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Componentwise Distance to Singularity

G. Rex
(Institute of Mathematics, University of Leipzig
rex@mathematik.uni-leipzig.de)

Abstract: A perturbation matrix $\mathcal{A} = A \pm \Delta$ is considered, where $A \in I\!R^{n,n}$ and $0 \leq \Delta \in I\!R^{n,n}$. The matrix $\mathcal{A}$ is singular iff $\mathcal{A}$ contains a real singular matrix. A problem is to decide if $\mathcal{A}$ is singular or nonsingular, a NP-hard problem. The decision can be made by the computation of the componentwise distance to the nearest singular matrix defined on the basis of the real spectral radius, and by the solution of 4^n eigenvalue problems.
Theorem 6 gives a new computation basis, a natural way to the "componentwise distance ..." definition, and a motivation to rename this in *radius of singularity* denoted by $sir(A, \Delta)$.
This new way shows: (i) - sir results from a real nonnegative eigensolution of a nonlinear mapping, (ii) - sir has a norm representation, (iii) - sir can be computed by 2^{n-1} nonnegative eigensolutions of the nonlinear mapping, (iv) - for the special case $\Delta = pq^T, 0 \leq p, q \in I\!R^n$ a formula for a computation of sir is given, also a trivial *algorithm* for the computation, and some examples as demonstration.

Key Words: perturbation matrix, interval matrix, componentwise distance to the nearest singular matrix, radius of singularity, NP-hard

1 Introduction

In this paper a perturbation matrix

$$\mathcal{A} = A \pm \Delta$$

with $A \in I\!R^{n,n}$ and $0 \leq \Delta \in I\!R^{n,n}$ or equivalent to this an interval matrix

$$\mathcal{A} = [A \Leftrightarrow \Delta \, , \, A + \Delta]$$

is considered.
The following two problems are the subject of this paper:

- to decide where is the nearest singular matrix with regard to the matrix A;
- to decide if $\mathcal{A}$ is singular or regular.

A matrix of the type above is called singular iff a real singular matrix is included in $\mathcal{A}$, and a matrix is said to be regular iff it is nonsingular.

It is well-known that the given decision problems can be solved by the computation of $d(A, \Delta)$ denoted as "radius of regularity" [see Poljak , Rohn (93)] or as "componentwise distance to the nearest singular matrix" [see Demmel (92)] and [Rump (97)].

On the other hand it has been shown by [Poljak , Rohn (93)] that the computation of $d(A, \Delta)$ is a NP-hard problem; there are 4^n linear eigenvalue problems to solve (see [Demmel], [Higham], [Chaitin-Chatelin , Frayssé]).

Theorem 6 in [Section 2] gives a new computation basis and "a natural way" to the definition of $d(A, \Delta)$, and a motivation to rename this in *radius of singularity* denoted by $sir(A, \Delta)$, also to make a distinction between both computation formulae.

In [Section 4] it is shown that $d(A, \Delta) = sir(A, \Delta)$, but for a computation of sir are "only" 2^{n-1} nonnegative eigensolutions to compute.

In the following it should be explained what is to be understood by " a natural way": At first in [Section 1.1] it is shown by using of interval mathematics that the representation of $\mathcal{A}$ as perturbation matrix is equivalent to the midpoint-radius representation of an interval matrix. *Theorem 3* gives equivalent formulations for the singularity of an interval matrix. The assumption that the midpoint matrix A of $\mathcal{A}$ is regular allowed the study of a nonlinear mapping $F_S : I\!R^n_+ \to I\!R^n_+$ with $F_S(z) := \Delta|A^{-1}Sz|$ where $I\!R^n_+ := \{x \in I\!R^n \mid 0 \leq x_i \,, 1 \leq i \leq n\}$ denotes *the cone of the* $I\!R^n$, and S *a signature matrix* [see Section 1.1.1]. Note there are 2^{n-1} signature matrices, and this set is denoted by O^n.

The basis statement of this paper is (see *Theorem 6*):

$$\exists(\lambda; z) \in I\!R_+ \times I\!R^n_+ \ (\ F_S(z) = \lambda\, z \,, \ ||z|| = 1 \ ) \ .$$

The proof based on *Brouwers fixed point theorem* given in [Section 1.1.2]. Note $||\cdot||$ stands for any vector norm.

