The Least $\Sigma$-jump Inversion Theorem for $n$-families

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Abstract: Studying the $\Sigma$-reducibility of families introduced by [Kalimullin and Puzarenko 2009] we show that for every set $X \geq_T \emptyset'$ there is a family of sets $F$ which is the $\Sigma$-least countable family whose $\Sigma$-jump is $\Sigma$-equivalent to $X \oplus \overline{X}$. This fact will be generalized for the class of $n$-families (families of families of ... of sets).

Key Words: jump of structure, enumeration jump, $\Sigma$-jump, $\Sigma$-reducibility, countable family, $n$-family

Category: F.1.1., F.1.2., F.4.1.

1 Introduction

In this paper study a reducibility among families of sets introduced in [Kalimullin and Puzarenko 2009]. We will say that a family $\mathcal{F}_0 \subseteq 2^\omega$ is $\Sigma$-reducible to a family $\mathcal{F}_1 \subseteq 2^\omega$ if for every admissible set $\mathcal{A}$

$$\mathcal{F}_1 \text{ is } \Sigma\text{-definable in } \mathcal{A} \implies \mathcal{F}_0 \text{ is } \Sigma\text{-definable in } \mathcal{A}.$$ 

A family $\mathcal{F} \subseteq 2^\omega$ is $\Sigma$-definable in $\mathcal{A}$ if there is a $\Sigma$-formula $\Phi$ such that

$$\mathcal{F} = \{ \{ x \in \omega : \Phi(x, y) \} : y \in Y \},$$
for some $\Sigma$-definable subset $Y \subseteq A$. This reducibility can be reformulated in terms of enumeration operators:

**Theorem 1.** [Kalimullin and Puzarenko 2009] For families $F_0$ and $F_1$, the following conditions are equivalent:

1. $F_0 \leq_{\Sigma} F_1$;
2. $F_0 \cup \{\emptyset\} = \{\Theta(C \oplus B \oplus E(F)) : C \in K_{F_1}\}$ for some enumeration operator $\Theta$ and some set $B \in K_{F_1}$, where $E(F) = \{u : \exists X \in F[D_u \subseteq X]\}$, and $K_{F_1}$ is the class of sets of the form $\{(n,m)\} \oplus A_1 \oplus \ldots \oplus A_m$, $A_i \in F_1$.

On the other hand, $\leq_{\Sigma}$ is a natural extension of the enumeration and Turing reducibilities, since $A \leq_{e} B \iff \{A\} \leq_{\Sigma} \{B\}$.

Let us highlight that $\Sigma$-reducibility among families is equivalent to the $\Sigma$-definability relation between special structures $M_F$ [Kalimullin and Puzarenko 2009]. See the end of this section for the detailed definition of $M_F$. Following [Montalbán 2009], [Puzarenko 2009], [Stukachev 2009] we can view the $\Sigma$-jumps of families as the jumps of the corresponding structures.

**Definition 2.** For a structure $\mathcal{M}$, define the jump of $\mathcal{M}$ to be the structure $J(\mathcal{M}) = (HF(\mathcal{M}), U_{\Sigma})$, where $U_{\Sigma}$ is a ternary $\Sigma$-predicate on $HF(\mathcal{M})$ universal for the class of all binary $\Sigma$-predicates on $HF(\mathcal{M})$, is called a $\Sigma$-jump.

For any $n$-family $\mathcal{F}$ instead of $\mathcal{J}(\mathcal{M}:\tau)$ we simply write $\mathcal{J}(\mathcal{F})$. The $\Sigma$-jump does not depend on the choice of a universal $\Sigma$-predicate, up to $\Sigma$-equivalence. Furthermore, this $\Sigma$-jump on structures having Turing (enumeration) degrees acts in the same way as a Turing (enumeration) jump (see [Puzarenko 2009]). As in the classical case, the $\Sigma$-jump operation satisfies the following:

1. $\mathcal{A} \leq_{\Sigma} \mathcal{J}(\mathcal{A})$;
2. $\mathcal{A} \leq_{\Sigma} \mathcal{B} \Rightarrow \mathcal{J}(\mathcal{A}) \leq_{\Sigma} \mathcal{J}(\mathcal{B})$.

