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Rigorous Numerical Studies of the Existence of Periodic Orbits for the Hénon Map

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Abstract: In this paper we perform a rigorous study of the Hénon map. We prove with computer assistance the existence of symbolic dynamics for h^2 and h^7 and the existence of periodic orbits of all periods but 3 and 5.

Key Words: chaos, computer assisted proof, interval arithmetic.

1 Introduction

In this paper we consider the Hénon map defined by the following equation:

$$h(x,y) = (1 + y - ax^2, bx), \tag{1}$$

where $a = 1.4$ and $b = 0.3$ are the "classical" parameter values. Although the definition of the Hénon map is very simple it displays very complicated dynamics. A typical trajectory of the map is shown in Fig. 1.

In Section 2 we recall the technique of TS-maps and formulate two theorems used in the following sections. In Section 3 we prove the existence of symbolic dynamics for h^2 and what follows the existence of periodic points of h for all even periods.

In [Zgliczyński 97b] the dynamics of topological horseshoe was proved for h^7. From this follows the existence of symbolic dynamics for h^7 and the existence of periodic orbits of h of period $7n$ for all natural n. In Section 4 we repeat the proof described in [Zgliczyński 97b] using interval arithmetic. We show that using this tool the number of points for which we must check certain conditions can be significantly reduced. Then checking some more conditions we prove the existence of periodic points with period 8 and all periods greater or equal to 10.

Finally by means of the interval Newton method we prove that within the region $[-5,5] \times [-5,5]$ there exists no periodic point with period 3 or 5 and we prove that there exist periodic points with period 9.

During all the computer–assisted proofs we use the procedures for interval computations form BIAS and PROFIL packages [Knüppel 93]. Programs were compiled using gnu C++ compiler (gcc version 2.7.2.1) and run on Sun Ultra 1 computer. The source code of the programs is available at the following www location: `http://fractal.zet.agh.edu.pl/~galias/int.html`. Additionally all the results were checked using the package for interval computations prepared by the author in Turbo-Pascal 7.0 programming environment and run on Pentium 166MHz.

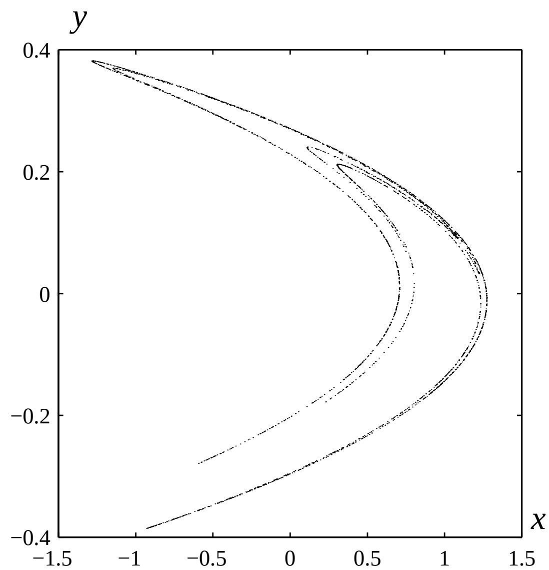

Figure 1: A trajectory of the Hénon map. 3000 points of the trajectory starting from the initial conditions: $x = 0$, $y = 0$ after a short transient (100 iterations) are plotted.

2 TS–Maps

One of the tools we use in our study is the technique of TS-maps (topological shifts) introduced in [Zgliczyński 97a, Zgliczyński 97b]. This technique can be used to prove the existence of an infinite number of periodic orbits for a given system. It combines existence results based on the fixed point index theory and computer–assisted computations, necessary to verify assumptions of the existence theorem.