On the set of the nonnegative eigensolutions is

$$\Lambda := \max_{S \in O^n} \max\{ \ \lambda(S) = ||\Delta|A^{-1}Sz||| \ \mid \ F_S(z) = \lambda(S)\, z \,, \ ||z|| = 1 \}$$

defined, see (17).

Let now, for a S, z and Λ the equation $F_S(z) = \Lambda\ z$ be fulfiled, then it can be shown that this is equivalent to $\mathcal{A}(\Lambda) := A \pm (1/\Lambda)\Delta$ is singular if $\Lambda \neq 0$ [see Section 3]. Furthermore in [Section 3] $\mathcal{A}(t) := A \pm t\,\Delta$ is discussed dependent of Λ. This way is new and leads also to the well-known definition $\min\{ \, t \geq 0 \mid \mathcal{A}(t) \ singular\} =: sir(A, \Delta)$, here denoted by $sir(A, \Delta)$ to make a distinction to $d(A, \Delta)$ and between both ways. That the ways are really different is shown in [Section 3] and [Section 4].

The term *radius of singularity* for $sir(A, \Delta)$ is motivated by the following: for $t = sir(A, \Delta)$ is $\mathcal{A}(t)$ the closure of $\mathcal{A}(t)$ regular for $t \in [0, sir(A, \Delta))$; a singular matrix $s \in \partial\mathcal{A}$ exists, where $\partial\mathcal{A}$ is the set of boundary matrices of $\mathcal{A}$, and last but not least $sir(A, \Delta)$ has with Λ a $||\cdot||$ - representation.

The special case $\Delta = pq^T$ is in [Section 5] considered. The application of *theorem 6* gives here a new representation formula for sir and leads to an *algorithm* for the computation; examples are given. Especially it is shown that p is an eigenvector of the mapping F_S and

$$sir(A, pq^T) = \frac{1}{max_{S \in O^{n-1}} \{q^T|A^{-1}Sp|\}} \ .$$

This formula contains also the special case where all elements $\Delta_{i,j} = 1$ considered by [Rohn (96)] , [Demmel (92)] , [Rump (97)]. A representation of sir in a subordinate matrix norm is also given by (32).

1.1 Definitions, notations, and preliminaries

In this first section brief survey connections between interval and perturbated matrices are given, since the interval analysis gives an easy access to the subject of this paper.

The symbol $I\!R := \{ X \mid X = [\underline{X}, \overline{X}] \; ; \; \underline{X}, \overline{X} \in I\!R \}$ denotes the set of all closed real intervals. The following equivalent notations for an $X \in I\!I\!R$ are used

$$\begin{aligned} X &= [\underline{X}, \overline{X}] \\ &= [mid(X) \Leftrightarrow rad(X), mid(X) + rad(X)] \\ &= \; mid(X) \pm rad(X) \end{aligned} \tag{1}$$

with the definitions: $mid(X) := (\underline{X} + \overline{X})/2$ for the *midpoint of* X and $rad(X) := (\overline{X} \Leftrightarrow \underline{X})/2$ for the *radius of* X. It is obvious to see that $rad(X)$ is always nonnegativ. The representation (1) is called the *midpoint-radius representation of an interval* X. In this context $rad(X)$ can be interpreted as a perturbation of $mid(X)$.

For the following some rules for intervals are used.

Lemma 1. *Let $x \in I\!R$ and $Y \in I\!I\!R$, then*

$$Yx = mid(Y)\, x \;\pm\; rad(Y)\, |x| \tag{2}$$

$$0 \in Y \Leftrightarrow |mid(Y)| \;\leq\; rad(Y) \tag{3}$$

Proof. To (2): The application of the multiplication for intervals [see Moore (79)], and (1) gives

$$\begin{aligned} Yx = [\, &\min\{(mid(Y) \Leftrightarrow rad(Y))\, x, (mid(Y) + rad(Y))\, x\}, \\ &\max\{(mid(Y) \Leftrightarrow rad(Y))\, x, (mid(Y) + rad(Y))\, x\}\,] \\ = \; & mid(Y)\, x \;\pm\; rad(Y)\, |x|. \end{aligned}$$

To (3):

$$0 \in Y \Leftrightarrow mid(Y) \Leftrightarrow rad(Y) \leq 0 \leq mid(Y) + rad(Y) \Leftrightarrow |mid(Y)| \leq rad(Y).$$