We define $\mathcal{J}^n(\mathcal{A})$ by induction on $n \in \omega$ as follows: $\mathcal{J}^0(\mathcal{A}) = \mathcal{A}$, $\mathcal{J}^{n+1}(\mathcal{A}) = \mathcal{J}(\mathcal{J}^n(\mathcal{A}))$. It was shown in [Puzarenko 2009] that for any structures $\mathcal{M}$ and $\mathcal{A}$ on a finite signature, $\mathcal{M}$ is $\Sigma_{m+1}$-definable in $\mathcal{A}$ iff $\mathcal{M} \leq_{\Sigma} \mathcal{J}^m(\mathcal{A})$.

In [Kalimullin and Puzarenko 2009] some unexpected properties of the family InfCE of all infinite c.e. sets under $\Sigma$-reducibility were found. In particular, for a family $\mathcal{F}$, $\mathcal{F} \leq_{\Sigma} \text{InfCE}$ iff the following conditions hold:

1. all sets in $\mathcal{F}$ are c.e.;
2. the index set $\{e : W_e \in \mathcal{F}\}$ is $\Sigma_0^0$;
3. there exists a computable cover $\hat{\mathcal{F}}$ of $\mathcal{F}$, i.e., a computable family $\hat{\mathcal{F}} \subseteq \mathcal{F}$ such that for any $W \in \mathcal{F}$, we have $W \subseteq V$ for some $V \in \hat{\mathcal{F}}$. 
In this paper, we show that the family InfCE has yet another natural property: InfCE is the $\Sigma$-least family among all countable families, whose $\Sigma$-jump computes $\emptyset''$, i.e., it is the least jump inversion of the Turing degree of $\emptyset''$. Moreover, each set $A \geq_T \emptyset'$ has such jump inversion. We show also, that each family $\mathcal{F} \geq_\Sigma \mathcal{V}$ has the $\Sigma$-least jump inversion in the extended class of $n$-families.

The notation and terminology follows from Rogers [Rogers 1967] and [Ershov 1996]. We now formally introduce the generalized notion of $n$-families and fix the precise way of their coding into the structures.

**Definition 3.** A 0-family is a subset of $\omega$. For an integer $n > 0$, an $n$-family is a countable set of $(n - 1)$-families. We consider the empty set as an 0-family.

According to [Kalimullin and Faizrahmanov 2016] the definition of computably enumerable $n$-families is inductive: an $n$-family $\mathcal{F}$ is computably enumerable if its elements, $(n - 1)$-families, are uniformly computably enumerable. We give this definition generalized to an arbitrary admissible set (see [Ershov 1996]):

**Definition 4.**
1. A $\Sigma_s$-formula $\Phi(\overline{z}, y), s \in \omega$, defines a 0-family $X \subseteq \text{Nat}(A)$ in an admissible set $A$ if there is a tuple $\overline{c} \in A^k$ such that
   
   $X = \{m \in \text{Nat}(A) : A |\Phi(\overline{c}, m)\}.$
   
   In this case, we will write $X = \mathcal{F}^{0,A}_{\Phi(\overline{c})}$.

2. A $\Sigma_s$-formula $\Phi(\overline{z}, x, y)$ defines an 1-family $\mathcal{F}$, if there are a nonempty $\Sigma_s$-subset $E \subseteq A$ and a tuple $\overline{c} \in A^k$ such that
   
   $\mathcal{F} = \{\mathcal{F}^{1,A}_{\Phi(\overline{c}, x)} : x \in E\}.$
   
   In this case, we will write $\mathcal{F} = \mathcal{F}^{1,A}_{\Phi(\overline{c})}$.

