A Representation Theorem for Monadic Pavelka Algebras

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Abstract: In this paper we define the monadic Pavelka algebras as algebraic structures induced by the action of quantifiers in Rational Pavelka predicate logic. The main result is a representation theorem for these structures.

Key Words: Pavelka algebra, monadic Pavelka algebra, MV-algebra, monadic MV-algebra.

Category: F.4.1.

1 Introduction

Rational Pavelka logic (RPL) is obtained from Lukasiewicz infinite valued propositional calculus (L) by adding the truth constants \( \tau \) for \( \tau \in [0, 1] \cap Q \). The corresponding algebraic structures (Pavelka algebras) will be MV-algebras that contain a set of constants \( \{ \tau | \tau \in [0, 1] \cap Q \} \) as a subalgebra. The quantifiers defined on an MV-algebra appear in [10, 11] reflecting the action of the quantifiers in Lukasiewicz infinite valued predicate calculus (L\( \forall \)). In this paper we start from the Rational Pavelka predicate logic (RPL\( \forall \)) in order to define the quantifiers on Pavelka algebras. This leads to the notion of monadic Pavelka algebra. If \( K \) is a non-empty set then the MV-algebra \( [0, 1]^K \) has a canonical structure of monadic Pavelka algebra. The main result of this paper is a representation theorem for monadic Pavelka algebras. In fact, our results can be viewed as algebraic versions of the results in [6] (see also [4], pp. 223-226).

2 Monadic MV-algebras

The MV-algebras were introduced in [1] as algebraic models for L. An MV-algebra is an algebraic structure \( \langle A, \oplus, \neg, 0 \rangle \) where \( \langle A, \oplus, 0 \rangle \) is an abelian monoid and \( \neg \) is an unary operation such that:

1. \( \neg x = x \) for any \( x \in A \),
2. \( x \oplus \neg 0 = \neg 0 \) for any \( x \in A \),
3. \( \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x \) for any \( x, y \in A \).

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1 C. S. Calude and G. Ștefănescu (eds.), Automata, Logic, and Computability. Special issue dedicated to Professor Sergiu Rudeanu Festschrift.
We also define \( 1 = -0, \ x \odot y = -(x \oplus y), \ x \multimap y = -(x \odot y), \ x \lor y = x \oplus -(x \odot y), \ x \land y = x \odot -(x \odot y) \). Thus \( \langle A, \lor, \land, 0, 1 \rangle \) is a bounded distributive lattice. If \( x \in A \) and \( n \) is a natural number we denote
\[
0x = 0, \ (n+1)x = nx \oplus x,
\]
\[
x^0 = 1, \ x^{n+1} = x^n \odot x.
\]
The interval \([0, 1]\) is an MV-algebra with respect to the operations \( x \oplus y = \min(1, x + y) \) and \( x \odot y = 1 - x + y \). In \([0, 1]\) we have that \( x \odot y = \max(0, x + y - 1) \) and \( x \multimap y = \min(1, 1 - x + y) \). If \( x < 1 \) then there exists a natural number \( n \) such that \( x^n = 0 \).

**Lemma 2.1** \([2]\) In every MV-algebra \( A \) the following equalities hold:
(i) \( a \odot \vee_{i \in I} x_i = \vee_{i \in I} (a \odot x_i), \ a \oplus \wedge_{i \in I} x_i = \wedge_{i \in I} (a \oplus x_i), \)
(ii) \( a \odot \bigvee_{i \in I} x_i = \bigvee_{i \in I} (a \odot x_i), \ a \oplus \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (a \oplus x_i), \)
(iii) if \( A \) is linearly ordered then
\[
\bigvee_{i \in I} (x_i \odot x_i) = 2 \left( \bigvee_{i \in I} x_i \right), \ \bigvee_{i \in I} (x_i \odot x_i) = \left( \bigvee_{i \in I} x_i \right)^2, \ \bigwedge_{i \in I} (x_i \odot x_i) = 2 \left( \bigwedge_{i \in I} x_i \right), \ \bigwedge_{i \in I} (x_i \odot x_i) = \left( \bigwedge_{i \in I} x_i \right)^2.
\]