Here we consider a special case of TS–maps defined on two sets N_0 and N_1. For the general case see [Zgliczyński 97b]. Let the sets N_0, N_1, E_0, E_1, E_2 be as depicted in Fig. 2. The important property of this sets is that E_0 lies on the left hand side of the sets N_0 and N_1, the set E_1 lies between N_0 and N_1 and E_2 lies on the right hand side of N_0 and N_1. Certain deformations of these sets are also possible (see [Zgliczyński 97b]). Let $W = N_0 \cup N_1 \cup E_0 \cup E_1 \cup E_2$. By $intW$ we denote the interior of W. We will say that the image of N_i *covers horizontally* the set N_j if the image of one of the vertical edges of N_i lies on the right hand side of N_j while the image of the second vertical edge lies on the left hand side of N_j. For example image of N_0 covers horizontally N_1 if $f(L(N_0)) \subset E_1$ and $f(R(N_0)) \subset E_2$ or $f(L(N_0)) \subset E_2$ and $f(R(N_0)) \subset E_1$, where $L(N_0)$ and $R(N_0)$ denote respectively the left and right vertical edges of N_0.

Let f be a continuous map defined on $N_0 \cup N_1$. In our analysis we consider two special cases of TS-maps. They are described in detail in [Galias 97]. The first case involves maps with topological horseshoe embedded (compare Fig. 2a).

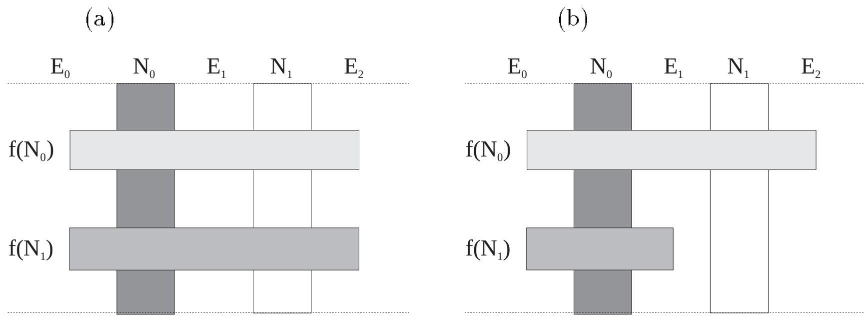

Figure 2: Images of sets N_0 and N_1 for the horseshoe map (a) and the deformed horseshoe map (b). For the horseshoe map the images of vertical edges of N_0 lie one in E_0 and the second in E_2 and similarly for N_1. For the deformed horseshoe the only difference is that the image of one of the vertical edges of N_1 is enclosed in E_1 instead of E_2

Theorem 1. *If $f(N_0), f(N_1) \subset intW$, the image of N_0 covers horizontally the sets N_0 and N_1 (vertical edges of N_0 are mapped by f in such a way that the image of one of the edges is enclosed in E_0, while the second one is enclosed in E_2), and the image of N_1 covers horizontally N_0 and N_1 then for any finite sequence $a_0, a_1, \ldots, a_{n-1} \in \{0,1\}^n$ there exists a point x satisfying*

$$f^i(x) \in N_{a_i} \quad for \quad i = 0, \ldots, n-1 \quad and \quad f^n(x) = x.$$

In this case one can also prove that the full shift on two symbols with the transition matrix [Robinson 95]

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2}$$

is embedded in the map f. Non-zero element a_{ij} of the transition matrix means that the image of N_i covers horizontally N_j (we can find a point $x \in N_i$ such that $f(x) \in N_j$).

The next theorem is important for maps with the deformed horseshoe embedded (compare Fig. 2b). From the set of n-element sequences with elements from the set $\{0,1\}$ let us choose sequences, which do not contain the subsequence $(1,1)$:

$$T_n = \{\, (a_0, \ldots, a_{n-1}) \in \{0,1\}^n \colon (a_j, a_{(j+1) \bmod n}) \neq (1,1) \text{ for } 0 \leq j < n \,\}. \tag{3}$$

Theorem 2. *If $f(N_0), f(N_1) \subset intW$, image of N_0 covers horizontally the sets N_0 and N_1, image of N_1 covers horizontally the sets N_0, then for any finite sequence $a = (a_0, a_1, \ldots, a_{n-1}) \in T_n$ there exists a point x satisfying*

$$f^i(x) \in N_{a_i} \quad for \quad i = 0, \ldots, n-1 \quad and \quad f^n(x) = x. \tag{4}$$

In this case the subshift on two symbols with the transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

is embedded in f.