3. A $\Sigma_s$-formula $\Phi(\overline{z}, x_1, \ldots, x_{n+2}, y)$ defines an $(n + 2)$-family $\mathcal{F}$, if there are a nonempty $\Sigma_s$-subset $E \subseteq A^{n+2}$ and a tuple $\overline{c} \in A^k$ such that
   
   $\mathcal{F} = \{\mathcal{F}^{n+1,A}_{\Phi(\overline{c}, x)} : x \in \text{Pr}_1(E)\},$
   
   where
   
   $\text{Pr}_1(E) = \{x : \exists y_1 \ldots \exists y_{n+1} (x, y_1, \ldots, y_{n+1}) \in E\}$,

   $E(x) = \{(y_1, \ldots, y_{n+1}) : (x, y_1, \ldots, y_{n+1}) \in E\}.$

   In this case, we will write $\mathcal{F} = \mathcal{F}^{n+2,A}_{\Phi(\overline{c})}$.

An $n$-family $\mathcal{F}$ is $\Sigma_s$-definable ($\Sigma$-definable for the case $s = 1$) in $A$ if some $\Sigma_s$-formula defines $\mathcal{F}$ in $A$. 
This definition extends the definition given in [Kalimullin and Puzarenko 2009].

We will see below that for the \( n \)-families it is enough to consider only special cases of admissible sets, namely, the hereditary finite structures \( \mathbb{HF}(\mathfrak{M}) \), where \( \mathfrak{M} \) is some algebraic structure. Let \( M \) be the domain of \( \mathfrak{M} \) and let \( \sigma \) be the language of \( \mathfrak{M} \). The domain of \( \mathbb{HF}(\mathfrak{M}) \) is the class of \( HF(M) \) of hereditarily finite sets over the set \( M \) is defined by induction as follows:

- \( H_0(M) = \{ \emptyset \} \);
- \( H_{n+1}(M) = H_n(M) \cup \mathcal{P}_\omega (H_n(M) \cup M) \);
- \( HF(M) = \bigcup_{n<\omega} H_n(M) \cup M \)

(\( P_\omega (X) \) denotes the set of all finite subsets of \( X \)).

The hereditarily finite superstructure over \( \mathfrak{M} \) is the algebraic structure \( \mathbb{HF}(\mathfrak{M}) \) in the signature \( \sigma \cup \{ U(1), \in \} \), where \( U^{\mathbb{HF}(\mathfrak{M})} = M, \in^{\mathbb{HF}(\mathfrak{M})} \subseteq (HF(M)) \times (HF(M) \setminus M) \) is the membership relation on \( \mathbb{HF}(\mathfrak{M}) \), the constant symbol \( \emptyset \) is interpreted as the empty set, and symbols in the signature \( \sigma \) are interpreted in the same way as on \( \mathfrak{M} \).

Following [Kalimullin and Puzarenko 2009], we can code every \( n \)-family \( \mathcal{F} \) into the admissible superstructure \( \mathbb{HF}(\mathfrak{M}_\mathcal{F}) \) over the special structure \( \mathfrak{M}_\mathcal{F} \) defined by induction as follows.

- For an arbitrary 0-family \( A \) let \( \mathfrak{M}_A \) be the structure in the signature \( \sigma = \{ r, I^1, R^2 \} \) with the domain \( M_\mathcal{F} = \omega \cup X, X = \{ x_n : n \in A \} \), such that \( R^{\mathfrak{M}_A} = \{(n,n+1) : n \in \omega \} \cup \{(x_n,n) : n \in A \}, r^{\mathfrak{M}_A} = 0 \) and \( I^{\mathfrak{M}_A} = \{ x^{\mathfrak{M}_A} \} \).