An extension of intervals to interval matrices, interpreted as a perturbation matrix, is useful in this context. A n-by-n interval matrix $(\mathcal{A}_{i,j})$ can be generated by a componentwise perturbation $(\Delta_{i,j})$ of a real matrix $(A_{i,j})$. The componentwise representation is given by using (1) with

$$\mathcal{A}_{i,j} = A_{i,j} \pm \Delta_{i,j}, \;\; 0 \;\leq\; \Delta_{i,j}$$

where $A_{i,j}$ is the midpoint, and $\Delta_{i,j}$ is the radius of $\mathcal{A}_{i,j}$ or

$$\mathcal{A} = A \pm \Delta\,, \;\; A \in I\!R^{n,n}\,, \;\; 0 \;\leq\; \Delta \in I\!R^{n,n} \tag{4}$$

a well-known notation for a perturbation matrix. This shows that because of (1) the representation (4) is equivalent to the following interval matrix

$$\mathcal{A} = [\, A \Leftrightarrow \Delta\,, \; A \;+\; \Delta\,]. \tag{5}$$

For the following the absolute value $|\cdot|$, and the relations $\leq$, $\in$, $\subseteq$, $\supset$ are to be used element- or componentwise, respectively.

As next a definition and some equivalent formulations for the singularity of an interval matrix (5) are given.

Definition 2.

$$\mathcal{A} \text{ singular} \iff \exists s \in I\!R^{n,n}\, \exists x \in I\!R^n \; (\; s \in \mathcal{A},\; s\,x = 0,\; x \neq 0 \;) \tag{6}$$

Theorem 3.

$$\mathcal{A} \textit{ singular} \iff \exists x \in I\!R^n \; (\; 0 \in \mathcal{A}\,x \,,\; x \neq 0 \;) \tag{7}$$
$$\iff \exists x \in I\!R^n \; (\; |A\,x| \;\leq\; \Delta\,|x| \,,\; x \neq 0 \;) \tag{8}$$

Proof. The elementewise application of (2) to $\mathcal{A}\,x$ with an $x \in I\!R^n$ and $\mathcal{A}$ from (5) gives

$$\mathcal{A}\,x = \{\, a\,x \mid a \in \mathcal{A} \,\} \tag{9}$$
$$= A\,x \;\pm\; \Delta\,|x|. \tag{10}$$

To (6) $\Leftrightarrow$ (7): With (9) it is obvious that (6) $\Leftrightarrow$ (7), then

$$\exists s \in I\!R^{n,n}\, \exists x \in I\!R^n \; (\; s \in \mathcal{A},\; s\,x = 0,\; x \neq 0 \;) \iff (\, 0 \in \mathcal{A}\,x \,).$$

To (7) $\Leftrightarrow$ (8): Consider $0 \in \mathcal{A}\,x$ from (7), use (10), and apply (3) with $A\,x = mid(\mathcal{A}\,x)$, $\Delta\,|x| = rad(\mathcal{A}\,x)$ then

$$0 \in \mathcal{A}\,x = A\,x \pm \Delta|x| \iff |A\,x| \leq \Delta|x|$$

proves the equivalence.

For the examinations in this paper it is important to note that the equivalence relation (8) can be specified with the midpoint matrix $A = mid(\mathcal{A})$; there are two cases in

Theorem 4.

$$A \textit{ singular} \;\Leftrightarrow\; \exists\, x \in I\!R^n \,(\, |A\,x| \;=\; 0 \;\leq\; \Delta\,|x|,\; x \neq 0\,) \tag{11}$$
$$(\mathcal{A} \textit{ singular},\, A \textit{ regular}) \;\Leftrightarrow\; \exists\, y \in I\!R^n \,(\, |y| \;\leq\; \Delta|A^{-1}y|,\; y \neq 0\,) \tag{12}$$

Proof. To (11): This is obvious with (8), and $A = mid(\mathcal{A})$ is singular. To (12): This is equivalent to (8) because of the regularity of A and $y := Ax \;\Leftrightarrow\; x = A^{-1}y$.

1.1.1 Signature matrices

Let $x \in I\!R^n$, then $x = S|x|$ with

$$S := \operatorname{diag}(\operatorname{sign}(x_1), \ldots, \operatorname{sign}(x_n))$$

where S is called *the signature matrix of the vector x.*
The following properties are obvious with the definition of S:

$$S = S^T = S^{-1}\,;\; S^2 = I\,;\; |S| = I\,,\; I \text{ denotes the n-by-n identity matrix.}$$

The set of all these signature matrices is denoted with O^n . Note the cardinality of O^n is 2^n.