If f is one-to-one one has to check only the conditions concerning the edges of sets N_i. Instead of proving that $f(N_i) \subset intW$ it is sufficient to prove that $f(bd(N_i)) \subset intW$, where $bd(N_i)$ denotes the border of the set N_i. This is a conclusion from Jordan's theorem (compare [Galias 97]).

3 Symbolic Dynamics for h^2 — Deformed Horseshoe

In this section we show using the technique described previously that the subshift on two symbols with the transition matrix (5) (the deformed topological horseshoe) is embedded in h^2.

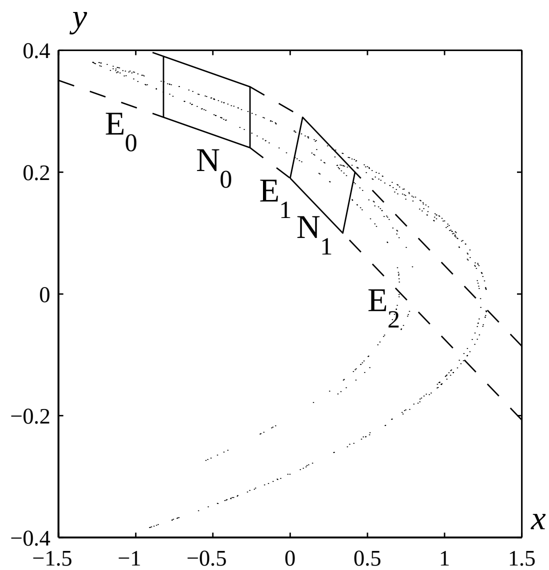

Figure 3: The definition of the sets N_0 and N_1 for the proof of symbolic dynamics for h^2

Let us define the sets N_i as follows: N_0 is a quadrangle $\overline{A_1A_2A_3A_4}$ and N_1 is the quadrangle $\overline{A_5A_6A_7A_8}$, where $A_1 = (-0.82, 0.29)$, $A_2 = (-0.82, 0.39)$, $A_3 = (-0.26, 0.34)$, $A_4 = (-0.26, 0.24)$, $A_5 = (0, 0.19)$, $A_6 = (0.08, 0.29)$, $A_7 = (0.42, 0.2)$ and $A_8 = (0.34, 0.1)$ (compare Fig. 3). We also define sets E_0, E_1 and E_2 lying respectively to the left, between and to the right of the sets N_0 and N_1. The set E_0 is a half–stripe lying on the left hand side of N_0 defined by straight lines A_2A_3, A_4A_1 and A_1A_2. E_1 is the quadrangle $\overline{A_4A_3A_6A_5}$. E_2 is a half–stripe lying on the right hand side of N_1, defined similarly as E_0. Let $W = N_1 \cup N_2 \cup E_0 \cup E_1 \cup E_2$.

(a)

(b)