- For an \( n \)-family \( \mathcal{F} = \{ S_i : i \in \omega \}, n > 0 \), let \( \mathfrak{M}_A \) be the structure in the signature \( \sigma = \{ r, I^1, R^2 \} \) with the domain \( \bigcup_{k=1}^{n} \mathfrak{M}_{S_k} \cup \{ r^{\mathfrak{M}_S} \} \) (each \( \mathfrak{M}_{S_k} \) is an isomorphic copy of \( \mathfrak{M}_{S_k} \) with a new domain) such that \( I^{\mathfrak{M}_S} = \bigcup_{k=1}^{n} I^{\mathfrak{M}_{S_k}} \) and

\[
R(x, y) \iff x = (\exists k, i) [x = r^{\mathfrak{M}_{S_k}} & y = r^{\mathfrak{M}_{S_k}} \lor R^{\mathfrak{M}_{S_k}} (x, y)]
\]

for each \( x, y \in [\mathfrak{M}_\mathcal{F}] \).

Through this inductive definition, the elements of \( I^{\mathfrak{M}_\mathcal{F}} \) are precisely the elements originally denoted as \( r^{\mathfrak{M}_S} \) for 0-families \( A \in \cdots \in \mathcal{F} \). For \( i \in I^{\mathfrak{M}_S} \) we denote the corresponding 0-family by \( A_i \).

It is easy to check that every \( n \)-family \( \mathcal{F} \) is \( \Sigma \)-definable in \( \mathbb{HF}(\mathfrak{M}_\mathcal{F}) \). For example, if \( n = 0 \) then a 0-family \( A \subseteq \omega \) is defined by the formula saying that there is a sequence

\[
r^{\mathfrak{M}_\mathcal{F}} = n_0, n_1, n_2, \ldots, n_{x+1}, p
\]

such that \( R(n_i, n_{i+1}) \) for all \( i \leq x \), and \( R(p, n_{x+1}), p \neq n_x \). Moreover, it follows from [Kalimullin and Puzarenko 2009] that the \( \Sigma \)-definability of \( \mathcal{F} \) is equivalent to the \( \Sigma \)-definability of \( \mathfrak{M}_\mathcal{F} \) itself.
Proposition 5. [Kalimullin and Puzarenko 2009] An n-family $F$ is $\Sigma$-definable in a countable admissible set $A$ iff the structure $M_F$ (and, therefore, $HF(M_F)$) is $\Sigma$-definable in $A$.

Under $\Sigma$-interpretation of a structure $M$ in a signature $\sigma$ we understand a $\Sigma$-definable structure $N$ in the language $\sigma \cup \{\sim\}$, where $\sim$ is a new congruence relation on $N$ such that $N/\sim = M$.

Definition 6. Let $F$ be an n-family and $M$ be a structure. We say that $F$ is $\Sigma$-reducible to $M$ (written $F \leq_{\Sigma} M$) if $M_F$ is $\Sigma$-definable in $HF(M)$. Similarly, $M \leq_{\Sigma} F$ if $M$ is $\Sigma$-definable in $HF(M_F)$. If $F$ and $S$ are $n$- and $m$-families correspondingly we say that $F$ is $\Sigma$-reducible to $S$ if $F \leq_{\Sigma} M_S$. As usual, the relation $\equiv_{\Sigma}$ holds in the case of $\Sigma$-reductions from the left to the right and from the right to the left.

Note that for an $n$-family $F$ and the $(n+1)$-family $\{F\}$ we have $\{F\} \equiv_{\Sigma} F$.

By this reason we can view an $n$-family $F$ as an $m$-family for $m > n$.

Recall that for the case $n = 0$ the standard notation is

$$Y \oplus A = \{2x : x \in Y\} \cup \{2x + 1 : x \in A\}.$$ 

If $Y$ is an arbitrary set and $F$ is an $n$-family, $n > 0$, then we define the join of $Y$ and $F$ inductively by letting

$$Y \oplus F = \{Y \oplus S : S \in F\}.$$ 

For an $n$-family $F$ and an integer $k$, denote by $F^k$ the $n$-family $\{k\} \oplus F$. Clearly, for every integer $k$ and an $n$-family $F$, we have $F \equiv_{\Sigma} F^k$. For $n$-families $F, G$ define the $n$-family

$$F \oplus G = F^0 \cup G^1.$$ 

It is easy to see that $F \leq_{\Sigma} F \oplus G$, $G \leq_{\Sigma} F \oplus G$, and

$$F \leq_{\Sigma} M, G \leq_{\Sigma} M \implies F \oplus G \leq_{\Sigma} M$$

for every structure $M$.

