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Notes on Partially-Ordered Structures in Computer Science: I. PA-Ordered Semirings and Some Related Structures[1]

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Abstract: While the existence of inverses is a natural condition in Algebra it is seldom satisfied in Computer Science applications. Since the group-theoretical orientation has to be abandoned we consider an advantage when the non-conventional structures needed are linked to an already existing knowledge. We propose semirings as a candidate and we aim at the Computer Science applications such as processes semantics, parallel composition, Fuzzy Theory, images ordering or MV-algebras. After the definition of pa-ordered semiring four typical examples are given. Some results concerning additively idempotent semirings are extended to monoids considered as their natural background. A direct sum representation is given for lower semilattice-ordered Gelfand semirings s-ordered. A sufficient condition is given for having the natural quasi-order an s-order. A multiplicative ordering is built up and its application to Visual Data is indicated. Wrt complements in pa-semirings we give sufficient conditions for the existence of some sums and for commutativity.

Key Words: Semiring, quasi-order, s-order.

Category: F.4.1.

1 Introduction. Sources and Some Related Basic References

There is no lack of motivation for considering partially-ordered structures in Computer Science. It is difficult to provide even an introduction because of the diversity of applications not yet unified. We can only give some examples.

Two important recent monographs on semirings written by *J.S.Golan* [JG92] and *U.Hebisch* and *H.J.Weinert* [HW99] refer extensively to Computer Science. The algebraic approaches to semantics of *E.G.Manes* and *M.A.Arbib* [MA86] are based on *pa (partially additive)-monoids* in which the semantics of a process is viewed as a partially defined *sum* of functions with disjoint domains, the multiplication being defined as the functional composition. *W.Kuich* systematically investigated formal power series over semirings. (see the reference book with *A.Salomaa* [KS86] on the semiring approach to automata and languages).

Semirings have been applied to parallel composition of processes [GM96]. Multilattice-ordered monoids [MB] and semirings have been applied in the theory of visual languages [JJ96], [PB97], [PB99]. It should be noted that an interesting family of semirings has been recently introduced in studying *trajectories* [MRS98]; their specific properties remain for a further research.

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In his pioneering article *L.A.Zadeh* [LZ65] (reproduced in [YO87], p.30) explicitly suggested two possible extensions of his theory: (I) to have as *codomain* of a membership function a suitable *partially-ordered set* $(P, \leq)$ instead of $I = [0, 1]$; (II) to consider *partially defined* functions [JG81], [MF96]. Actually, the considered *De Morgan's* rules are typical for lattice-ordered groups [GB73]. Fuzzy logic has been presented in the context of Lattice Theory, e.g., *H.Rasiowa* (see [LZ92], p.5-25) and *J.A.Goguen* [JG81] with an additional link to semirings.

The fundamental result of *D.Mundici* [DM86] has opened an avenue of research with rich literature. Substantial results are linking MV-algebras with lattice-ordered groups or rings [GB73], [LF63] in many interesting ways [DN95]. Instead of being dedicated to these links, further notes present some results concerning partially-ordered semirings and rings which might be of some interest in the previously suggested contexts.

It is difficult to provide even a general view on the rapid development of the algebraic approach in different fields of Theoretical Computer Science [WW92], [BP94], [GS90], [AM89].

2 Basic Terminology, Concepts, Notations and Introductory Examples

Terminology The basic concept we consider in this part is that of a *pa-ordered semiring* (see further definitions), i.e., a *partially-ordered partially-additive semiring* being possible to have for some results only *semirings-like structures* (see further explanations). A partial semiring or *pa-semiring* is a semiring such that *addition* is only *partially defined* with the natural modification of some axioms.

In the following, *order* means *partial order*. If *commutativity* is not (explicitly) specified then *multiplication* can be *non-commutative*. When not (explicitly) stated we assume that multiplication is *associative*. If $< M, +, 0, \leq >$ is a commutative ordered monoid, we say that M is *s-ordered (sum-ordered/difference ordered* (*J.S.Golan* [JG92])/*naturally ordered* (*A.H.Clifford*)) iff $a \leq b$ with $a, b \in M$ is equivalent to $\exists c \in M$ such that $b = a + c$. We say that $< M, +, 0, \wedge, \leq >$ is a *lower semilattice-ordered* iff $< M, \wedge >$ is a lower semilattice, $< M, \leq >$ is the associated poset and the operation $+$ is *distributive* wrt to $\wedge$. The prefix $l-$ to an algebraic structure denotes that the structure is *lattice-ordered*. The monoids and semirings additive operations are distributive wrt the lattice operations when the involved sums are defined.