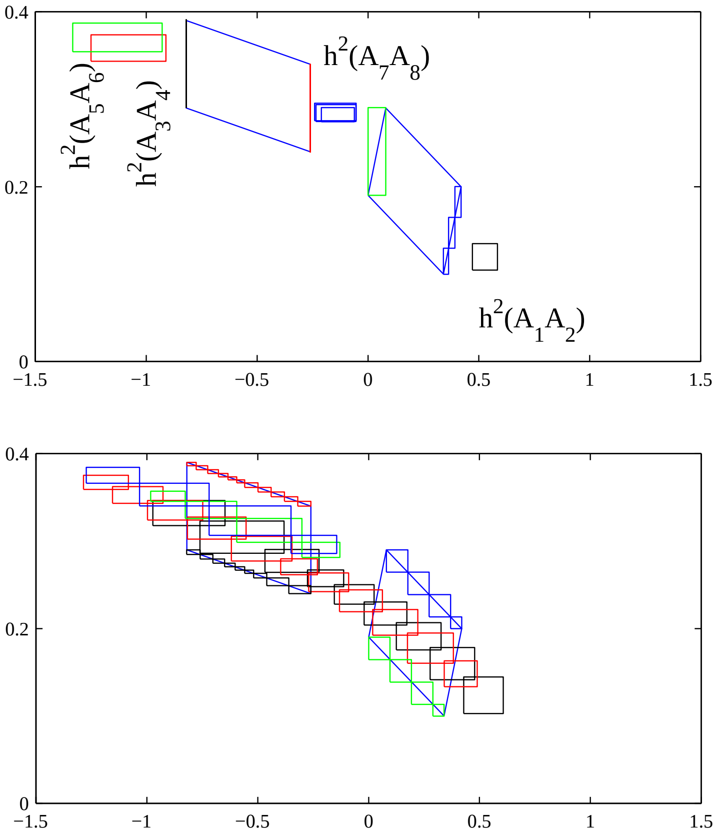

Figure 4: (a) the covering of the vertical edges of N_0 and N_1 with rectangles (notice that the rectangles covering edges of N_0 are very thin as these edges are parallel to the y axis) and its image under h^2, (b) the covering of horizontal edges of N_0 and N_1 and its image under h^2 obtained in computer assisted proof

With the computer assistance we have proved that the image of N_0 covers horizontally N_0 and N_1 and the image of N_1 covers horizontally N_0. This is formally written in the following lemma.

Lemma 3.

1. $h^2(\overline{A_1A_2}) \subset E_2$ *and* $h^2(\overline{A_3A_4}) \subset E_0$,
2. $h^2(\overline{A_5A_6}) \subset E_0$ *and* $h^2(\overline{A_7A_8}) \subset E_1$,
3. $h^2(\overline{A_1A_4})$, $h^2(\overline{A_2A_3})$, $h^2(\overline{A_5A_8})$, $h^2(\overline{A_6A_7}) \subset intW$.

Proof. For the proof of 1 and 2 we have covered the vertical edges $\overline{A_1A_2}$, $\overline{A_3A_4}$, $\overline{A_5A_6}$ and $\overline{A_7A_8}$ by 1, 1, 1 and 3 rectangles respectively. Using interval arithmetic we have proved that their images under h^2 are enclosed in the appropriate sets

E_i. The covering of vertical edges with rectangles (two-dimensional intervals) and their images under h^2 computed during the proof are shown in Fig. 4a. One can clearly see that $h^2(\overline{A_1A_2})$ lies on the right hand side of N_1, $h^2(\overline{A_3A_4})$ and $h^2(\overline{A_5A_6})$ lie on the left hand side of N_0 and $h^2(\overline{A_7A_8})$ lies between N_0 and N_1.

For the proof of 3 we have covered the horizontal edges $\overline{A_1A_4}$, $\overline{A_2A_3}$, $\overline{A_5A_8}$ and $\overline{A_6A_7}$ by 9, 11, 4 and 4 rectangles respectively. The covering of horizontal edges with rectangles and their images under h^2 are shown in Fig. 4b. We have checked that the images are enclosed within the set $intW$.

For the whole proof of the existence of symbolic dynamics for h^2 it was sufficient to compute images of 34 rectangles under h^2.

From Lemma 3 and Theorem 2 it follows that for every sequence of symbols $a = (a_0, a_1, \ldots, a_{n-1}) \in T_n$ there exists a point x such that

$$h^{2i}(x) \in N_{a_i} \quad \text{for} \quad i = 0, \ldots, n-1 \quad \text{and} \quad h^{2n}(x) = x.$$

In particular for every positive integer n there exists a periodic point of h^2 with period n. Hence for every even integer n there exists a periodic point of the Hénon map with period n. In this way we have also proved that the subshift on two symbols with the transition matrix (5) is embedded in h^2.