**Lemma 2.2** \([1, 2]\) The implication operation \( \multimap \) has the following properties:
(i) \( x \leq y \iff x \multimap y = 1, \)
(ii) \( y \odot z \leq x \iff y \leq z \multimap x, \)
(iii) \( (x \lor y) \multimap z = (x \multimap z) \land (y \multimap z). \)

A non-empty subset \( F \) of \( A \) is an MV-filter (filter) if for every \( x, y \in A \) the following are satisfied:
4. \( x, y \in F \implies x \odot y \in F, \)
5. \( x \leq y, \ x \in F \implies y \in F. \)

For \( X \subseteq A \) the filter generated by \( X \) is given by
\[
\mathcal{F}(X) = \{ a \in A \mid x_1 \oplus \cdots \oplus x_n \leq a \text{ for some } n < \omega \text{ and } x_1, \ldots, x_n \in X \}.
\]
If \( F \) is a filter and \( b \in A \) then
\[
\mathcal{F}(X \cup \{b\}) = \{ a \in A \mid x \oplus b^n \leq a \text{ for some } n < \omega \text{ and } x \in F \}.
\]

With any filter \( F \) of \( A \) we can associate a congruence \( \sim_F \) on \( A: \)
\[
x \sim_F y \iff (x \multimap y) \land (y \multimap x) \in F.
\]
Denote by \( A / F \) the quotient MV-algebra \( A / \sim_F \) and denote by \( a / F \) the class of \( a \in A \).

A proper filter \( P \) is prime if \( x \lor y \in P \) implies \( x \in P \) or \( y \in P \). One can prove that a proper filter \( P \) is prime iff \( x \multimap y \in P \) or \( y \multimap x \in P \) for any \( x, y \in A \) iff \( A / P \) is a linearly ordered MV-algebra.

**Definition 2.3** An existential quantifier on an MV-algebra \( A \) is a mapping \( \exists : A \longrightarrow A \) which satisfies the following axioms:
M0. \( \exists \emptyset = 0, \)
M1. \( x \leq \exists x, \)
M2. \( \exists (a \odot y) = \exists a \odot \exists y, \)
M3. \( \exists (a \oplus y) = \exists a \oplus \exists y, \)
M4. \( \exists (ax) = \exists x \odot \exists x, \)
M5. \( \exists (ax) = \exists x \odot \exists x. \)

If we define \( \forall x = -\exists -x \) for any \( x \in A \) then the mapping \( \forall : A \longrightarrow A \) fulfills the following properties:
M0\(\forall\). \( \forall 1 = 1, \)
M1\(^{\circ}\): \(\forall x \leq x\),
M2\(^{\circ}\): \(\forall (x \lor y) = \forall x \lor \forall y\),
M3\(^{\circ}\): \(\forall (x \land y) = \forall x \land \forall y\),
M4\(^{\circ}\): \(\forall (x \rightarrow x) = \forall x \rightarrow \forall x\),
M5\(^{\circ}\): \(\forall (x \rightarrow y) = \forall x \rightarrow \forall y\).

A mapping \(\forall: A \rightarrow A\) satisfying the properties M0\(^{\circ}\) - M5\(^{\circ}\) will be called a universal quantifier on \(A\). A monadic MV-algebra is a pair \(\langle A, \exists \rangle\) where \(A\) is an MV-algebra and \(\exists\) is an existential quantifier on \(A\). One can also define a monadic MV-algebra as a pair \(\langle A, \forall \rangle\) where \(A\) is an MV-algebra and \(\forall\) is an universal quantifier on \(A\).

**Lemma 2.4** [10] In every monadic MV-algebra the following properties are satisfied:
(i) \(\exists 1 = 1\),
(ii) \(\exists x = x\),
(iii) \(\exists (-x) = -\exists x\),
(iv) \(\exists (x \lor y) = \exists x \lor \exists y\),
(v) \(\exists (x \land y) = \exists x \land \exists y\),
(vi) \(\exists (a \land b) = a \land \exists b\),
(vii) \(\exists (a \lor b) = \exists a \lor \exists b\),
(viii) \(x \leq y \Rightarrow \exists x \leq \exists y\) and \(\forall x \leq \forall y\),
(ix) \(\exists \forall x = \forall x\), \(\forall \exists x = \exists x\).