Definition 1. A pa-ordered semiring $R = (R, +, 0, 1, \leq)$ is a nonempty set R on which the operations of (partial) addition $+$ and multiplication $\cdot$ (omitted as usual) and a (partial) order structure have been defined such that the following conditions are satisfied:

1. $(R, +, 0, \leq)$ is a partial(ly-additive) ordered monoid with identity element 0 (see further);
2. $(R, \cdot, 1, \leq)$ is a (partially) ordered monoid with identity element 1 (see further);
3. Multiplication distributes over (partial) addition from either side when the involved sum exists;
4. $0r = 0 = r0$, for all r in R.

In (1) we assume that if one side of the associativity exists then the other exists as well and we have the equality. Moreover, $r+0=0+r=r$ always exists. The compatibility between the partial addition and the order has the following form: if $a \leq b$ and if $b+x \in R$ exists then $a+x \in R$ exists as well and we have $a+x \leq b+x$. Further if $u, v \in R_+ = \{w \in R | w \geq 0\}$, then $a \leq b$ imply $uav \leq ubv$.

Example 1. Let $Pfn(X,X)$ be the set of partial functions from X to X (it is possible to reduce the case $Pfn(X_1, X_2)$ to the considered one without loosing generality). We consider the pa-ordered semiring $Pfn(X,X) = (Pfn(X,X), +, 0, \cdot, 1, \subseteq)$ in the following sense: $f+g$ is defined iff $dom(f) \cap dom(g) = \emptyset$ and in general $\sum(f_c : c \in C)$ is defined iff the domains are disjoint and in this case $\sum(f_c : c \in C)(x) :=$ if $\exists c_0 \in C, x \in dom(f_{c_0})$ then $f_{c_0}(x)$ else undefined.

0 is the function with the empty domain, $dom(0) = \emptyset$; $\cdot$ is the functional composition; $1 = 1_X$ is the identity function; $f \subseteq g$ iff f is a restriction of g (dually, g is an extension of f), i.e., $dom(f) \subseteq dom(g)$ and $f(x) = g(x)$, for any $x \in dom(f)$.

Obviously, $dom(\sum(f_c : c \in C)) = \cup(dom(f_c) : c \in C)$ (see [MA86], [AM89] for the partially-additive monoids introduced by *E.G.Manes* and *M.A.Arbib* and M.E. Steenstrup [MS85]).

Example 2. Let Σ be an *alphabet*, i.e., a nonempty set, and Σ^* the free monoid generated by Σ with the identity e. A *language* over Σ is a subset of Σ^*. The set of all languages over Σ is denoted by $P(\Sigma^*)$. Let o be an associative operation $o : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^*$ such that e is an identity element, e.g., *catenation*, *anti-catenation* or *shuffle* $\sqcup\!\sqcup$ defined recursively by

$$(au \sqcup\!\sqcup bv) = a(u \sqcup\!\sqcup bv) \cup b(au \sqcup\!\sqcup v).$$

The considered operation defined on *words* $u \in \Sigma^*$ can be extended to languages by taking

$$L_1 o L_2 = \cup(u_1 o u_2 : u_i \in L_i, i = 1, 2).$$

The ordered system

$$S = (P(\Sigma^*), \cup, \emptyset, o, \{e\}, \subseteq)$$

is an upper complete semi-lattice semiring with least element $\emptyset$ with distributivity over infinite sums (totally defined). This semiring is *a (dditively)-idempotent, zerosumfree*, i.e., such that $a+b=0$ implies $a=b=0$, with *last/infinite* element (see [JG92] for the algebraic terms and [GM96] for the development of the considered example and applications to parallel computation).

Example 3. For *Fuzzy Theory* (FT) we have the semiring of functions

$$(X \longrightarrow Y = [0,1], \vee, 0, \wedge, 1, \leq, c)$$

where X is a *crisp* set, Y the real unit interval, $\vee = max$, $\wedge = min$ and $\leq$ are defined component-wise. In addition we consider the functions $f_0 = 0$, $f_1 = 1$ and $f_c = 1 - f$. The basic initial result of *L.A.Zadeh* [LZ65], [YO87] was that the considered structure viewed as a *De Morgan bounded distributive lattice* (DMBDLT) is the natural framework for FT.

