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Too Many Minor Order Obstructions (For Parameterized Lower Ideals)[1]

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Abstract: We study the growth rate on the number obstructions (forbidden minors) for families of graphs that are based on parameterized graph problems. Our main result shows that if the defining graph problem is $\mathcal{NP}$-complete then the growth rate on the number of obstructions must be super-polynomial or else the polynomial-time hierarchy must collapse to Σ_3^P. We illustrate the rapid growth rate of parameterized lower ideals by computing (and counting) the obstructions for the graph families with independence plus size at most k, $k \leq 12$.

Key Words: graph minors, obstruction sets, polynomial hierarchy

Category: F.4, F.m, G.2

1 Introduction

Several results in structural graph theory concern the characterization of graph families by obstruction sets, such as Kuratowski's famous characterization of planar graphs by the two forbidden graphs $K_{3,3}$ and K_5. Kuratowski's result indicates the form of all obstruction set characterizations of graph families; for some fixed graph family, denoted by $\mathcal{F}$, a graph G is a member of $\mathcal{F}$ if and only if G does not contain (as a substructure) any member of some set of graphs $\mathcal{O}(\mathcal{F}) = \{O_1, O_2, \ldots\}$.

We are interested in "substructures" based on the *minor order*. Specifically, a graph H is a *minor* of a graph G, denoted $H \leq_m G$, if H can be obtained by contracting some (possibly zero) edges in a subgraph of G. The celebrated **Graph Minor Theorem (GMT)**, formerly known as Wagner's Conjecture, by Robertson and Seymour states that any family of graphs $\mathcal{F}$ that is *closed* under minors (i.e., if $H \leq_m G$ and $G \in \mathcal{F}$ then $H \in \mathcal{F}$) has an obstruction set $\mathcal{O}(\mathcal{F})$ of **finite** cardinality [RS85]. More generally, the term *lower ideal* is used to indicate that the family of graphs $\mathcal{F}$ is closed relative to a partial order.

An algorithmic consequence of the GMT is that many graph families (e.g. all the minor-order lower ideals) are easily shown to have polynomial-time membership algorithms. These membership algorithms are based on a known algorithm, which runs in time $O(n^3)$, that can check if a fixed graph is a minor of another graph [RS85]. Hence, in theory, to check for membership in an applicable graph family $\mathcal{F}$, one just runs this polynomial time minor checking algorithm once for each obstruction in $\mathcal{O}(\mathcal{F})$. Here an input graph G is a member of $\mathcal{F}$ if and only if G has none of these obstructions as a minor. Thus, once the "promised" finite

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set of obstructions $\mathcal{O}(\mathcal{F})$ is found for the lower ideal $\mathcal{F}$, a constructive membership algorithm exists. However, it is still open whether these new algorithms can be made practical. This is because there are astronomically large hidden constants in this cubic-time minor-containment result [FRS87]. Also the number of obstructions may be prohibitively large, as indicated in this paper.

Many graph families are related by some type of integer parameter. For example, a natural extension of Kuratowski's Theorem is to characterize orientable surfaces with fixed genus [Tru93]. (There is also a non-orientable analog to Kuratowski's characterization; for example, Glover et. al. [GHW79] use 35 obstructions to characterize the real projective plane, which is topologically viewed as the Möbius band + lid.) It is interesting to note that graphs embeddable on the surface of the torus (genus 1), the simplist orientable surface after the plane (sphere = genus 0), has not yet been characterized by obstructions. The toroidal obstruction set, currently a grand prize in the field, has a projected size of over 3000 graphs (e.g. see [Neu93]). Current folklore says that it is unfeasible to characterize by forbidden minors any orientable surface with genus 2 or more. This illustrates the rapid growth in the number of obstructions for these types of lower ideals, even for small parameter values.

We want to study the growth rate of various graph families related by an integer parameter, where the family indexed by an integer k is contained in the family indexed by $k+1$. Also, many minor-order lower ideals are defined by considering yes-instances to an integer parameterized graph problem. For example, the planar and toroidal graph families are surface-embeddable families, where graphs embeddable on the plane (sphere) are clearly embeddable on the torus. Formally, a *k-parameterized lower ideal* $\mathcal{F}$ is defined by some graph invariant function λ that maps graphs to integers such that whenever $H \leq_m G$ we also have $\lambda(H) \leq \lambda(G)$. In this general framework, a graph family $\mathcal{F}$ equals $\{G \mid \lambda(G) \leq k\}$ for some integer constant k. Several studied graph families such as k–Vertex Cover (graphs with a vertex cover of cardinality at most k) and k–Feedback Vertex Set are typical examples, for which a few of these families have recently been characterized by obstructions [CD94, CDF95].