1.1.2 Brouwer's fixed point theorem

In this context we use the following version of *Brouwers fixed point theorem*

Theorem 5. *Let ∂B be the sphere of a closed unit ball $B := \{ x \in I\!R^n \mid ||x|| \leq 1 \}$ of $I\!R^n$, and let $T : B \to I\!R^n$ be a continuous mapping on B with $T(\partial B) \subseteq B$, then*

$$\exists x \in B \ (\ Tx = x \ ).$$

Proof. See [Riedrich (76)] .

2 Basic theorem

On the basis of (12) a nonlinear mapping is defined and studied in

Theorem 6. *Let $S \in O^n$ and $F_S : I\!R^n_+ \to I\!R^n_+$ with $F_S(z) := \Delta|A^{-1}Sz|$, then*

$$\exists(\lambda; z) \in I\!R_+ \times I\!R^n_+ \ (\ F_S(z) = \lambda\, z \,, \ ||z|| = 1 \ ) \tag{13}$$

and

$$\lambda = ||\, \Delta \,|\, A^{-1}\, S\, z \,|\, || \tag{14}$$

Such a pair $(\lambda; z)$ is called a nonnegative eigensolution of F_S.

Proof. For the nonlinear mapping F_S it is useful to define *the kernel of* F_S by

$$ker(F_S) := \{ \, z \in I\!R^n_+ \ \mid \ F_S(z) \ = \ 0 \,, \ z \neq 0 \, \}.$$

The proof of this theorem is divided in two parts:
Part 1: Assume $0 < \Delta \in I\!R^{n,n}$, then it is obvious to see that $ker(F_S) = \emptyset$ (where $\emptyset$ denotes the *empty set*). The intersection of a closed unit ball with $I\!R^n_+$ is defined by

$$B_+ := \{ \, z \in I\!R^n_+ \ \mid \ ||\, z \,|| \ \leq \ 1 \, \}.$$

Now, a nonlinear functional $f_S : \ B_+ \ \to \ I\!R^n_+$ can be defined by

$$f_S \ := \ ||\, \Delta|A^{-1}\, S\, z \,|\, ||$$

and since $ker(F_S) = \emptyset$ f_S has the property

$$0 \ < \ f_S(\, z \,) \quad \forall z \in B_+.$$

This property allows the definition of a nonlinear mapping $G_S \ : \ B_+ \ \to \ I\!R^n_+$ with

$$G_S(z) := \frac{1}{f_S(z)} \ F_S(z).$$

Properties of G_S are

$$\text{(i):}\quad G_S \in C^0(B_+) \quad \text{and} \quad \text{(ii):}\quad G_S(\partial B_+) \subseteq \partial B_+ \subseteq B_+$$

where C^0 denotes the set of all continuous functions. (ii) is clearly by construction.

Put now $B := B_+$ and $T := G_S$, and apply *theorem 5* then

$$\exists z \in B_+ \,(\, G_S(z) = z \,)$$

$(G_S$ has a fixed point) and furthermore for the mapping F_S is

$$F_S(z) = \lambda\, z$$

with

$$\lambda := \|\, |\Delta|\, A^{-1}\, S\, z\,| \,\|\,,\ \|\, z\,\| = 1$$

satisfied.

This means the nonlinear mapping F_S has at least a real nonnegative eigensolution $(\,\lambda\,;\,z\,)$.

Part 2 (general case): Let $0 \le \Delta$. Then define with an arbitrary $0 < C \in I\!R^{n,n}$

$$\Delta(\,t\,) := \Delta + t\,C\,,\ t \in I\!R_+\ .$$

Since $0 < \Delta(\,t\,)$ for $t > 0$ part 1 of this proof can be applied, and

$$(\,\lambda(\,t\,)\,;\,z(\,t\,)\,) \in I\!R_+ \times I\!R_+^n \quad \text{exists with} \quad \|\,z(\,t\,)\,\| = 1$$

for

$$F_S(\,z(\,t\,)\,) = \lambda(\,t\,)\,z(\,t\,)$$

with

$$\lambda(\,t\,) := \|\, |\,\Delta(\,t\,)\,|\,A^{-1}\,S\,z(\,t\,)\,| \,\|\ .$$

Since ∂B_+ is compact, there is an accumulation point $z \in \partial B_+$ for each zero sequence $\{\,t_k\,\} \to +0$. Because of the convergence of the sequences

$$\{F_S(\,z(\{\,t_k\,\})\,\} \to F_S(\,z\,) \quad \text{and} \quad \{\,\lambda(\,\{\,t_k\,\}\,)\,\} \to \lambda$$

for each $\{\,t_k\,\} \to +0$, any such accumulation point satisfied $F_S(\,z\,) = \lambda\,z$ with $\lambda := \|\, |\,\Delta\,|\,A^{-1}\,S\,z\,| \,\|$ and $\|\,z\,\| = 1$.

