 \ldots v_{u_m}$ of consecutive nodes of the m-th generation of the tree T_s we denote by

$$T_s(v_1)_k T_s(v_2)_k \ldots T_s(v_{u_m})_k \tag{3}$$

the concatenation of the strings of nodes $T_s(v_1)_k, \ldots, T_s(v_{u_m})_k$. The string (3) is a concatenation of consecutive nodes of the tree T_s of the same generation.

Lemma 2

Let $T_s(1)_{k+1} = v_1 \ldots v_{u_{k+1}}$, and let the length of the word $\mu^k(0)$ be a_k. Then

- $T_s(1)_{k+1} = T_s(2)_k \ldots T_s(s-3)_k T_s(10)_k T_s(11)_k$;
- $T_s(10)_k = T_s((s-3)2)_{k-1} \ldots T_s((s-3)(s-4))_{k-1} T_s(100)_{k-1} T_s(101)_{k-1}$.

Moreover, for the codes of the nodes $v_1 \dots v_{u_{k+1}}$ *we have:*

$$c(v_1) = 1^k 2, \dots,\ c(v_{u_k}) = 21^k, \dots, c(v_{(s-4)u_k}) = (s-3)1^k,$$

$$c(v_{(s-4)u_k+(s-5)u_{k-1}}) = (s-3)(s-4)1^{k-1},$$
$$c(v_{(s-4)u_k+(s-5)u_{k-1}+a_{k-1}}) = 1001^{k-1},$$

$$c(v_{(s-4)u_k+a_{k-1}}) = 101^k,\ c(v_{u_{k+1}}) = 1^{k+2}.$$

Proof

It follows directly from the definitions, see Fig. 6.

□

Let $v_1, v_2 \in V_s$. The subtrees $T_s(v_1)$ and $T_s(v_2)$ are isomorphic iff the nodes v_1 and v_2 have the same status, i.e., $\sigma(v_1) = \sigma(v_2)$.

We consider the isomorphism

$$\iota = \iota_{v_1, v_{v_2}} : T_s(v_1) \longrightarrow T_s(v_2),$$

which preserves the order of the numeration in T_s, i.e., for $v', v'' \in T_s(v_1)$ satisfying $\nu(v') < \nu(v'')$ then $\nu(\iota(v')) < \nu(\iota(v''))$ holds.

Let $2 \leq j \leq s-3$. Then the tree $T_s(j)$ is isomorphic with the tree $T_s = T_s(1)$. Denote by

$$\iota_j = \iota_{j,1} : T_s(j) \longrightarrow T_s(1)$$

the isomorphism, which preserves the order of the numeration.

For all the lemmas which follow, see Fig. 3, 4, 5, 6.

Lemma 3

Let v *be a node of the* k*-th generation of the tree* $T_s(j)$*, for* $2 \leq j \leq s-3$*. If the code of the image* $\iota_j(v)$ *of the node* v*, w.r.t. the isomorphism* ι_j*, is*

$$c(\iota_j(v)) = \alpha_{k+1}\alpha_k \dots \alpha_0,$$

with possibly $\alpha_{k+1} = 0$*, then*

$$c(v) = (\alpha_{k+1} + j - 1)\alpha_k \dots \alpha_0.$$

Proof

Since v is a node of the k-th generation of the tree $T_s(j)_k$, then v is a node of the $(k+1)$-th generation of the tree $T_s(1)$, and $\iota_j(v) \in T_s(1)_k$.

Lemma 2 implies that for the values of the numeration function on v and $\iota_j(v)$ we have

$$\overline{j1^{k-1}2} \leq \nu(v) \leq \overline{(j+1)1^k},$$

$$\overline{1^{k-1}2} \leq \nu(\iota_j(v)) \leq \overline{1^{k+1}}$$

(remind that we write $m = \overline{\alpha_k \ldots \alpha_0}$ if $m = \alpha_k u_k + \cdots + \alpha_0 u_0$).

Then

$$\nu(v) = \nu(\iota_j(v)) + (j-1)u_{k+1},$$

which implies the assertion.

□

The tree $T_s(11)$ is isomorphic with the tree $T_s(1)$. Let

$$\iota_{11} : T_s(11) \longrightarrow T_s(1)$$

be the isomorphism, which preserves the order of the numbering in T_s.