**Example 2.5** [3] If \(K\) is a non-empty set then \([0, 1]^K\) becomes a monadic MV-algebra by defining \(\exists: [0, 1]^K \rightarrow [0, 1]^K\) in the following way:

\[(\exists p)(k) = \bigvee \{p(l) \mid l \in K\}\] for any \(p \in [0, 1]^K\) and \(k \in K\).

The axioms M0-M5 can be proved by using Lemma 2.1.

### 3 Monadic Pavelka algebras

Let us denote \(L\) the MV-algebra \([0, 1] \cap \mathbb{Q}\).

**Definition 3.1** A Pavelka algebra is a structure \(\langle A, \{F : r \in L\} \rangle\) where \(A\) is an MV-algebra and \(\{F : r \in L\} \subseteq A\) such that:

P0. \(\emptyset = 0\),
P1. \(r \equiv s = r \equiv s\) for any \(r, s \in L\),
P2. \(\neg \neg r = \neg r\) for any \(r \in L\),
P3. \(F \neq \exists\) for any distinct \(r, s \in L\).

Thus, the mapping \(\tau \mapsto \tau\) is an injective morphism of MV-algebras. The Lindenbaum - Tarski algebra of Rational Pavelka logic (RPL) is a Pavelka algebra.

The notion of morphism of Pavelka algebras is introduced as usual.

**Lemma 3.2** Let \(\langle A, \{F : r \in L\} \rangle\) be a Pavelka algebra, \(P\) a proper filter of \(A\) and \(r, s \in L\). Then the following hold:
(i) \(\tau \in P\) iff \(\tau = 1\),
(ii) \(r \leq s\) iff \(\tau / p \leq \tau / p\).

*Proof.* (i) If \(\tau \neq 1\) then there is \(n < \omega\) such that \(\tau^n = 0\), so \(\tau^n = 0\). But \(\tau \in P\) implies \(\tau^n \in P\) for each \(m < \omega\). We get \(0 \in P\). Contradiction.
(ii) \(r \leq s\) iff \(\tau \rightarrow \tau \rightarrow s = 1\) iff \(\tau \rightarrow \tau \rightarrow \pi \in P\) iff \(\tau \rightarrow \tau \rightarrow \pi \in P\) iff \(\tau / p \leq \tau / p\). \(\Box\)
Definition 3.3 A monadic Pavelka algebra is a structure \( \langle A, \exists, \{ r : r \in L \} \rangle \) where \( \langle A, \exists \rangle \) is a monadic MV-algebra and \( \langle A, \{ r : r \in L \} \rangle \) is a Pavelka algebra such that \( \exists r = r \) for any \( r \in L \).

The notion of morphism of monadic Pavelka algebras is introduced as usual.

Example 3.4 Let \( F \) be the set of formulas of Rational Pavelka predicate logic (RPLV) and \( \sim \) the following equivalence relation on \( F \): \( \varphi \sim \psi \) iff \( \vdash \varphi \leftrightarrow \psi \). If \( x \) is a variable then we denote \( (\exists x)([\varphi]) = [\exists x \varphi] \) where \( \varphi \) is a formula and \([\varphi]\) its class in \( F/\sim \). Then \( \langle F/\sim, \exists x : F/\sim \rightarrow F/\sim \rangle \) is a monadic MV-algebra. If \( r, s \in L \) are distinct then \( [r] \neq [s] \). If \( [r] = [\bar{s}] \) then \( \vdash \bar{r} \rightarrow \bar{s} \) and \( \vdash \bar{s} \rightarrow \bar{r} \). We get \( r \leq s \) and \( s \leq r \) (see [4]), so \( r = s \). It is easy to show that in this way \( F/\sim \) becomes a monadic Pavelka algebra.

Example 3.5 Let \( K \) be a non-empty set. For \( r \in L \) denote \( \tau : K \rightarrow [0,1] \) the constant function \( k \mapsto r \). Thus \( \langle [0,1]^K, \{ r : r \in L \} \rangle \) is a Pavelka algebra, so, by Example 2.5, \( [0,1]^K \) is endowed with a structure of monadic Pavelka algebra.