4 Symbolic Dynamics for h^7 — Topological Horseshoe

In [Zgliczyński 97b] the author introduced the quadrangles $N_0 = \overline{A_1A_2A_3A_4}$, $N_1 = \overline{A_5A_6A_7A_8}$ shown in Fig. 5 (notice that they are different to the sets defined in the previous section), where $A_1 = (0.460, 0.01)$, $A_2 = (0.595, 0.28)$, $A_3 = (0.691, 0.28)$, $A_4 = (0.556, 0.01)$, $A_5 = (0.588, 0.01)$, $A_6 = (0.723, 0.28)$, $A_7 = (0.755, 0.28)$ and $A_8 = (0.62, 0.01)$. He also defined the set E_0 as a part of the plane lying above the straight line A_1A_4 and on the left hand side of line A_1A_2, $E_1 = \overline{A_4A_3A_6A_5}$ and the set E_2 consisting of points lying below line A_5A_8 or below line A_6A_7 and one the right hand side of line A_7A_8. The set W is defined as before as $W = N_0 \cup N_1 \cup E_0 \cup E_1 \cup E_2$. For these sets using the technique of TS-maps he proved the existence of the topological horseshoe. He proved that the full shift on two symbols with the transition matrix (2) is embedded within the map h^7. Zgliczyński did not use the interval arithmetic. Instead he computed the 7^{th} iteration of the Hénon map at some points and estimated the position of nearby points after seven iterations by means of Lipschitz constant of the Hénon map. The proof required computation of h^7 for approximately 60000 points.

Using the same sets N_i and E_i we have repeated the proof. In order to prove the existence of symbolic dynamics associated with the full shift we have to prove that the images of N_0 and N_1 under h^7 cover horizontally the set $N_0 \cup N_1$.

Lemma 4. *The image of N_0 under h^7 covers horizontally N_0 and N_1, i.e.,*

$$h^7(A_1A_2) \subset E_2 \text{ and } h^7(A_3A_4) \subset E_0,$$
$$h^7(\overline{A_1A_4}), h^7(\overline{A_2A_3}) \subset intW.$$

The image of N_1 under h^7 covers horizontally N_0 and N_1, i.e.,

$$h^7(A_5A_6) \subset E_0 \text{ and } h^7(A_7A_8) \subset E_2,$$
$$h^7(\overline{A_5A_8}), h^7(\overline{A_6A_7}) \subset intW.$$

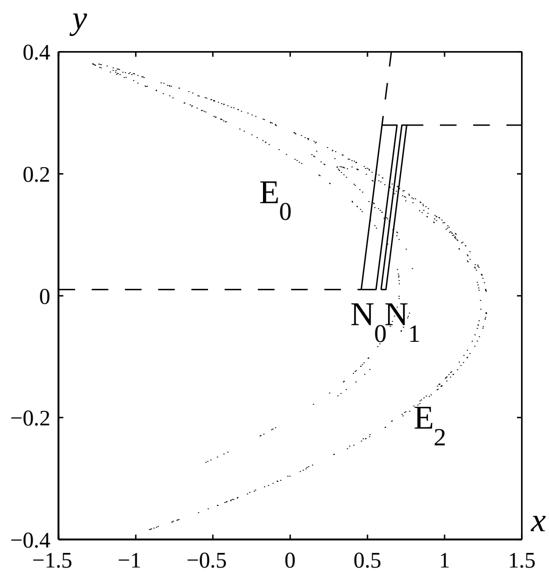

Figure 5: The definition of the sets N_0 and N_1 for the proof of symbolic dynamics for h^7