2 Jump and jump inversion on n-families

Example 1. ([Puzarenko 2009]). For a 0-family $A$ the jump $J(A)$ is $\Sigma$-equivalent to $M_{J(A)}$, where $J(A)$ is the the enumeration jump of $A$:

$$J(A) = K(A) \oplus \overline{K(A)}$$

and

$$K(A) = \{n : n \in \Phi_n(A)\},$$

where $\{\Phi_n\}_{n \in \omega}$ is an effective enumeration of the enumeration operators.
Example 2. It is easy to check that for the family InfCE of all infinite c.e. sets we have \( J(\text{InfCE}) \equiv J(J(\emptyset)) \equiv J(\emptyset) \). Indeed, \( \emptyset \) is computably isomorphic to \( \{ n : W_n \text{ is infinite} \} \), and a c.e. set \( W_n \) is infinite if and only if the (uniformly) computable set

\[ V_n = \{ s : W_{n,s} \neq W_{n,s+1} \} \]

is infinite, and so, if and only if \( F \subseteq V_n \) for some \( F \in \text{InfCE} \). The predicate \( F \subseteq V_n \) can be recognized by \( J(F) \).

The inverse reduction \( J(\text{InfCE}) \leq J(J(\emptyset)) \) is obvious.

Therefore, the family InfCE is a jump inversion of \( J(J(\emptyset)) \), i.e., \( J(\text{InfCE}) \equiv J(J(\emptyset)) \).

Proposition 7. The 1-family InfCE is the least jump inversion for the 0-family \( J(J(\emptyset)) \) among countable structures, i.e., \( J(J(\emptyset)) \leq \Sigma M \) implies \( \text{InfCE} \leq \Sigma M \).

Proof. Suppose \( J(J(\emptyset)) \leq \Sigma M \) for some countable \( M \). Then the index set \( \{ n : W_n \text{ is infinite} \} \) is \( \Sigma_2 \)-definable in \( \mathbb{HF}(M) \). Then there is \( \Delta_0 \)-formula \( \Phi \) such that

\[ W_n \text{ is infinite} \iff \mathbb{HF}(M) \models \exists a \forall b \Phi(n,a,b). \]

Then the sequence

\[ V_{n,a} = \begin{cases} W_n, & \text{if } \mathbb{HF}(M) \models \forall b \Phi(n,a,b); \\ \omega, & \text{otherwise,} \end{cases} \]

exhausting all infinite c.e. sets can be determined by the \( \Sigma \)-predicate

\[ x \in V_{n,a} \iff x \in W_n \lor x \in \omega \land \exists b \neg \Phi(n,a,b). \]

This allows us to prove the reducibility \( \text{InfCE} \leq \Sigma M \) for every countable \( M \) such that \( J(J(\emptyset)) \leq \Sigma M \).

Now, our goal is to extend Proposition 7 for arbitrary \( n \)-family \( \mathcal{F} \). For each \( n \)-family \( \mathcal{F} \), recursively define an \((n+1)\)-family \( \mathcal{E}(\mathcal{F}) \):

\[ \mathcal{E}(\mathcal{F}) = \begin{cases} \mathcal{K}_1 \cup \{ 2x \} : x \in A \}, & \text{if } n = 0 \text{ and } \mathcal{F} = A \subseteq \omega, \\ \mathcal{K}_{n+1} \cup \{ \mathcal{E}(S) : S \in \mathcal{F}^0 \}, & \text{if } n > 0, \end{cases} \]

where \( \mathcal{K}_1 = \{ 2x, 2x + 1 : x \in \omega \} \) and \( \mathcal{K}_{n+1} = \{ \mathcal{K}_n \} \). This is similar to some definitions that appear in [Kalimullin and Puzarenko 2009] and [Faizrahmanov and Kalimullin 2016 (a), (b)].