If \( A \) is a monadic Pavelka algebra then a morphism of monadic Pavelka algebras \( \Phi : A \rightarrow [0,1]^K \) will be called a representation of \( A \).

Lemma 3.6 In a monadic Pavelka algebra \( A \) the following equalities hold for any \( r \in L \) and \( a \in A \):
(i) \( \exists (r \oplus a) = r \oplus \exists (a) \),
(ii) \( \exists (r \odot a) = r \odot \exists (a) \),
(iii) \( \forall (r \oplus a) = r \oplus \forall (a) \),
(iv) \( \forall (r \odot a) = r \odot \forall (a) \),
(v) \( r \rightarrow \exists a = \exists (r \rightarrow a) \),
(vi) \( \exists a \rightarrow r = \forall (a \rightarrow r) \),
(vii) \( \forall a \rightarrow r = \forall (r \rightarrow a) \),
(viii) \( \exists a \rightarrow r = \exists (a \rightarrow r) \).

Proof. (i) \( \exists (r \oplus a) = \exists (\exists (r \oplus a)) = \exists (r \oplus \exists a) = r \oplus \exists a \).
(ii), (iii), (iv) follow similarly.
(v) \( \exists (r \rightarrow a) = \exists ((r \rightarrow a) \oplus a) = \exists (r \oplus a) = r \oplus \exists a = r \rightarrow \exists a \).
(vi) \( \forall (a \rightarrow r) = \forall (\neg a \oplus r) = \forall (r \oplus \neg a \oplus a) = \forall a \oplus r = \forall a \rightarrow r = \exists a \rightarrow r \).
(vii), (viii) follow similarly. \( \square \)

One remark that \( B = \exists (A) = \forall (A) \) is a Pavelka subalgebra of \( A \).

For the rest of the paper let \( \langle A, \exists, \{ r : r \in L \} \rangle \) be an arbitrary monadic Pavelka algebra and \( B = \exists (A) \).

Lemma 3.7 If \( s \in L, a \in A \) and \( r \not\leq a \) then there exists \( X \subseteq B \) such that:
(i) \( \text{filt} (X \cup \{ a \rightarrow r \}) \) is proper,
(ii) for any \( b \in B \) and \( r \in L \), \( r \rightarrow b \in X \) or \( b \rightarrow \bar{r} \in X \).

Proof. We shall prove that the \( \text{filt} (a \rightarrow r) \) is proper. If not, then \( (a \rightarrow r)^n = 0 \) for some \( n < \omega \). But \( (a \rightarrow r)^n \cap (\bar{r} \rightarrow a)^n = 1 \) so \( (a \rightarrow a)^n = 1 \). This yields \( \bar{r} \rightarrow a = 1 \), hence \( r \leq a \). Contradiction. Thus there exists \( b \not\in \text{filt} (a \rightarrow r) \).

Consider an enumeration \( \{ (a_\xi, r_\xi) : \xi < \kappa \} \) of the set \( B \times L \). We shall construct by induction a sequence \( \{ X_\xi \}_{\xi < \kappa} \) such that \( b \not\in \text{filt} (X_\xi) \) for any \( \xi < \kappa \).
• $X_0 = \{a \rightarrow \bar{s}\}$

• $\xi = \xi + 1$. The induction hypothesis is $b \notin \text{filt}(X_\xi)$. Assume $b \in \text{filt}(X_\xi \cup \{a_\xi \rightarrow \bar{r}_\xi\}) \cap \text{filt}(X_\xi \cup \{\bar{r}_\xi \rightarrow a_\xi\})$ so there is $n < \omega$ such that $(a_\xi \rightarrow \bar{r}_\xi)^n \rightarrow b \in \text{filt}(X_\xi)$ and $(\bar{r}_\xi \rightarrow a_\xi)^n \rightarrow b \in \text{filt}(X_\xi)$. But