Lemma 4

Let v be a node of the k-th generation of the tree $T_s(11)$. If the code of the image $\iota_{11}(v)$ of the node, w.r.t. the isomorphism ι_{11}, is

$$c(\iota_{11}(v)) = \alpha_{k+1} \ldots \alpha_0,$$

with possibly $\alpha_{k+1} = 0$, then the code of the node v is

$$c(v) = 1\alpha_{k+1} \ldots \alpha_0.$$

Proof

Since v is a node of the k-th generation of the tree $T_s(11)_k$, then $v \in T_s(1)_{k+1}$ and $\iota_{11}(v) \in T_s(1)_k$. From Lemma 2 it follows that

$$\overline{101^{k-1}2} \leq \nu(v) \leq \overline{1^{k+2}},$$

$$\overline{1^{k-1}2} \leq \nu(\iota_{s-1}(v)) \leq \overline{1^{k+1}}.$$

Then

$$\nu(v) = \nu(\iota_{s-1}(v)) + u_{k+2},$$

which implies the assertion.

□

The tree $T_s((s-3)j),\ 2 \leq j \leq s-4$, is isomorphic with the tree $T_s(2)$. Let

$$\kappa_j = \iota_{(s-3)j,2} : T_s((s-3)j) \longrightarrow T_s(2)$$

be the isomorphism, which preserves the order of the numeration in T_s.

Lemma 5

Let $v \in T_s((s-3)j)_{k-1},\ 2 \leq j \leq s-4$. If the code of the node $\kappa_j(v)$ is $c(\kappa_j(v)) = \alpha_k \ldots \alpha_0,\ \alpha_k \in \{1,2\}$, then the code of v is

$$c(v) = (s-3)(\alpha_k + j - 2)\alpha_{k-1} \ldots \alpha_0.$$

Proof

Since $v \in T_s((s-3)j)_{k-1}$, then $v \in T_s(1)_{k+1},\ \kappa_j(v) \in T_s(2)_{k-1}$.
From Lemma 2, it follows that:

$$\overline{(s-3)(j-1)1^{k-2}2} \leq \nu(v) \leq \overline{(s-3)j1^{k-1}},$$

$$\overline{1^{k-1}2} \leq \nu(\kappa_j(v)) \leq \overline{21^{k-1}},$$

$$\nu(v) = \nu(\kappa_j(v)) + (s-3)u_{k+1} + (j-2)u_k.$$

This implies the assertion.

$\square$

The tree $T_s(100)$ is isomorphic with the tree $T_s(10)$. Let

$$\lambda = \iota_{100,10} : T_s(100) \longrightarrow T_s(10)$$

be the isomorphism preserving the order of the numbering in T_s.

Lemma 6

Let $v \in T_s(100)_{k-1}$.

(i) If

$$\overline{(s-3)(s-4)1^{k-2}2} \leq \nu(v) \leq \overline{(s-3)(s-4)^k},$$

and the code of the node $\lambda(v)$ is

$$c(\lambda(v)) = (s-3)\alpha_{k-1} \ldots \alpha_0,$$

then the code of the node v is

$$c(v) = (s-3)(s-4)\alpha_{k-1} \ldots \alpha_0.$$

(ii) If

$$\overline{10^{k+2}} \leq \nu(v) \leq \overline{1001^{k-1}},$$

and the code of $\lambda(v)$ is

$$c(\lambda(v)) = 10\alpha_{k-1} \ldots \alpha_0,$$

then

$$c(v) = 100\alpha_{k-1} \dots \alpha_0.$$

Proof

Since $v \in T_s(100)_{k-1}$, then $v \in T_s(1)_{k+1}$ and $\lambda(v) \in T_s(10)_{k-1}$.

Case (i):

$$\overline{(s-3)(s-4)1^{k-2}2} \leq \nu(v) \leq \overline{(s-3)(s-4)^k}. \tag{4}$$

Then it follows that:

$$\overline{(s-3)1^{k-2}2} \leq \nu(\lambda(v)) \leq \overline{(s-3)(s-4)^k}. \tag{5}$$

The codes of the nodes v satisfying (4) are:

$$c(v) = (s-3)(s-4)\alpha_k \dots \alpha_0. \tag{6}$$

The codes of the nodes w satisfying (5) are:

$$c(w) = (s-3)\beta_k \dots \beta_0. \tag{7}$$

Since (4) and (5) have the same length, then (6) and (7) imply the assertion.