The statement of this theorem is, that for F_S exists at least a real nonnegative eigensolution. This statement is very important for the following definition of the radius of singularity.

3 Radius of singularity - $sir(A, \Delta)$

Theorem 6 gives the basis for the definition of the radius of singularity.
Consider

$$F_S(z) \;=\; \lambda\, z \;=\; \Delta\, |\, A^{-1}\, S\, z\, | \tag{15}$$

and (15) can be transformed with $x \,:=\, A^{-1}\, S\, z$ and $\lambda \neq 0$ into

$$|\, A\, x\, | \;=\; \frac{1}{\lambda}\, \Delta\, |\, x\, |\;; \tag{16}$$

on the other hand use (8):

$$A \text{ singular } \iff \exists x \in I\!R^n\, (\, |A\, x| \;\le\; \Delta\, |x|\,,\;\; x \neq 0\,)\;.$$

Equation (16) shows that $|\, A\, x\, | \le \Delta\, |\, x\, |$ is sharp for the "smallest pair" $(1/\lambda; x)$. That leads with (16), and (14) under the consideration that $\lambda = \lambda(S)$ to the following definition

$$\Lambda := \max_{S \in O^n} \max\{\, \lambda(S) = ||\Delta|A^{-1} S z|||\;\; |\;\; F_S(z) = \lambda(S)\, z\,,\; ||z|| = 1\,\} \tag{17}$$

Remark 1: This definition depends on the eigensolutions of F_S. Note that also z depends on S, $z = z(S)$.

Put Λ in (16) then is because of (8) the perturbation matrix $\mathcal{A}(\frac{1}{\Lambda})$ singular. That is for

$$\mathcal{A}(t) \;:=\; A \;\pm\; t\,\Delta \begin{cases} \text{regular if } \; t \in [\,0\,,\, \frac{1}{\Lambda}) \\ \text{singular if } t \in [\,\frac{1}{\Lambda}\,,\, \infty] \end{cases} \tag{18}$$

where $\mathcal{A}(t)$ be closed by $\mathcal{A}(\infty)$ this will be convenient in this context, see also below.

Furthermore there are two special cases to be considered:
(a):

$$A \text{ singular } \iff \exists x \in I\!R^n\, (\, |A\, x| \;= 0 \;=\; \frac{1}{\Lambda} \Delta\, |x|,\;\; x \neq 0\,)$$

That is for $\mathcal{A}(t)$: set $\frac{1}{\Lambda} = 0$ then $\mathcal{A}(t)$ is singular $\forall t \in [0, \infty]$.

(b): Define

$$ker(F) := \cup_{S \in O^n} \{ker(F_S)\} = \{x \in I\!R^n (\Delta |A^{-1}\, x| = 0\,,\;\; x \neq 0\}.$$

Let $ker(F) \neq \emptyset$ and

$$\nexists(\lambda, x) \in I\!R_+ \times I\!R^n \backslash ker(F)\;\; (\,\lambda |x| = \Delta\, |\, A^{-1}\, x\, |\,,\;\; x \neq 0\,) \tag{19}$$

then

$$\lambda |x| = 0 = \Delta |A^{-1} x| \quad \forall x \in ker(F) \tag{20}$$

is solvable only for $\lambda = 0$, that is $\Lambda = 0$.

It can be seen immediately that (19) and (20) are also true if Δ is replaced by $t\Delta$. That means

$$\forall t \in [0,\infty)(\nexists x \in I\!R^n(|x| \le t\Delta|A^{-1}x|\,,\; x \ne 0) \Longleftrightarrow \mathcal{A}(t)\; regular\,)\,. \quad (21)$$

The equivalence (21) follows from the negation of (12).

The statement of (b) for $\mathcal{A}(t)$ is: $\mathcal{A}(t)$ is singular for $t = \infty \quad (\Lambda = 0)$.