Given a minor-order lower ideal $\mathcal{F}$, often (but not always) another class $\{k\text{–}\mathcal{F} \mid k > 0\}$ of parameterized and finitely-characterizable graph families is obtained by defining lower ideals that are within k vertices (or edges) of $\mathcal{F}$. That is, for a fixed family $\mathcal{F}$, the following parameterized families are lower ideals (e.g., see [FL88] for proof):

$$k\text{–}\mathcal{F}_v = \{G = (V, E) \mid G \setminus V' \in \mathcal{F} \text{ where } V' \subseteq V \text{ and } |V'| \leq k\} \quad .$$

And the following families are sometimes lower ideals:

$$k\text{–}\mathcal{F}_e = \{G = (V, E) \mid G \setminus E' \in \mathcal{F} \text{ where } E' \subseteq E \text{ and } |E'| \leq k\} \quad .$$

2 Growth Rate of Obstructions

There is a tendency for the number of obstructions for natural parameterized families to grow explosively as a function of the parameter k. For example, the number of minor-order obstructions for k–Pathwidth (i.e., graphs with pathwidth at most k) is 2 for $k = 1$, 110 for $k = 2$, and provably more than 60 million for $k = 3$ [KL94]. It is known that there are at least $k!^2$ obstructions that are trees for each k [TUK91].

In this section we study common families of parameterized minor-order lower ideals and provide (1) a practical constructive result that allows one to compute the disconnected obstructions of k–$\mathcal{F}$ from the connected obstructions of its sub-family lower ideals $\{k'$–$\mathcal{F} \mid k' < k\}$ and (2) a complexity result that indicates when one should expect super-polynomial growth in the number of obstructions as the parameter k increases.

2.1 Considering disconnected obstruction

Let λ be a function that maps graphs to non-negative integers such that:

1. For graphs G_1 and G_2, $\lambda(G_1 \cup G_2) = \lambda(G_1) + \lambda(G_2)$,
2. For any minor H of G, $\lambda(H) \leq \lambda(G)$, and
3. For any graph G there exists a minor H such that $\lambda(H) \geq \lambda(G) - 1$.

The family of graphs $\mathcal{F}[k] = \{G \mid \lambda(G) \leq k\}, k \geq 0$ has an obstruction set since it is a lower ideal (from property 2 above) in the minor order. Also $\mathcal{O}(\mathcal{F}[k])$ has finite cardinality because of the Graph Minor Theorem. A concrete example is $\lambda(G) = genus(G)$, where $genus(G)$ denotes the smallest genus of all orientable surfaces on which G can be embedded (property 1 follows from [BHKY62]). In fact, there are many examples of lower ideals that satisfy all three properties such as all of the "within k vertices" families (e.g., within one vertex of outer-planar) that are discussed in [Din95].

Lemma 1 *If $\mathcal{F}$ is a lower ideal such that*

$$G_1 \in \mathcal{F} \text{ and } G_2 \in \mathcal{F} \iff (G_1 \cup G_2) \in \mathcal{F}$$

then the function

$$\lambda(G) = \min(|V'| \ : \ G \setminus V' \in \mathcal{F} \text{ where } V' \subseteq V)$$

can be used to define the above parameterized lower ideals $\mathcal{F}[k]$, $k \geq 0$.

Proof. Clearly, $\lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$ since if $G_1 \setminus V_1 \in \mathcal{F}$ and $G_2 \setminus V_2 \in \mathcal{F}$ then $(G_1 \cup G_2) \setminus (V_1 \cup V_2) \in \mathcal{F}$. Likewise, $\lambda(G_1) + \lambda(G_2) \leq \lambda(G_1 \cup G_2)$ since if $(G_1 \cup G_2) \setminus V_{1,2} \in \mathcal{F}$ then $G_1 \setminus (V(G_1) \cap V_{1,2}) \in \mathcal{F}$ and $G_2 \setminus (V(G_2) \cap V_{1,2}) \in \mathcal{F}$. So property 1 holds for λ. Property 2 follows from the fact that $\mathcal{F}[k]$ is defined as "a within k vertices" family of a lower ideal (see [FL88]). Property 3 follows from the fact that if H is any minor obtained from G by deleting a single vertex then $\lambda(H)$ can be at most one less than $\lambda(G)$. □

For example, the k–Vertex Cover and k–Feedback Vertex Set parameterized lower ideals are defined from the base lower ideals $\mathcal{F}_{VC} = \{$graphs with no edges$\}$ and $\mathcal{F}_{FVS} = \{$graphs with no cycles$\}$, respectively. Here the families $\mathcal{F}_{VC}$ and $\mathcal{F}_{FVS}$ are closed under graph unions, so Lemma 1 is valid.

