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Sequential Continuity of Linear Mappings in Constructive Mathematics[1]

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Abstract: This paper deals, constructively, with two theorems on the sequential continuity of linear mappings. The classical proofs of these theorems use the boundedness of the linear mappings, which is a constructively stronger property than sequential continuity; and constructively inadmissable versions of the Banach-Steinhaus theorem.

Key Words: sequential continuity, pointwise continuity, linear mappings, constructive mathematics

Category: F.m

1 Introduction

The classical validity of many important theorems of analysis, such as the Banach-Steinhaus theorem, the open mapping theorem and the closed graph theorem, depends on Baire's theorem about complete metric spaces, which is an indispensable tool in this area. A form of Baire's theorem has a constructive proof [Bridges and Richman 87, Theorem 1.3], but its classical equivalent,

> *If a complete metric space is the union of a sequence of its subsets, then the closure of at least one set in the sequence must have nonempty interior,*

which is used in the standard argument to prove the above theorems has no known constructive proof.

In [Ishihara 91, Ishihara 92, Ishihara 93], we dealt with the constructive distinctions between certain types of continuity, such as sequential and pointwise continuity; we subsequently proved, in [Ishihara 94], constructive versions of Banach's inverse mapping theorem, the open mapping theorem, and the closed graph theorem for sequentially continuous linear mappings. In this paper we prove, within Bishop's constructive mathematics, two theorems on the sequential continuity of linear mappings which are classically proved using the boundedness (pointwise continuity) of linear mappings, the Banach-Steinhaus theorem, and the closed graph theorem [Rudin 91, Theorem 2.8 and Theorem 5.1]. We assume that the reader has access to [Bishop and Bridges 85] or [Bridges and Richman 87], but very little background in constructive analysis is required for the understanding of our proofs.

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2 Main results

We begin by proving two lemmas which, despite having been referenced on occasion, have only appeared in a preprint [Ishihara 90].

Lemma 1. *Let T be a linear mapping of a Banach space X into a normed space Y, and let (x_n) be a sequence converging to 0 in X. Then for all positive numbers a, b with $a < b$, either $\|Tx_n\| > a$ for some n or else $\|Tx_n\| < b$ for all n.*

Proof. Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\|x_j\| < \frac{1}{(k+1)^2} \quad (j \geq n_k),$$

and, taking $n_0 := 0$, set

$$s_k := \max\{\|Tx_j\| : n_{k-1} < j \leq n_k\}.$$

Construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_k = 0 &\Rightarrow s_k < b, \\ \lambda_k = 1 &\Rightarrow a < s_k. \end{aligned}$$

We may assume that $\lambda_1 = 0$. Define a sequence (y_k) in X as follows: if $\lambda_k = 0$, set $y_k := 0$; if $\lambda_k = 1 - \lambda_{k-1}$, choose j with $n_{k-1} < j \leq n_k$ and $\|Tx_j\| > a$, and set $y_i := kx_j$ for all $i \geq k$. Then (y_k) is a Cauchy sequence: in fact, $\|y_i - y_k\| \leq 1/k$ whenever $i \geq k$. So (y_k) converges to a limit y in X. Choosing a positive integer N such that $\|Ty\| < Na$, consider any integer $k \geq N$. If $\lambda_k = 1 - \lambda_{k-1}$, then $y = kx_j$ for some j with $\|Tx_j\| > a$, so

$$Na \leq ka < \|kTx_j\| = \|Ty\| < Na,$$

a contradiction. Hence $\lambda_k = \lambda_{k-1}$ for all $k \geq N$. It follows that either $\lambda_k = 1$ for some k or else $\lambda_k = 0$ for all k. $\square$

Lemma 2. *Let T be a linear mapping of a Banach space X into a normed space Y, and let (x_n) be a sequence converging to 0 in X. Then for all positive numbers a, b with $a < b$, either $\|Tx_n\| > a$ for infinitely many n or else $\|Tx_n\| < b$ for all sufficiently large n.*

Proof. Let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that

$$\|x_j\| < \frac{1}{k^2} \quad (j \geq n_k).$$

By successively applying Lemma 1 to the sequences $(x_{n_k}, x_{n_k+1}, \ldots)$, construct an increasing binary sequence (λ_k) such that

$$\begin{aligned} \lambda_k = 0 &\Rightarrow \exists j \geq n_k (a < \|Tx_j\|), \\ \lambda_k = 1 &\Rightarrow \forall j \geq n_k (\|Tx_j\| < b). \end{aligned}$$

We may assume that $\lambda_1 = 0$. Define a sequence (z_k) in X as follows: if $\lambda_k = 0$, choose $j \geq n_k$ such that $\|Tx_j\| > a$ and set $z_k := kx_j$; if $\lambda_k = 1 - \lambda_{k-1}$, set $z_i := kx_{k-1}$ for all $i \geq k$. Then (z_k) is a Cauchy sequence, and so converges to a limit $z \in X$. Choosing a positive integer N such that $\|Tz\| < Na$, consider any integer $k > N$. If $\lambda_k = 1 - \lambda_{k-1}$, then $z = (k-1)x_j$ for some j with $\|Tx_j\| > a$, so

$$Na \leq (k-1)a < \|(k-1)Tx_j\| = \|Ty\| < Na,$$

a contradiction. Hence $\lambda_k = \lambda_{k-1}$ for all $k \geq N$. It follows that either $\lambda_k = 1$ for some k or else $\lambda_k = 0$ for all k. In the former case, for each k there exists $j \geq n_k$ such that $\|Tx_j\| > a$; in the latter, there exists k such that $\|Tx_j\| < b$ for all $j \geq n_k$. □

A mapping $f : X \to Y$ between metric spaces is said to be

1. **discontinuous** if there exist a sequence (x_n) in X, a point $x \in X$, and a positive number δ such that $x_n \to x$ and $d(f(x_n), f(x)) \geq \delta$ for all n; and
2. **strongly extensional** if $f(x) \neq f(y)$ implies $x \neq y$ (where $x \neq y$ means $0 < d(x, y)$).