In consideration of these three cases (18), (a), and (b) it is evident to define

$$\min\{\, t \in [0,\infty] \mid \mathcal{A}(t)\, singular\,\} =: sir(A,\Delta) \quad (22)$$

here denoted as $\underline{r}adius\ of\ \underline{si}ngularity\ of\ a\ matrix$ $\mathcal{A}$ with

$$sir(A,\Delta) = \begin{cases} 0 \text{ if} & \text{A is singular } (see\,(a)) \\ \infty \text{ if} & \Lambda = 0 \qquad\quad (see\,(b)) \\ \frac{1}{\Lambda} \text{ otherwise} & \qquad\qquad\;\; (see\,(18)) \end{cases} \quad (23)$$

For the following is sir as abbreviation of $sir(A,\Delta)$ to be understood.

The way to sir has shown:

- sir results from a real nonnegative eigensolution of F_S;
- sir has with (17) a normrepresentation for any vectornorm on $I\!R^n$;
- because of $|A^{-1}Sz| = |A^{-1}(-S)z| \quad \forall S \in O^n$ are "only" 2^{n-1} eigensolutions of F_S to compute;
- a singular matrix $s \in \partial\mathcal{A}(sir)$ exists where $\partial\mathcal{A}(sir)$ denotes the boundary set of $\mathcal{A}(sir)$ (see section 5 examples) and further

  $$\mathcal{A}(sir) = int\{\, \mathcal{A}(sir)\,\} \cup \partial\mathcal{A}(sir)$$

  with $\quad int\{\, \mathcal{A}(sir)\,\} = \{\, a \in I\!R^{n,n} \;\mid\; a \in \mathcal{A}(t)\;\, t \in [0, sir)\}$,

Remark 2: From (b) follows

$ker(F) \ne \emptyset$ is a necessary condition for $\Lambda = 0$ or $sir = \infty$, respectively.

An example is given in the next section.
For a better understanding of sir some examples are given in [Section 5].

3.1 A nontrivial example for $sir = \infty$

Let

$$\mathcal{A} := D \pm \Delta \tag{24}$$

with D is a regular real diagonal matrix, and

$$\Delta := \begin{pmatrix} 0 & * & \cdot & \cdot & \cdot & * \\ 0 & 0 & * & \cdot & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & 0 & * & * \\ 0 & \cdot & \cdot & \cdot & 0 & * \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

where $*$ stands for arbitrary nonnegative real elements.

With (24) is $F_S(z) = \Delta \,|\, D^{-1}\, S\, z \,|$ and the application of the *Perron/Frobenius theorem*, see e.g. [Riedrich (76)], gives

$$F_S(z) = \Delta \,|\, D \,|^{-1}\, z = \rho(\, \Delta \,|\, D \,|^{-1}\,)\, z = 0\, z \quad \forall S \in O^{n-1}$$

where $\rho(\cdot)$ denotes the spectral radius. It is obviously to see that $\Lambda = 0$, and therefore is $sir(A, \Delta) = \infty$.

Remark 3: It is easy to see that $ker(F) \neq \emptyset$ then $e^1 := (1, 0, \ldots, 0)^T \in ker(F)$.

4 $sir(A, \Delta)$ is equivalent to $d(A, \Delta)$

J.Rohn defined in [Rohn (89)], and [Poljak , Rohn (93)] the *radius of regularity* by

$$d(A, \Delta) := inf\{\, t \geq 0 \mid [\, A \Leftrightarrow t\, \Delta\,,\; A + t\, \Delta\,]\; singular\,\} \,. \tag{25}$$

With (22) is shown that the inf is achieved.

The computation formula given by J. Rohn [see Rohn (89)] for $d(A, \Delta)$ is equivalent to $sir(A, \Delta)$.

Then an equivalent transformation of

$$F_S(z) \;=\; \Delta \,|\, A^{-1}\, S\, z \,| \;=\; \lambda\, z \tag{26}$$

with

$$x \;=\; A^{-1}\, S\, z$$

gives

$$\lambda \,|\, A\, x \,| \;=\; \Delta \,|\, x \,| \;,$$

and with

$$T_1\, A\, x \;=\; |\, A\, x \,|\,,\; T_2\, x \;=\; |\, x \,|\,,\; T_1\,,\, T_2 \in O^n$$

is (26) transformed into

$$A^{-1}\, T_1\, \Delta\, T_2\, x \;=\; \lambda\, x \;, \tag{27}$$

a real eigenvalue problem.