Theorem 2 *If $G = C_0 \cup C_1$ is an obstruction for $\mathcal{F}[k]$, $k \geq 0$, then each C_i is an obstruction for $\mathcal{F}[\lambda(C_i) - 1], i = 0, 1$.*

Proof. First note that $\lambda(C_i) \neq 0$ for otherwise removing that component from G is a contradiction to the fact that G is an obstruction. Thus, we claim that each C_i is an obstruction to a smaller family (i.e., some $\mathcal{F}[k']$ where $k' < k$).

Assume that C_i is not an obstruction for $\mathcal{F}[\lambda(C_i) - 1]$, the smallest family not containing C_i. Not being an obstruction implies that there is a minor C' of C_i such that $\lambda(C') = \lambda(C_i)$. But this implies that for the minor $G' = C' \cup C_{1-i}$ of G,

$$\begin{aligned} \lambda(G') = \lambda(C' \cup C_{1-i}) &= \lambda(C') + \lambda(C_{1-i}) \\ = \lambda(C_1) + \lambda(C_2) &= \lambda(C_1 \cup C_2) \\ = \lambda(G) &= k + 1 \quad . \end{aligned}$$

The existence of this minor contradicts the assumption that G is an obstruction for $\mathcal{F}[k]$. So the connected graph C_i must also be an obstruction. □

Corollary 3 *If $G = \bigcup_{i=0}^{r} C_i$ is an obstruction for $\mathcal{F}[k]$, $k \geq 0$, then each C_i is an obstruction for $\mathcal{F}[\lambda(C_i) - 1]$, $0 \leq i \leq r$.*

Proof. This follows by repeatedly applying the above theorem. □

Many obstruction set characterizations are based on k-parameterized lower ideals. It would be surprising if the growth rate in the number of obstructions per each k is slow. To conclude this subsection we present an observations to substantiate this claim. (The next subsection studies the growth rate in a different setting.)

Corollary 4 *If the number of connected obstructions of $\mathcal{F}[k]$ is at least one, for all $k \geq 0$, then the total number of obstructions of $\mathcal{F}[k]$ is greater than or equal to the number of integer partitions of $k+1$.*

Proof. Because of property (3) of $\mathcal{F}[k]$, we know that any obstruction $O \in \mathcal{O}(\mathcal{F}[k])$ has $\lambda(O) = k+1$. By Corollary 3 there exist disconnected obstructions $O_1 \cup O_2 \cup \cdots \cup O_m$, where each O_i is connected and $\sum_{i=1}^{m} \lambda(O_i) = k+1$. There is a one-to-one correspondence with these (non-isomorphic) obstructions and the number of integer partitions of $k+1$. □

We can easily generate the number of integer partitions for various n. Hence, that following gives lower bounds on the total number of obstructions for $\mathcal{F}[k]$.

n =	1	2	3	4	5	6	7	8	9	10	11	12	13	...	20	30
counts =	1	2	3	5	7	11	15	22	30	42	56	77	101		627	5604

n =	40	50	60	70	80	90	100
counts =	37338	204226	966467	4087968	15796476	56634173	190569292

2.2 When to expect lots of obstructions

To lead up to our main result, let $\mathcal{F}_k = \{G \mid \gamma(G) \leq k\}$ be any k-parameterized lower ideal that is defined by some integer function $\gamma(G) \leq p(|G|)$, where p is a any bounding polynomial function. Also let $p_\gamma(G,k)$ denote the corresponding graph problem that determines if $\gamma(G) \leq k$ where both G and k are part of the input.

Recall that the *polynomial time hierarchy* is the structure formed by the classes Σ_k^P, Π_k^P and Δ_k^P for each $k \geq 0$, where

1. $\Sigma_0^P = \Pi_0^P = \Delta_0^P = P$,
2. $\Sigma_{k+1}^P = \mathcal{NP}(\Sigma_k^P)$,
3. $\Delta_{k+1}^P = \mathcal{P}(\Sigma_k^P)$,
4. $\Pi_{k+1}^P = \text{co-}\mathcal{NP}(\Sigma_k^P)$, and
5. $\mathcal{PH} = \bigcup_{k\geq 0} \Sigma_k^P = \bigcup_{k\geq 0} \Pi_k^P = \bigcup_{k\geq 0} \Delta_k^P \quad .$

The above complexity notation $S(A)$ denotes the complexity class whose instances can be solved in time S with the help of an oracle for A. Also, for the next result, note that $S/poly$ denotes the non-uniform complexity class whose instances can be solved in time S with access to polynomial-length advice (i.e. with respect to the input size, not the input instance).