Example 4. The matrix theory over the semiring

$$(R_+ \cup \{\infty\}, \wedge, \infty, +, 0, \leq)$$

is used when computing *minimal paths* over a finite directed graph (for the applications in Operations Research see the *minimax algebra* introduced and studied by *R.A. Cuninghame-Green*, e.g., [CG91] and the applications of ordered algebraic structures including semirings investigated by *U.Zimmermann* [UZ81]).

There are different possible sources for extending the concept of a semiring or *partially-ordered semiring* (po-semiring) by suppressing some of the conditions, e.g., 0 is an *annihilator*, two-sided distributivity, the existence of an identity element (a semiring without identity element is called *hemiring*), the result of an operation is a single element (not a *set* as for *hypersemirings*) (see [JG92] for basic notions and notations). The considered extensions will be still called semirings.

It was not possible to provide here more than a review of some results obtained. It is recognized its partial character and consequently the difficulty in getting the corresponding integrated image.

Some proofs are omitted because of the existing limitations.

3 Idempotent Monoids and Additivelly Idempotent Semirings

We extend the results from [JG92] on *a-idempotent semirings* showing that the appropriate framework is that of *idempotent monoids*.

Proposition 2. *If $(M, +, 0)$ is a partially-additive commutative idempotent monoid then the relation $a_1 \leq a_2$ defined by $a_1 + a_2 = a_2$ is such that $(M, +, 0, \leq)$ is a pa-ordered.*

Proof. Readily idempotence implies the reflexivity. The antisymmetry follows from commutativity. For transitivity suppose $a_1 \leq a_2$ and $a_2 \leq a_3$. Therefore $a_1 + a_2 = a_2$ and $a_2 + a_3 = a_3$ and thus $a_3 = (a_1 + a_2) + a_3 = a_1 + (a_2 + a_3) = a_1 + a_3$.

Obviously, $0 \leq a$, for any $a \in M$, i.e., M is *positive* and therefore *zerosumfree*, i.e., $a + b = 0$ implies $a = b = 0$.

Let now $a_1 \leq a_2$, i.e., $a_1 + a_2 = a_2$. If $a + a_2 \in M$ exists then we have

$$a + a_2 = a + (a_1 + a_2) = a + (a_1 + a_2) + a = (a + a_1) + (a + a_2).$$

□

See [JG92], pg. 205, proposition 18.18 concerning *additively idempotent semirings* previously extended.

Proposition 3. *If $(M, +, 0, \leq')$ is a commutative pa-monoid idempotent and positive then previous $\leq$ and $\leq'$ coincide.*

Corollary 4. *A partially-additive commutative and idempotent monoid $(M, +, 0)$ can be ordered up to a positive pa-ordered monoid $(M, +, 0, \leq)$ in an unique way (defining $a \leq b$ iff $a + b = b$ for $a, b \in M$). See [KS86].*

Corollary 5. *A partially-additive a-idempotent hemiring* $(R, +, 0, \cdot)$ *- not necessarily associative - can be ordered up to a positive pa-ordered hemiring* $(R, +, 0, \cdot, \leq)$ *in an unique way.*

The previous results provide extensions wrt [KS86] and [JG92] as indicated.

Proposition 6. *If* $(M, +, 0)$ *is a partially-additive commutative idempotent monoid and* $a, b \in M$ *then* $a + b \in M$ *exists iff* $\exists c \in M$ *such that* $a, b \leq c$ *and in this case* $a + b = sup(a, b) = a \vee b$*.*

See again [JG92], p. 205, proposition 18.18.

Note. We can prove a criterion for *a-idempotency*: for $(R, +, \cdot, 1)$ if $(R, +)$ is a semigroup, $(R, \cdot, 1)$ is a groupoid with identity 1 and if $1 + r$ is an unit, $\forall r \in R$, then $n1 = m1$, for some $n, m \in N$, $n > m$ implies *a-idempotency* (see [FS66] for a particular case).

Proposition 7. *If* R *is a pa-hemiring - not necessarily associative - then for any* $a \in R$ *m(multiplicatively)-idempotent,* $S_a = \{s \in R \mid 0 \leq s \leq a\}$ *is a subhemiring of* R *such that* $S_a = (S_a, +, 0, \cdot, \leq, a)$ *is a bounded upper semilattice.*

See [JG92], p. 209, proposition 18.28 on additivity idempotent partially-ordered semirings.

4 Some Related Order Conditions and Structures

It is well known that the lattice of an l-group is (automatically) distributive [GB73], [LF63]. We obtain the following extension in the commutative case.