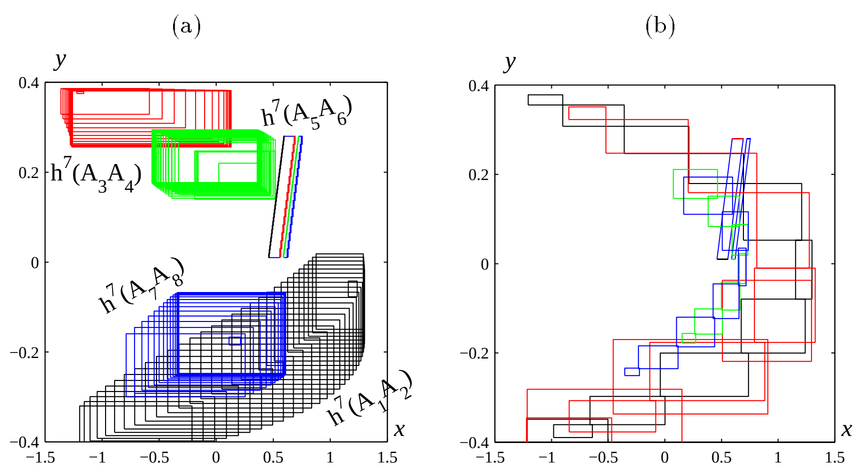

Figure 6: (a) the covering of the vertical edges of N_0 and N_1 with rectangles and their images under h^7, (b) the covering of horizontal edges of N_0 and N_1 and their images under h^7

Proof. The proof was carried out using interval arithmetic. The covering of vertical edges with rectangles and their images under h^7 are shown in Fig. 6a. Similar covering of horizontal edges and its image are shown in Fig. 6b. We have checked that they are enclosed in appropriate sets. For the proof of the existence of topological horseshoe it was sufficient to compute the images of 131 rectangles under h^7.

Notice that the number of rectangles for which the image is computed is significantly reduced when compared to the original proof. Probably Zgliczyński overestimated the error (he did not use the interval arithmetic).

5 Periodic Points with Periods $n \geq 7$, $n \neq 9$

Lemma 4 states that the images of sets N_i under h^7 covers horizontally the sets N_0 and N_1. It follows that for every natural n there exists a periodic point of h with period $7n$. In order to prove the existence of periodic points with other periods we have checked the positions of N_0 and N_1 under h^i, for $i = 1, \ldots, 6$.

Lemma 5.

1. *The set $h^1(N_0)$ covers N_1, i.e.,*

$$h^1(\overline{A_1A_2}) \subset E_2, h^1(\overline{A_3A_4}) \subset E_1 \cup N_0 \cup E_0, \tag{6}$$

$$h^1(\overline{A_2A_3}), h^1(\overline{A_4A_1}) \subset intW. \tag{7}$$

The set $h^2(N_0)$ covers N_0, i.e.,

$$h^2(\overline{A_1A_2}) \subset E_0, h^2(\overline{A_3A_4}) \subset E_1 \cup N_1 \cup E_2, \tag{8}$$

$$h^2(\overline{A_2A_3}), h^2(\overline{A_4A_1}) \subset intW. \tag{9}$$

The set $h^i(N_0)$ for $i = 3, \ldots, 6$ covers both of the sets N_0 and N_1, i.e.,

$$h^3(\overline{A_1A_2}), h^4(\overline{A_3A_4}), h^5(\overline{A_1A_2}), h^6(\overline{A_3A_4}) \subset E_2, \tag{10}$$

$$h^3(\overline{A_3A_4}), h^4(\overline{A_1A_2}), h^5(\overline{A_3A_4}), h^6(\overline{A_1A_2}) \subset E_0, \tag{11}$$

$$h^i(\overline{A_2A_3}), h^i(\overline{A_4A_1}) \subset intW \text{ for } i = 3, \ldots, 6. \tag{12}$$

2. *Images of edges of N_1 under h^i (for $i = 1, \ldots, 6$) have empty intersection with the sets N_0 and N_1.*

$$h^1(L), h^3(L), h^5(L) \subset E_0 \text{ and } h^2(L), h^4(L), h^6(L) \subset E_2,$$

where L is any of the edges $\overline{A_5A_6}$, $\overline{A_6A_7}$, $\overline{A_7A_8}$, $\overline{A_8A_5}$.