Theorem 5 *If $p_\gamma(G,k)$ is $\mathcal{NP}$-complete then $f_k = |\mathcal{O}(\mathcal{F}_k)|$ must be super-polynomial else the polynomial time hierarchy ($\mathcal{PH}$) collapses to Σ_3^P.*

Proof. Yap (see [Yap83]) has shown that the following are equivalent for $i > 0$:
(a) $\Pi_i^P \subseteq \Sigma_i^P/\text{poly}$ or $\Sigma_i^P \subseteq \Pi_i^P/\text{poly}$
(b) $\Sigma_i^P/\text{poly} = \Pi_i^P/\text{poly}$
(c) $\mathcal{PH}/\text{poly} = \Sigma_i^P/\text{poly}$ or $\mathcal{PH}/\text{poly} = \Pi_i^P/\text{poly}$

Now if f_k is bounded by a polynomial then co-$p_\gamma(G,k) \in \mathcal{NP}/\text{poly}$ by having polynomial advice consisting of all the obstructions within $\mathcal{O}(\mathcal{F}_k)$ of size at most $|G|$, for each $1 \leq k \leq p(|G|)$. That is, we can nondeterministically verify in polynomial time (for input G and k) that some obstruction $O \in \mathcal{O}(\mathcal{F}_k)$ is a minor of G. Since $p_\gamma(G,k)$ is $\mathcal{NP}$-complete, we get

$$\text{co-}\mathcal{NP} = \Pi_1^P \subseteq \mathcal{NP}/\text{poly} = \Sigma_1^P/\text{poly} \quad .$$

An application of Yap's result, given above, shows that

$$\Sigma_1^P/\text{poly} = \Pi_1^P/\text{poly} = \mathcal{PH}/\text{poly} \quad .$$

And this implies $\Sigma_3^P = \Pi_3^P$ by another theorem of Yap that states that for $i > 0$,

$$\Sigma_i^P/\text{poly} = \Pi_i^P/\text{poly} \Rightarrow \Sigma_{i+2}^P = \Pi_{i+2}^P \quad .$$

So using the well-known fact that if $\Sigma_i^P = \Pi_i^P$ for any $i > 0$ then the polynomial time hierarchy collapses to level i, we get $\mathcal{PH} = \Sigma_3^P = \Pi_3^P$ unless there are a super-polynomial number of obstructions for $\mathcal{F}_k$. $\square$

This result can be applied to most of the k-parameterized lower ideals of interest. For example, the k–Vertex Cover and k–Feedback Vertex Set graph families satisfy the preconditions of the above result. Also since determining the genus of a graph is $\mathcal{NP}$-complete, the lower ideal k-Genus $= \{G \mid genus(G) \leq k\}$ also belongs in the above class.

3 Graphs with small Independence plus Size

To illustrate the results of the previous section, we investigate a contrived (but non-trivial) graph family. Recall that the *independence* of a graph is the maximum number of mutually non-adjacent vertices (i.e., the cardinality of the largest independent set). It is easy to show that any minor of a graph with independence + size $\leq k$ also satisfies this inequality. For example, after deleting an edge from a graph the independence can increase by at most one, implying that the sum will not increase. For any fixed k, let us call this graph family k–EDGE BOUNDED INDSET.

For each small fixed k, an obstruction set for k–EDGE BOUNDED INDSET is easily found by the following logic: We know that $k+1$ isolated vertices is an obstruction, so $k+1$ also bounds the order of the largest obstruction. We display all the connected obstructions for 0–EDGE BOUNDED INDSET through 9–EDGE BOUNDED INDSET in Figure 1. By applying Theorem 2 the disconnected obstructions are easily obtained from the connected ones. For example, the union $P_2 \cup K_1$ of a path of length 2 and an isolated vertex is an obstruction for 4–EDGE BOUNDED INDSET.

Looking at the first few k–EDGE BOUNDED INDSET graph families ($0 \leq k \leq 12$), the counts given in Table 1 indicate, as expected, exponential growth in the number of obstructions.