Proposition 8. *If* $(G, +, \vee, \wedge)$ *is a commutative lattice-ordered grupoid (not necessarily associative or with neutral element) with the cancellation property then the support lattice is distributive.*

An important identity used in Fuzzy Modeling - also Maxmin algebras [CG91], [JG92] and assumed by *G.Ciobanu* [CD85] in his definition of the considered family of lattice-ordered semigroups - can be obtained in rather general conditions.

Proposition 9. *If* $(S, +, \vee, \wedge)$ *is such that* $(S, +)$ *is a commutative groupoid,* $(S, \vee, \wedge)$ *is a lattice and such that* $(S, +, \wedge, \vee)$ *is lattice-ordered then*

$$x + y = (x \wedge y) + (x \vee y), \forall x, y \in S.$$

The relevance of the previous identity has been noticed [CD85]. If $(S, +, \vee, \wedge)$ is a lattice-ordered groupoid satisfying this identity then S is commutative. If S is cancellative then $(S, \vee, \wedge)$ is distributive.

The next conclusion relates to *orthogonality* and therefore to decomposing an f-ring or semiring into subdirectly irreducible factors.

Proposition 10. *If* $(M, +, 0, \wedge, \leq)$ *is a lower semilattice-ordered monoid then* $a \wedge b = 0$ *implies* $a \wedge (b + c) = a \wedge c$*, for* $a, b, c \in M$ *and* $c \geq 0$*.*

A semiring R is called *Gelfand* [JG92] if $1+r$ is an unit, i.e., $(1+r)^{-1} \in R, \forall r \in R$.

A Φ-algebra is an Archimedean *Riesz* space that is an l-ring with an identity element 1 that is *weak* order unit, i.e., $1 \wedge x = 0$ implies $x = 0$. We have the following sufficient condition.

Proposition 11. *If* $(S, +, 0, \cdot, 1, \leq, \wedge)$ *is a partial structure such that* (i) $(S, +, 0, \leq)$ *is a partially-ordered groupoid with neutral element* 0*,* (ii) $(S, \cdot, 1, \leq)$ *is a partially-ordered monoid with identity* $1 > 0$*,* (iii) *every element* $1+u$*,* $u \geq 0$*, is an unit and* (iv) $(S, \leq, \wedge)$ *is a lower semi-lattice then* $x \wedge y = 0$ *implies* $xy = 0$.

J.S.Golan [JG92] defines the lattice-ordered semiring with the additional conditions $x + y = x \vee y$ and $x \cdot y \leq x \wedge y$ following the model of the ideal theory for rings. We do not assume these restrictions. Results of the type previously illustrated converge to an extension to the noncommutative case of a theorem of *F.A.Smith* [FS66].

Notation. For a semiring R we define the set

$$K(R) = \{x \in R \mid x + a = x + b \Longrightarrow a = b\}.$$

If R is a lower semilattice-ordered semiring and $A \subseteq R$ then the set

$$ort(A) = \{x \in R \mid x \wedge a = 0, \forall a \in A\}$$

is called the *orthogonal complement* of A.

Proposition 12. *If* $(R, +, 0, \cdot, 1, \wedge)$ *is a lower semilattice-ordered Gelfand semiring and if the order is s-order then* R *is the direct sum of* K *and* $ort(K)$ *iff the set*

$$U = \{t \in R \mid \exists k \in K : t = 1 \wedge k\}$$

has a supremum $k_0 = sup(U) \in K$ $(K = K(R))$*.*

The proof is omitted with excuses; it has some chapter-like length.

Proposition 13. *If* R *is a lower semilattice-ordered semiring such that* $1 + r$ *is an unit,* $\forall r \in R_+$*,* $1 > 0$ *- not necessarily commutative wrt addition or with* 0 *annihilator - then* R *is an* f*-semiring (in the same form as for* l*-rings, i.e., iff* $a \wedge b = 0$ *and* $c \geq 0$ *then* $a \wedge bc = a \wedge cb = 0$*).*

Proof. We have $0 \leq a \wedge bc \leq (a \wedge b)(1 + c)$. □

5 Sum-Ordering

Proposition 14. *If* $(M, +, 0)$ *is a commutative pa-monoid then the following two conditions are equivalent:*

(1) M *is s-ordered;*

(2) *If* $a, b, c \in M$ *satisfy the equality* $a = a + b + c$*, then* $a = a + b$*.*

See [JG92], p. 209, proposition 18.30 on semirings.