Proof. For the proof the edges $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, $\overline{A_4A_1}$, $\overline{A_5A_6}$, $\overline{A_6A_7}$, $\overline{A_7A_8}$ and $\overline{A_8A_5}$ were covered by 19, 11, 42, 11, 35, 7, 16, and 7 rectangles respectively. The images of these rectangles under h^i for $i = 1, \ldots, 6$ were computed. We have checked that the conditions (6)...(12) are fulfilled.

The results proved in lemmas 4 and 5 are summarized in Table 1. Using these results one can easily prove the existence of periodic points for all periods greater or equal to 7 with the exception of period 9.

i	$h^i(N_0)$	$h^i(N_1)$
1	N_1	—
2	N_0	—
3	N_0,N_1	—
4	N_0,N_1	—
5	N_0,N_1	—
6	N_0,N_1	—
7	N_0,N_1	N_0,N_1

Table 1: Images of N_0 and N_1 under h^i ($i = 1, \dots, 7$). In the second and third columns the sets which are covered horizontally by $h^i(N_0)$ and $h^i(N_1)$ are given

Lemma 6. *For every integer $n \geq 7$, $n \neq 9$ there exist periodic point of h with period n.*

Proof. As an example we show how to prove the existence of period-8 orbit. Let us consider the set N_1. As it follows from lemma 4 the image of N_1 under h^7 covers N_0. From lemma 5 it follows that $h(N_0)$ covers N_1. Hence it is clear that $h^8(N_1)$ covers N_1. Using similar argument as for the TS-maps one can prove that there exists a point x within N_1 such that $h^8(x) = x$. Now it is sufficient to prove that 8 is the minimal period of x. But this is clear as $h^i(N_1)$ has empty intersection with N_1 for $i = 1, \dots, 6$.

6 Periodic Points with Periods 1, 3, 5 and 9

So far we have shown that there exist periodic points with all periods but 1, 3, 5 and 9. The existence of a fixed point can be proved analytically. There exist two such points (x_1, bx_1) and (x_2, bx_2) where

$$x_{1,2} = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2a}.$$

One of the fixed points is embedded within the numerically observed strange attractor.

In order to decide the existence of periodic points with periods 3, 5 and 9 within the set $M = [-5, 5] \times [-5, 5]$ we have used the interval Newton method [Alefeld 94, Götz 94]. This method allows to prove the existence and uniqueness of fixed points within specific interval. It also allows to exclude the existence of a fixed point within a given interval. The idea is to divide the set M into small subsets for which assumptions of the interval Newton method can be checked. Using this technique we have proved the following lemma.

Lemma 7. *Let $M = [-5, 5] \times [-5, 5]$.*

1. *There exists no periodic point with period 3 within the set M.*
2. *There exists no periodic point with period 5 within the set M.*
3. *There exist 6 period-9 orbits within M.*

Proof. To prove part 1 we have covered the set M by 493 rectangles. Using the interval Newton method we have proved that there are no period-3 orbits within any of these rectangles. Similarly using 4241 rectangles for the covering of M we have proved that there are no period-5 orbits within M. For the proof of part 3 the set M was covered by 2974053 rectangles. We have proved the existence of exactly 54 periodic points with period 9 within M which correspond to 6 different period-9 orbits.

7 Conclusions

In this paper we have shown rigorously with computer assistance that

A. the subshift on two symbols corresponding to the deformed horseshoe is embedded in h^2,
B. the full shift on two symbols corresponding to the topological horseshoe is embedded in h^7,
C. h has periodic points of all periods but 3 and 5,
D. h has no periodic points with periods 3 and 5 within the set $[-5,5]\times[-5,5]$.

The symbolic dynamics for h^2 and h^7 is proved for invariant sets embedded in the strange attractor observed numerically. Also all the periodic orbits the existence of which is proved (apart from one of the fixed points) lie in the region where the strange attractor is observed. This indicates that the dynamics of the system is very complicated. However the existence of a strange attractor for classical values of parameters still remains an open problem.

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