Table 1: Number of connected and disconnected minor-order obstructions for k–EDGE BOUNDED INDSET, for $0 \leq k \leq 12$.

$k =$	0	1	2	3	4	5	6	7	8	9	10	11	12
connected	1	0	0	1	0	2	2	3	5	12	21	42	86
disconnected	0	1	1	1	2	2	4	7	10	17	31	58	105
total obstructions	1	1	1	2	2	4	6	10	15	29	52	100	191

We end with a brief look at the computational complexity of determining membership in k–EDGE BOUNDED INDSET. Without using a lookup table (noting that this family has a finite number of members to check with an isomorphism algorithm), recognizing graphs in k–EDGE BOUNDED INDSET requires a running time on the order $O(n^{k-m})$ with a brute-force algorithm. That is, where the size m has been previously computed, an algorithm could check to see if any of the vertex subsets of cardinality $k - m$ is an independent set. An application of the GMT yields an improved cubic-time algorithm, based on checking a finite set of forbidden minors. Furthermore, since the treewidth of members of k–EDGE BOUNDED INDSET are at most k and since we can express graphs with these properties in second-order monadic logic, we can build a linear-time membership algorithm (e.g. see [ACPS91]).

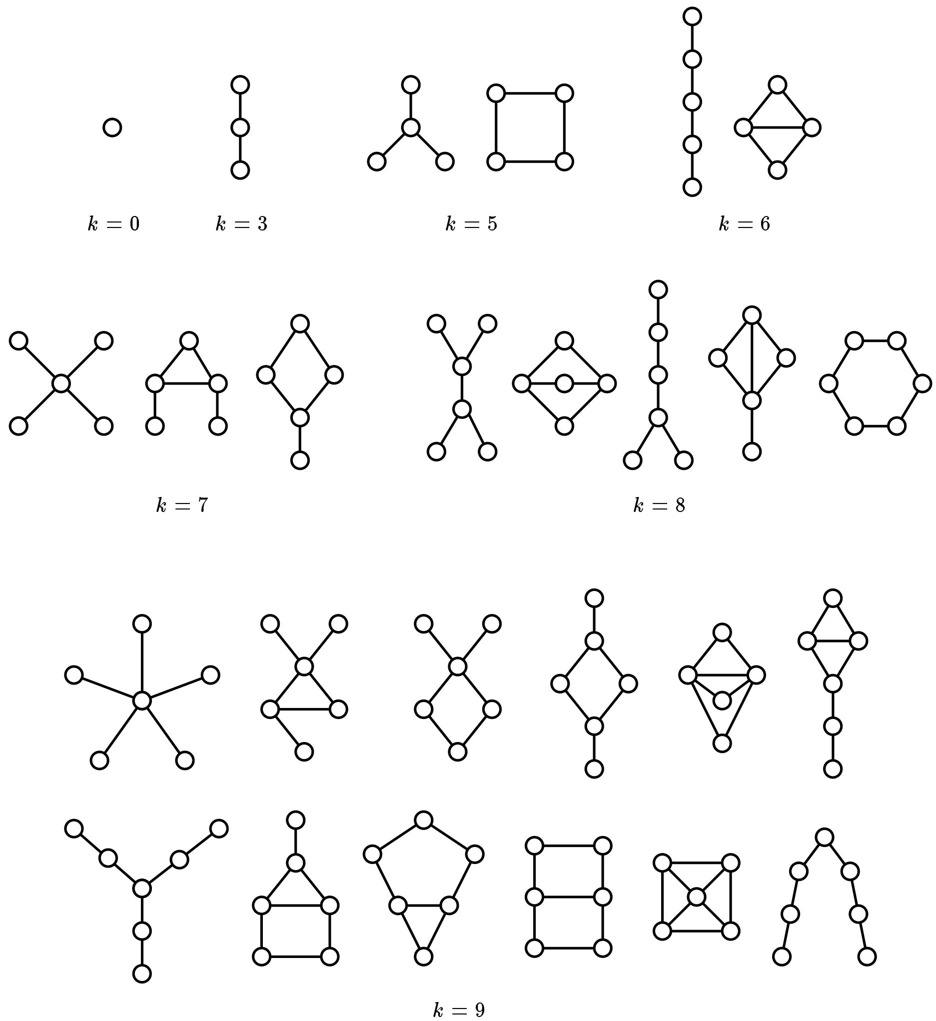

Figure 1: All connected minor-order obstructions for k–Edge Bounded IndSet, for $0 \leq k \leq 9$.

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