$$b = 1 \rightarrow b$$

$$\equiv [(a_\xi \rightarrow \bar{r}_\xi)^n \lor (\bar{r}_\xi \rightarrow a_\xi)^n] \rightarrow b$$

$$\equiv [(a_\xi \rightarrow \bar{r}_\xi)^n \rightarrow b] \land [(\bar{r}_\xi \rightarrow a_\xi)^n \rightarrow b]$$

hence $b \in \text{filt}(X_\xi)$. Contradiction. It follows that $b \notin \text{filt}(X_\xi \cup \{a_\xi \rightarrow \bar{r}_\xi\})$ or $b \notin \text{filt}(X_\xi \cup \{\bar{r}_\xi \rightarrow a_\xi\})$. Thus one can define

$$X_\xi = \begin{cases} X_\xi \cup \{a_\xi \rightarrow \bar{r}_\xi\} & \text{if } b \notin \text{filt}(X_\xi \cup \{a_\xi \rightarrow \bar{r}_\xi\}) \\ X_\xi \cup \{\bar{r}_\xi \rightarrow a_\xi\} & \text{otherwise.} \end{cases}$$

• If $\xi$ is a limit ordinal then $X_\xi = \bigcup_{\xi' < \xi} X_{\xi'}$.

It follows that $b \notin \text{filt}(\bigcup_{\xi < k} X_\xi)$ and we define $X = \bigcup_{\xi < k} X_\xi \cup \{a \rightarrow \bar{s}\}$. □

**Lemma 3.8** Let $X$ be a the set constructed in Lemma 3.7. If $\exists a \in X$ then there exists a prime filter $P$ such that $X \cup \{a\} \subseteq P$.

**Proof.** By the dual of [2], Proposition 1.2.13 it suffices to prove that $\text{filt}(X \cup \{a\})$ is a proper filter. If not, then there exist $m < \omega$ and $x_1, \ldots, x_n \in X$ such that $x_1 \circ \cdots \circ x_n \circ a^m = 0$. Denote $x = x_1 \circ \cdots \circ x_n$ so $c \leq \neg a^m$, hence $\forall c \leq \neg a^m \neg \exists (a^m)$. But $c \in \exists (A)$ because $X \subseteq \exists (A)$ so $c \leq \neg \exists (a^m)$, hence $\neg \exists (a^m) \in \text{filt}(X)$. By hypothesis, $\exists (a^m) = (\exists a)^m \in \text{filt}(X)$, contradicting that $\text{filt}(X)$ is proper. □

## 4 Representation theorem

In this section we shall prove a representation theorem for monadic Pavelka algebras.

**Theorem 4.1** Let $\langle A, \exists, \{ \tau : \tau \in L \} \rangle$ be a monadic Pavelka algebra. If $a \in A$ and $s \in L$ such that $s \notin a$ then there exist a non-empty set $K$, a representation $\Phi : A \rightarrow [0, 1]^k$ and $k \in K$ such that $\Phi(a)(k) \leq s$.

**Proof.** Let $X$ be the set constructed in Lemma 3.7 and $K$ the set of prime filters of $A$ including $X$. For any $x \in A$ and $P \in K$ denote

$$[x]_P = \sup \{\tau \in L \mid x \rightarrow \tau \in P\}.$$

In order to define $\Phi$ we have to prove some properties.

(i) $[x]_P = \inf \{\tau \in L \mid x \rightarrow \tau \in P\}$.

If $\tau \rightarrow x \in P$ and $x \rightarrow \tau \in P$ then, by Lemma 3.2, $\tau \rightarrow x \in P$, so $\tau \leq s$. It follows that $[x]_P \leq \inf \{\tau \in L \mid x \rightarrow \tau \in P\}$. If we assume $[x]_P < \inf \{\tau \in L \mid x \rightarrow \tau \in P\}$ then there is $q \in L$ such that $[x]_P < q < \inf \{\tau \in L \mid x \rightarrow \tau \in P\}$, so $\sigma \rightarrow x \notin P$ and $x \rightarrow \sigma \notin P$. This contradicts the fact that $P$ is a prime filter.

(ii) $[x \circ y]_P = [x]_P \oplus [y]_P$.