According to the following theorem we will call \( \mathcal{E}(\mathcal{F}) \) as the least \( \Sigma \)-jump inversion for \( \mathcal{F} \) (meaning that in fact it is an inversion of \( J(\emptyset) \oplus \mathcal{F} \)).
Theorem 8. For any $n$-family $\mathcal{F}$ the $(n + 1)$-family $\mathcal{E}(\mathcal{F})$ is the least jump inversion of $\mathcal{F}$. Namely,

1) $\mathcal{F} \leq_{\Sigma} \mathcal{J}(\mathcal{E}(\mathcal{F}))$;

2) for each countable structure $\mathcal{B}$ of a finite signature, $\mathcal{E}(\mathcal{F}) \leq_{\Sigma} \mathcal{B}$ if $\mathcal{F} \leq_{\Sigma} \mathcal{J}(\mathcal{B})$.

3) $\mathcal{J}(\mathcal{E}(\mathcal{F})) \leq_{\Sigma} \mathcal{J}(\emptyset) \oplus \mathcal{F}$.

Proof. 1) Since we can view each $n$-family as an $m$-family for $m > n$, without loss of generality we assume that $n > 0$. Let $\mathcal{A} = \mathcal{HF}(\mathcal{M}_{\mathcal{E}(\mathcal{F})})$.

It is easy to see that there is a $\Sigma_2$-formula $\Phi$ such that

$$\mathcal{A} \models \Phi(x_1, \ldots, x_n, m) \iff \exists i [R^A(x_n, i) \& I^A(i) \& A_i = \{2m\}] \iff$$

$$\exists i [R^A(x_n, i) \& I^A(i) \& 2m \in A_i \& 2m + 1 \notin A_i],$$

where each $A_i$, for $i \in I^n(x)$, is from the definition of $\mathcal{M}_{\mathcal{E}(\mathcal{F})}$. Then for the $\Sigma$-subset

$$E = \{(x_1, \ldots, x_n) : R^A(x_1, x_1) \& R^A(x_i, x_{i+1}) \text{ for } 1 \leq i < n\}$$

of $\mathcal{A}^n$ we will have

$$\mathcal{F} = \{\mathcal{F}^n_{\Phi(x), E(x)} : x \in \text{Pr}_1(E)\}.$$ 

Hence $\mathcal{F} \leq_{\Sigma} \mathcal{E}(\mathcal{F})$ so that $\mathcal{F} \leq_{\Sigma} \mathcal{J}(\mathcal{E}(\mathcal{F}))$.

2) Let an $n$-family $\mathcal{F}$ is $\Sigma$-reducible to $\mathcal{J}(\mathcal{B})$ for some structure $\mathcal{B}$. Hence $\mathcal{F}^0 \leq_{\Sigma} \mathcal{J}(\mathcal{B})$. Fix a $\Sigma_2$-subset $E \subseteq HF(\mathcal{B})$, $\Sigma_2$-formula $\Theta$ and a tuple $\bar{r} \in HF^m(\mathcal{B})$ such that

$$\mathcal{F}^0 = \{\mathcal{F}^n_{\Theta(\bar{r}, x), E(x)} : x \in \text{Pr}_1(E)\}.$$ 

Let $\Psi$ be a $\Delta_0$-formula such that the $\Sigma_2$-formula $\exists a \forall b \Psi(a, b, \bar{r}, x_1, \ldots, x_n, k)$ defines the $\Sigma_2$-predicate

$$\{(x_1, \ldots, x_n) \in E^n : \Theta(\bar{r}, x_1, \ldots, x_n, k)\}$$

in $\mathcal{HF}(\mathcal{B})$. Then there is a $\Sigma$-formula $\Phi$ such that for every $x_1, \ldots, x_n, a \in HF(\mathcal{B})$ and $k \in \omega$ we have $\mathcal{HF}(\mathcal{B}) \models \Phi(x_1, \ldots, x_n, (a, k), 2k)$ and

$$\mathcal{HF}(\mathcal{B}) \models \Phi(x_1, \ldots, x_n, (a, k), 2k+1) \iff \mathcal{HF}(\mathcal{B}) \models \exists b \Psi(a, b, \bar{r}, x_1, \ldots, x_n, k).$$