Case (ii):

$$\overline{10^{k+2}} \leq \nu(v) \leq \overline{1001^{k-1}}. \tag{8}$$

Then it follows that:

$$\overline{10^{k+1}} \leq \nu(\lambda(v)) \leq \overline{101^{k-1}}. \tag{9}$$

The codes of the nodes v satisfying (8) are:

$$c(v) = 100\alpha_{k-1} \dots \alpha_0. \tag{10}$$

The codes of the nodes w satisfying (9) are:

$$c(w) = 10\beta_{k-1} \dots \beta_0. \tag{11}$$

Since (8) and (9) have the same length, then (10) and (11) imply the assertion. $\square$

The tree $T_s(101)$ is isomorphic with the tree $T_s(1)$. Let

$$\rho : T_s(101) \longrightarrow T_s(1)$$

be the isomorphism, preserving the order of the numeration in T_s.

Lemma 7
Let $v \in T_s(101)_{k-1}$. If the code of the node $\rho(v)$ is

$$c(\rho(v)) = \alpha_k \dots \alpha_0$$

with possibly $\alpha_k = 0$, then the code of the node v is

$$c(v) = 10\alpha_k \dots \alpha_0.$$

Proof
Since $v \in T_s(101)_{k-1}$, then $v \in T_s(1)_{k+1}$ and $\rho(v) \in T_s(1)_{k-1}$. From Lemma 2 it follows that:

$$\overline{1001^{k-2}2} \leq \nu(v) \leq \overline{101^k} \tag{12}$$

and

$$\overline{1^{k-2}2} \leq \nu(\rho(v)) \leq \overline{1^k}. \tag{13}$$

The code of the node v satisfying (12) is of the type

$$c(v) = 10\alpha_k \dots \alpha_0 \tag{14}$$

and for the code of the node w, satisfying (13) it follows that:

$$c(w) = \beta_k \dots \beta_0. \tag{15}$$

Since (12) and (13) have the same length, then (14) and (15) imply the assertion. □

Acknowledgements

Both authors are much in debt to the University of Bremen and the University of Metz and to Professor Heinz-Otto Peitgen. His interest and support simulated the work. We are also grateful to Dr. I. Stoyanova for the remarks and the suggestions.

References

1. C. Goodmann-Strauss, A strongly aperiodic set of tiles in the hyperbolic plane, (submitted)
2. F. Herrmann, M. Margenstern, K. Morita, Cellular automata in the hyperbolic plane: implementing a polynomial algorithm to solve SAT problem, CA 2000, Intern. Workshop on Cellular Automata, Osaka, Japan, (2000), 25–26.
3. M. Hollander, Greedy numeration systems and regularity, Theory of Comput. Systems, **31** (1998), 111–133.
4. J. E. Hopcroft, J. D. Ullman, *Introduction to Automata Theory, Languages and Computations, Addison-Wesley Publ. Co, 1979.*

5. C. Mann, *On Heesch's problem and other tiling problems, PHD Thesis, University of Arkansas, 2001.*
6. M. Margenstern, K. Morita, NP problems are tractable in the space of cellular automata in the hyperbolic plane, Theor. Comp. Sci., **259** (2001), 99–128.
7. M. Margenstern, K. Morita, A polynomial solution for 3-SAT in the space of cellular automata in the hyperbolic plane, J. Universal Computations and Systems, **5-9**, (1999), 563–573.
8. M. Margenstern, An introduction to cellular automata in the hyperbolic plane. CA 2000, Intern. Workshop on Cellular Automata, Osaka, Japan, (2000), 1–2.
9. M. Margenstern, A package of algorithms to devise cellular automata in the hyperbolic plane and related questions, Computer Science Journal of Moldova, **8-2**, (2000), 95-115.
10. M. Margenstern, New tools for cellular automata in the hyperbolic plane, J. Univ. Comp. Sci. **6** 12(2000), 1226–1256.
11. M. Margenstern, G. Skordev, Rectangular grids in the hyperbolic plane for cellular automata, CA 2000, Intern. Workshop on Cellular Automata, Osaka, Japan, (2000), 40–41.
12. M. Margenstern, G. Skordev, Tools for devising cellular automata in the 3D hyperbolic space, I - The geometric algorithm, Proceedings of SCI'2002, Orlando, (2002), (to appear).
13. M. Margenstern, G. Skordev, Tools for devising cellular automata in the 3D hyperbolic space, II - The numeric algorithms, Proceedings of SCI'2002, Orlando, (2002), (to appear).
14. G. A. Margulis, S. Mozes, Aperiodic tilings of the hyperbolic plane by convex polygons, Israel J. Math., **107** (1998), 319–325.
15. E. B. Vinberg (Ed.), *Geometry II, Spaces of Constant Curvature, Springer-Verlag, Berlin, 1993.*