Proposition 15. *Let R be a zerosumfree pa-semiring, not necessarily associative, with 0 annihilator and $\leq$ its natural quasi-order.*

If every element in $R - \{0\}$ admits a left inverse in R then R is a pa-ordered semiring wrt its natural quasi-order.

Proof. If $a \leq b$ and $b \leq a$, $\exists c, d \in R$ so that $b = a + c$ and $a = b + d$. Since R is zerosumfree if $a = 0$ then $b = 0$.

Assume now $a \neq 0$. Therefore $\exists a^{-1} \in R$ (left inverse) with

$$a^{-1}b = 1 + a^{-1}c \text{ and } 1 = a^{-1}b + a^{-1}d \qquad (*)$$

Consequently,

$$1 = (1 + a^{-1}c) + a^{-1}d = 1 + a^{-1}c + a^{-1}d$$

by the associativity of partial addition. Let $e = a^{-1}c + a^{-1}d$ (e exists in R). From $(*)$ follows $1 + e = 1$. Let $f \in R$ with $fe = 1$ and therefore $f + 1 = f$.

We have

$$1 = f(a^{-1}c + a^{-1}d) = f(a^{-1}c) + f(a^{-1}d) =$$
$$= (f + 1)(a^{-1}c) + f(a^{-1}d) = 1 + a^{-1}c = a^{-1}b \text{ (see (*))}.$$

The natural quasi-order is therefore an order and it is easy to check the compatibility with the partial addition and multiplication. □

See [JG92], pg. 207, proposition 18.24 on zerosumfree division semirings.

6 Complemented Elements and the Multiplicative Ordering

We provide an approach to building order structures by using the multiplicative structure and the *complemented elements*. We extend the method given in [JG92] in the sense of considering pa-semirings and eliminating the condition that they are zerosumfree.

Proposition 16. *If $c(S) \subseteq S$ are such that*

(i) $re \in S, \forall r \in S$ and $e \in c(S)$,
(ii) $\forall r \in S, \exists e_r \in c(S)$ with $re_r = r$,
(iii) If $e \in c(S)$ then $e^2 = e$,
(iv) If $e, f \in c(S)$ then $ef \in c(S)$,
(v) If $e, f \in c(S)$ then $ef = fe$

and such that

(vi) $r(ef) = (re)f, \forall r \in S$ and $\forall e, f \in c(S)$,

then the relation $r \leq s$ in S defined by $\exists e \in c(S)$ satisfying $r = se$ is a partial order on S.

Proof. The expression of the form se with $s \in S$ and $e \in c(S)$ are meaningful because of (i). The relation $\leq$ is *reflexive* because of (ii). For *transitivity* suppose $r \leq s$ and $s \leq t$, i.e., $r = se$ and $s = tf$ with $r, s, t \in S$ and $e, f \in c(S)$. Hence $r = (tf)e$ and because of (vi), $r = t(fe)$. Since $f, e \in c(S)$ we have $r = tg$ with $g = fe \in c(S)$ because of (iv).

For *antisymmetry* assume $r \leq s$ and $s \leq r$, i.e., $r = se$ and $s = rf$, with $r, s \in S$ and $e, f \in c(S)$. We have $r = (rf)e$. Because of (v) and (vi) we can write $r = (re)f$. Because of (iii) and (vi) since $r = se$ we have $r = re$ and therefore $r = rf$. But $s = rf$ and therefore $r = s$. □

Proposition 17. *If R is a pa-semiring and $comp(R) = \{e \in R \mid \exists \bar{e} \in R$ with $e + \bar{e} = 1, e\bar{e} = \bar{e}e = 0\}$ is the center of R and if $ef\bar{e} = 0$, $\forall e, f \in comp(R)$, then $c(R) = comp(R)$ satisfies the previous $(i) - (vi)$ conditions.*

Proof. (partial) Consider $e, f \in c(R)$ and $1 = e1 + \bar{e} = e(f + \bar{f}) + \bar{e} = ef + g$, with $g = \bar{e} + e\bar{f}$ (proving that $g \in R$ exists).

Further $0 = ef\bar{e} + efe\bar{f} = (ef)g$ and similarly $g(ef) = 0$. To obtain the *commutativity* we proceed as following:

$$ef = 1(ef) = (f + \bar{f})ef = f(ef) + \bar{f}(ef) = f(ef)$$
$$fe = (fe)1 = (fe)(f + \bar{f}) = (fe)f + f(e\bar{f}) = (fe)f = f(ef) = ef.$$

□

The center $comp(Pfn)$ coincides with the unit interval $(I_{\subseteq} = [0,1], \subseteq)$.