The following lemma is proved in [Ishihara 94].

Lemma 3. *If there exists a mapping f of a complete metric space into a metric space such that f is strongly extensional and discontinuous, then* ∃-PEM*:*

$$\forall \alpha \in \mathbb{N}^{\mathbb{N}} (\exists n(\alpha(n) \neq 0) \vee \forall n(\alpha(n) = 0))$$

holds. □

Many well-known theorems in classical analysis could be proved with ∃-PEM. For example, one can see that the following lemmas are immediate consequences of ∃-PEM and the constructive least-upper-bound principle [Bishop and Bridges 85, (4.4.3)].

A linear mapping $T : X \to Y$ between normed spaces is said to be

1. **sequentially continuous** if $x_n \to 0$ implies that $Tx_n \to 0$;
2. **bounded** if there exists $c > 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in X$.

Lemma 4. *Under* ∃-PEM*, every sequentially continuous linear mapping from a separable normed space into a normed space is bounded.* □

A bounded linear mapping $T : X \to Y$ between normed spaces is said to be **normable** if its **operator norm**

$$\|T\| := \sup\{\|T(x)\| : x \in X, \|x\| \leq 1\}$$

exists.

Lemma 5. *Under* ∃-PEM*, every bounded linear functional on a separable normed space is normable.* □

Recall that the following form of the **uniform boundedness theorem** has a constructive proof [Bishop and Bridges 85, page 392, Problem 20].

Theorem 6. *Let (T_n) be a sequence of bounded linear mappings from a Banach space X into a normed space Y, and let (x_n) be a sequence of unit vectors in X such that $\|T_n x_n\| \to \infty$. Then there exists $x \in X$ such that $(\|T_n x\|)$ is unbounded.* □

We now prove a constructive version of [Rudin 91, Theorem 2.8].

Theorem 7. *Let (T_m) be a sequence of sequentially continuous linear mappings from a separable Banach space X into a normed space Y such that*

$$Tx := \lim_{m\to\infty} T_m x$$

exists for all $x \in X$. Then T is sequentially continuous.

Proof. Since X is complete, T is strongly extensional, by [Bridges and Ishihara 90, Corollary 2]. Let (x_n) be a sequence converging to 0 in X. Then by Lemma 2, for each $\epsilon > 0$ either $\|Tx_n\| > \epsilon/2$ for infinitely many n or else $\|Tx_n\| < \epsilon$ for all sufficiently large n. In the former case, by passing to an appropriate subsequence, we may assume that $\|Tx_n\| > \epsilon/2$ for all n. Therefore $\exists$-PEM holds, by Lemma 3; so, by Lemma 4, every sequentially continuous linear mapping of X into Y is bounded. Construct a sequence (y_n) in X and a strictly increasing sequence (m_n) of positive integers such that $\|y_n\| = 1$ and $\|T_{m_n} y_n\| \to \infty$. Then applying the uniform boundedness theorem to the sequence $(T_{m_n})_{n=1}^{\infty}$ of bounded linear mappings on X, we can now produce $z \in X$ such that the sequence $(\|T_{m_n} z\|)_{n=1}^{\infty}$ is unbounded. This contradicts the hypotheses that

$$Tz = \lim_{m\to\infty} T_m z = \lim_{n\to\infty} T_{m_n} z$$

exists, and so ensures that $\|Tx_n\| < \epsilon$ for all sufficiently large n. Since $\epsilon > 0$ is arbitrary, it follows that T is sequentially continuous. □

Next we prove a constructive version of [Rudin 91, Theorem 5.1].

Theorem 8. *Let T be a linear mapping from a Banach space X into a separable normed space Y with the following property: if f is a normable linear functional f on Y, and (x_n) converges to 0 in X, then $f(Tx_n) \to 0$. Then T is sequentially continuous.*

Proof. Let (x_n) be a sequence converging to 0 in X. Then, as in the proof of Theorem 7, for each $\epsilon > 0$ either $\|Tx_n\| > \epsilon/2$ for infinitely many n or else $\|Tx_n\| < \epsilon$ for all sufficiently large n. In the former case, we can construct a sequence (y_n) in X such that $y_n \to 0$ and $0 < \|Ty_n\| \to \infty$. Therefore $\exists$-PEM holds, by Lemma 3; so every linear functional on Y is normable, and Y^* (the dual of Y) is a Banach space relative to the usual norm, by Lemma 5. By [Bishop and Bridges 85, (7.4.5)], for each n there exists $f_n \in Y^*$ such that $\|f_n\| = 1$ and $f_n(Ty_n) > \|Ty_n\|/2$. Applying the uniform boundedness theorem to the bounded linear functionals $f \mapsto f(Ty_n)$ on Y^*, we can now produce $g \in Y^*$ such that the sequence $(g(Ty_n))$ is unbounded. This contradicts the hypotheses of our theorem. □

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