(iii) $[x \circ y]_P = [x]_P \oplus [y]_P$.

In order to prove (ii) we have

$$[x \circ y]_P = \inf \{\tau \mid x \circ y \rightarrow \tau \in P\}$$

and

$$[x]_P \oplus [y]_P = \sup \{\tau \oplus q \mid \tau \rightarrow x \in P, \sigma \rightarrow y \in P\}.$$

By Lemma 3.2, $\tau \rightarrow x \in P$, $\sigma \rightarrow y \in P$ and $x \circ y \rightarrow \tau \in P$ implies
\[ \tau'/p \leq x/p, \bar{q}/p \leq y/p \text{ and } x/p \oplus y/p \leq \bar{q}/p \text{ so } \tau \oplus q \leq t. \]

We proved that \([x]_p \oplus [y]_p \leq [x \oplus y]_p\). The converse inequality and (iii) follow similarly.

(iv) \([\tau]_p = \tau\) for any \(\tau \in L\).

By Lemma 3.2, \([\tau] = \sup\{q \in L \mid q \leq \tau\} = \tau\).

Let us define \(\Phi : A \rightarrow [0,1]^K\) by \(\Phi(x)(P) = [x]_P\) for any \(x \in A\) and \(P \in K\). In accordance to (ii)-(iv), \(\Phi\) is a morphism of Pavelka algebras. Now we shall prove that

(v) \(\Phi([x]_P) = ([\Phi(x)](P))_{\forall x \in A\text{ and } P \in K}\).

If \(\tau \in L\) and \(P, Q \in K\) we have, in accordance to Lemma 3.2, \(\tau \rightarrow \exists x \in P\) if \(\tau \rightarrow x \in Q\), therefore \([\exists x]_P = [\exists x]_Q\). Then \([\exists x]_P = [\exists x]_Q \geq [x]_Q\) for every \(Q \in K\), hence

\(\Phi([x]_P) = [\exists x]_P \geq \sup\{[x]_Q \mid Q \in K\} = \sup\{\Phi(x)(Q) \mid Q \in K\} = (\exists \Phi(x))(P)\).

The following implications:

\(\tau < [\exists x]_P \Rightarrow \exists x \rightarrow \tau \not\in P\) (cf. (i))
\(\Rightarrow \exists \exists x \rightarrow \tau \not\in X\) (cf. \(X \subseteq P\))
\(\Rightarrow \exists (\exists \rightarrow x) \in X\) (cf. Lemma 3.7)
\(\Rightarrow \exists \tau \rightarrow x \in Q\) (cf. Lemma 3.6)
\(\Rightarrow \tau \rightarrow [x]_Q\)

for some \(Q \in K\), establish the converse inequality in (v). Indeed, if we assume \([\exists x]_P > \sup\{[x]_Q \mid Q \in K\}\) then there is \(\tau \in L\) such that \([\exists x]_P > \tau > \sup\{[x]_Q \mid Q \in K\}\) contradicting the above implications. Therefore, \(\Phi\) is a representation of \(A\).

Finally, by Lemma 3.7, there exists \(P_0 \in K\) such that \(X \cup \{a \rightarrow \tau\} \subseteq P_0\) so \(\Phi(a)(P_0) \leq s\) and \(\Phi(a)(P_0) = \inf\{\tau \mid a \rightarrow \tau \in P_0\}\).

For any \(a \in A\) let us define

\([a] = \sup\{\tau \mid \tau \leq a\}\)
\(\|a\| = \inf\{\Phi(a)(k) \mid \Phi : A \rightarrow [0,1]^K\text{ representation and } k \in K\}\).

Corollary 4.2 \([a] = \|a\| \text{ for any } a \in A\).

Proof. The inequality \([a] \leq \|a\|\) is obvious. Assume there exists \(s \in L\) such that \([a] < s < \|a\|\). Thus \(\tau \not\leq a\), so, by Theorem 4.1, there exist a representation \(\Phi : A \rightarrow [0,1]^K\) and \(k \in K\) such that \(\Phi(a)(k) \leq s\). Therefore \(\|a\| \leq \Phi(a)(k) \leq s\). Contradiction.

References