The given method for building up order structures by using complemented elements in $comp(R)$ was used in [PB99] for unifying the different order structures introduced and studied by *P.Bottoni* et al. [PB97] for visual images (VIs) on a bi-dimensional media. The basic idea of [PB99] consists in an attempt to have VIs handled as a new data type called Visual Data (VD) and in enabling VD with an order structure following the method of the D.S.Scott's theory [DS76], [GS90] and its application in pattern recognition [RB88].

7 Complements in Partial Semirings: Existence of Sums and Commutativity

Definition 18. For a pa-semiring - not necessarily commutative or associative and without supposing that 0 is an annihilator - we define

$$C_1 = \{r \in R \mid \exists r' \in R, r + r' = 1\}$$

(r' is called an a(additive)-complement of r).

Proposition 19. *Let R be a pa-semiring - not necessarily commutative or associative and without supposing that 0 is an annihilator. For $n \in N$, $n \geq 2$, let $r_1, \ldots r_n$ be in C_1 such that $r_h r_i = 0$, for any $h \neq i$. Then $s = r_1 + \ldots r_n$ exists and is in C_1.*

Proof. The proof is by induction on $n \in N$, $n \geq 2$.

Let $r_1, r_2 \in C_1$, i.e., $r_1 + r_1' = r_2 + r_2' = 1$.

Hence

$$r_1 = r_1(r_2 + r_2') = r_1 r_2 + r_1 r_2' = r_1 r_2',$$
$$r_2' = (r_1 + r_1')r_2' = r_1 + r_1' r_2'.$$

One has

$$1 = r_2 + r_2' = r_2 + (r_1 + r_1' r_2') = (r_1 + r_2) + r_1' r_2'.$$

which proves that $r_1 + r_2$ exists in C_1 (and $r_1' r_2'$ is an a-complement for $r_1 + r_2$ as well as $r_2' r_1'$).

The inductive step should be obvious. □

Proposition 20. *Let R be a pa-semiring - not necessarily commutative - $a \in comp(R)$ and*

$$C_a = \{r \in R \mid \exists s = r'_a, r + s = a\}$$

such that $\forall r_1, r_2 \in C_a$,

$$r_1\bar{a} + r_2\bar{a} = 0 \text{ implies } r_1\bar{a} = r_2\bar{a} = 0,$$

and dually (right/left)

$$\bar{a}r_1 + \bar{a}r_2 = 0 \text{ implies } \bar{a}r_1 = \bar{a}r_2 = 0.$$

In these conditions,

$$aw = wa = w \text{ and } \bar{a}w = w\bar{a} = 0, \forall w \in C_a$$

and

$$ax = xa, \forall x \in C_1.$$

Remark. Obviously, $0, 1 \in comp(R)$ (one needs 0 to be an annihilator) and $0, a \in C_a, \forall a \in R$. The previous proposition extends a result of [MS85] concerning the center of a so-ring (a sum-ordered partial ring with an infinitary addition partially defined). The detailed comparison of the present results wrt [MS85] remains outside present limits.

Proof. For $w \in R$, we have

$$w = w(a + \bar{a})(a \in comp(R)) = wa + w\bar{a}.$$

Since $w \in C_a, \exists w' = w'_a \in R$, such that $w + w' = a$. Hence

$$0 = a\bar{a} = (w + w')\bar{a} = w\bar{a} + w'\bar{a}$$

which implies $w\bar{a} = 0$ and therefore $w = wa$ and similarly $\bar{a}w = 0$ and $w = aw$.

Further if $x \in C_1$ then $\exists x' = x'_1 \in R$ such that $x + x' = 1$ and therefore $a = x_1 + ax'$ with $x_1 = ax$, $x_1 \in C_a$ (for $r = x_1$, $\exists s = ax' \in R$, such that $r + s = a$). Similarly, $a = x_2 + x'a$ with $x_2 = xa$, $x_2 \in C_a$. We have

$$xa = x_2 = ax_2(\text{ since } x_2 \in C_a) = a(xa) =$$
$$= (ax)a = x_1a = x_1(\text{ since } x_1 \in C_a) = ax.$$

□

In the previous conditions, $a^2 = a, ab = ba$ and $ab\bar{a} = 0, \forall a, b \in comp(R)$. If the complement exists then it is unique.

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