The definition of the *real spectral radius* for a $B \in I\!R^{n,n}$

$$\rho_0(B) := \begin{cases} \max\{\,|\,\lambda\,|\,\} & \text{if} \qquad \text{B has real eigenvalues} \\ \quad 0 & \text{otherwise} \end{cases}$$

applied to (27) gives

$$\max_{T_1,T_2 \in O^n} \{\, \rho_0(\, A^{-1}\, T_1\, \Delta\, T_2\,)\,\} \;=\; \Lambda \tag{28}$$

The left hand side of (28) was given by J. Rohn in [Rohn (89)], and is because of (27) equivalent to Λ from (17). Therefore is $d(A,\Delta) \;=\; sir(A,\Delta)$ as given in (23).

For a computation of Λ are 2^{n-1} eigensolutions of F_S to compute instead of 4^n linear eigenvalue problems for $\rho_0(\, A^{-1}\, T_1\, \Delta\, T_2\,)$.

The computation of *sir* is also a NP-hard problem since in [Poljak and Rohn (93)] was shown that the computation of $d(A,\Delta)$ is a NP-hard problem.

In the following section applications of *theorem 6* are given.

5 A dyad as a special perturbation matrix

Let

$$\Delta := p\, q^T \;,\;\; p,\, q \in I\!R^n \;,$$

Δ is called *dyad* and A be the regular midpoint matrix of the perturbation matrix $\mathcal{A} = A \pm \Delta$.

Then on the basis of the proof of *theorem 6* for a $S \in O^n$

$$G_s(z) = \frac{p\, q^T\, |\, A^{-1}\, S\, z\,|}{||\, p\, q^T\, |\, A^{-1}\, S\, z\,|\,||} \;=\; \frac{1}{||\, p\,||}\; p \;.$$

This means

$$z = \frac{1}{||\, p\,||}\; p\,,\;\; ||\, z\,|| = 1 \tag{29}$$

is a fixed point of G_S. On the other hand is (29) also an eigenvector for

$$F_S(\frac{1}{||\, p\,||}\; p) = q^T\, |\, A^{-1}\, S\, p\,|\, \frac{1}{||\, p\,||}\; p = \lambda\, \frac{1}{||\, p\,||}\; p$$

with the eigenvalue

$$\lambda = \lambda(S) = q^T \,|\, A^{-1}\, S\, p \,|\ .$$

Finally there is the following representation for sir

$$sir(A, pq^T) = \frac{1}{\max_{S \in O^{n-1}} \{ q^T \,|\, A^{-1}\, S\, p \,| \}}\ . \tag{30}$$

In the following there are some other representations for (30) given:

Define $e := (1, \ldots, 1)^T \in IR^n$, the diagonal matrices

$$D_p := diag(p)\,,\ \ D_q := diag(q) \quad \text{then} \quad p = D_p\, e\,,\ Q = D_q\, e\,,$$

and

$$H := D_q\, A^{-1}\, D_p\ .$$

With these definitions and $||\, x\,||_1 := e^T \,|\, x\,|$ is

$$sir(A, pq^T) = \frac{1}{\max_{S \in O^{n-1}} ||\, H\, S\, e\,||_1} \tag{31}$$

equivalent to (30).

sir can be represented also by a *subordinate norm* [see Golub , Van Loan (89)] on a finite set, especially the corners of the unit cube on IR^n, then

$$sir(A, pq^T) = \frac{1}{||\, H\,||_{\infty,1}} \tag{32}$$

where

$$||\, H\,||_{\infty,1} = \max_{S \in O^{n-1}} \{ ||\, H\, S\, e\,||_1 \} = \max_{||Se||_\infty = 1} ||\, H\, S\, e\,||_1\ .$$

Because $S \in O^{n-1}$ is the cardinality 2^{n-1}, furthermore it was shown in [Rohn (96)] that the computing of (30) is NP-hard.

It is worthy to stress, that the formulae (30) – (32) are suitable for a computation of a parallel computer [Rex].

The following examples are given for a demonstration of the statements of this paper. The basis for the computation gives the following algorithm.

5.1 ALGORITHM 1

given:	A, p, q	
set:	$\Lambda := 0$	
	$k := 1$	
repeat:		
	$S \in O^{n-1}$	formal notation
	$Aw = Sp$	solve a linear system of equations (exactly)
	$\lambda := q^T \lvert w \rvert$	
Test:	If $\lambda > \Lambda$ then $\Lambda := \lambda$	
	$k := k + 1$	
until	$k > 2^{n-1}$	

5.1.1 Example 1

Let $\mathcal{A} = A \pm p\,q^T$ with

$$A := \begin{pmatrix} 1 & \Leftrightarrow 1 \\ 1 & 1 \end{pmatrix}, \qquad p := \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad q := \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

then $\quad sir := sir(A, pq^T) = 2/15 \quad$ computed with *Algorithm 1.*

The matrix

$$\mathcal{A}(\mu) := A \pm \mu\, sir\, p\, q^T \quad \text{is regular for} \quad 0 \leq \mu < 1\,. \tag{33}$$

$$\mathcal{A}(\mu) = \begin{pmatrix} 1 & \Leftrightarrow 1 \\ 1 & 1 \end{pmatrix} \pm \mu \frac{2}{15} \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix}.$$

Furthermore

$$\mathcal{A}(1) = \frac{1}{15} \begin{pmatrix} [\,9\,,\,21] & [-23\,,\,-7] \\ [\,3\,,\,27] & [\,-1\,,\,31] \end{pmatrix} \subseteq \mathcal{A} := \begin{pmatrix} [\,-2\,,\,4] & [\,-5\,,\,3] \\ [\,-5\,,\,7] & [\,-7\,,\,9] \end{pmatrix}.$$

There is only one singular matrix s on $\partial\mathcal{A}$

$$s := \frac{1}{15} \begin{pmatrix} 21 & \Leftrightarrow 7 \\ 3 & \Leftrightarrow 1 \end{pmatrix} \in \partial\mathcal{A} := \frac{1}{15} \begin{pmatrix} \{\,9\,,\,21\} & \{-23\,,\,-7\} \\ \{\,3\,,\,27\} & \{\,-1\,,\,31\} \end{pmatrix}.$$

5.1.2 Example 2

Let $\mathcal{A} = A \pm p\,q^T$ with

$$A := \begin{pmatrix} 1 & 0 & -4 \\ 0 & 2 & 6 \\ 1 & -2 & 0 \end{pmatrix}, \qquad p := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad q := \begin{pmatrix} 0.5 \\ 0.25 \\ 2 \end{pmatrix},$$

then $\quad sir := sir(A, pq^T) = 40/38 \quad$ computed with *Algorithm 1.*

And analogous to (33) is

$$\mathcal{A}(\mu) = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 2 & 6 \\ 1 & -2 & 0 \end{pmatrix} \pm \mu \frac{40}{38} \begin{pmatrix} 0.5 & 0.25 & 2 \\ 0.5 & 0.25 & 2 \\ 0.5 & 0.25 & 2 \end{pmatrix}.$$

Furthermore

$$\mathcal{A}(1) = \frac{2}{38} \begin{pmatrix} [\ 9,29] & [\ -5\ ,\ \ 5] & [-116,\ 36] \\ [-10,10] & [\ 33\ ,\ 43] & [\ 74,154] \\ [\ 9,29] & [-43\ ,-33] & [\ -40,\ 40] \end{pmatrix} \supset \mathcal{A}$$

$$\mathcal{A} = \frac{2}{38} \begin{pmatrix} [\ 9.5\ ,\ 28.5] & [\ -4.75\ ,\ \ 4.75] & [-114\ ,\ \ 38] \\ [-9.5\ ,\ \ 9.5] & [\ 33.25\ ,\ 42.75] & [\ \ 76\ ,\ 152] \\ [-9.5\ ,\ 28.5] & [-42.75\ ,-33.25] & [\ -38\ ,\ \ 38] \end{pmatrix}.$$

There is a singular matrix s on $\partial\mathcal{A}$

$$s := \frac{2}{38} \begin{pmatrix} 9 & -5 & -116 \\ -10 & 33 & 74 \\ 9 & -43 & -40 \end{pmatrix} \in \partial\mathcal{A} := \frac{2}{38} \begin{pmatrix} \{\ 9\ ,\ 28\} & \{\ -5\ ,\ \ 5\} & \{-116\ ,\ \ 36\} \\ \{-10\ ,\ 10\} & \{\ 33\ ,\ 43\} & \{\ \ 74\ ,\ 154\} \\ \{\ 9\ ,\ 29\} & \{-43\ ,-33\} & \{\ -40\ ,\ \ 40\} \end{pmatrix}.$$

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