It is easy to see that for every $x_1, \ldots, x_n, a \in HF(\mathcal{B})$ and $k \in \omega$ we have

$$\mathcal{F}^n_{\Phi(x_1, \ldots, x_n, (a, k))} = \{2k\} \iff \mathcal{HF}(\mathcal{B}) \models \forall b \Psi(a, b, \bar{r}, x_1, \ldots, x_n).$$
Thus, $\mathcal{E}(\mathcal{F}) = \{\mathcal{F}^{n+1}_{x}(\mathcal{F}^n) : x \in \mathcal{P}_1(C)\} \cup \mathcal{N}_{n+1}$ for the $\Sigma$-set

$$C = \{(x_1, \ldots, x_n, (a, k)) \in \mathcal{H}_n(\mathcal{B}) : x_1, \ldots, x_n, a \in \mathcal{H}(\mathcal{B}), k \in \omega\}.$$ 

Therefore $\mathcal{E}(\mathcal{F}) \leq \Sigma \mathcal{B}$.

3) By Theorem 1 from [Stukachev 2009] there is a countable structure $\mathcal{B}$ such that $J(\emptyset) \oplus \mathcal{F} \equiv \Sigma \mathcal{J}(\mathcal{B})$. Since $\mathcal{F} \leq \Sigma \mathcal{J}(\mathcal{B})$ we have $\mathcal{E}(\mathcal{F}) \leq \Sigma \mathcal{B}$. Therefore, $\mathcal{J}(\mathcal{E}(\mathcal{F})) \leq \Sigma \mathcal{J}(\mathcal{B}) \leq \Sigma J(\emptyset) \oplus \mathcal{F}$. This ends the proof.

**Corollary 9.** For every pair of $\Sigma$-families $\mathcal{F}$ and $\mathcal{G}$

1. $\mathcal{F} \leq \Sigma \mathcal{G} \implies \mathcal{E}(\mathcal{F}) \leq \Sigma \mathcal{E}(\mathcal{G})$;

2. $\mathcal{E}(\mathcal{F} \oplus \mathcal{G}) \equiv \Sigma \mathcal{E}(\mathcal{F}) \oplus \mathcal{E}(\mathcal{G})$.

**Proof.** Part 1 follows from the fact that $\mathcal{F} \leq \Sigma \mathcal{G} \leq \Sigma \mathcal{E}(\mathcal{G})$. Part 2 follows from the fact that $\mathcal{E}(A \oplus B) = \mathcal{F}_1 \cup \{\{2x : x \in A \oplus B\} \cup \{\{4x\} : x \in A\} \cup \{\{4x+2\} : x \in B\} \equiv \Sigma \{X \oplus Y : X \in \mathcal{E}(A) \& Y \in \mathcal{E}(B)\} = \mathcal{E}(A) \oplus \mathcal{E}(B)$.

By the definition of $\mathcal{E}(\cdot)$ the least double jump inversion $\mathcal{E}^2(\mathcal{F}) = \mathcal{E}(\mathcal{E}(\mathcal{F}))$ of an $\mathcal{F}$-family $\mathcal{F}$ is an $(n+2)$-family. But we know from [Faizrahmanov and Kalimullin 2016 (a)] that under Turing reducibility of presentations of $\mathcal{F}$-families the least double jump is an $(n+1)$-family. For example, for the case of 0-family $A$ the least double jump $\mathcal{E}^2(A)$ has the same Turing degrees of presentations of $\mathcal{M}_{\mathcal{E}^2(A)}$ as the degrees of presentations of $\mathcal{M}_5$, where $\mathcal{G}$ is the 1-family

$$\mathcal{G} = \{F \subseteq \omega : F \text{ is finite}\} \cup \{\{x\} : x \in A\}.$$ 

Below we show that for the case of $\Sigma$-reducibility we can not have an equivalence between $\mathcal{E}^2(\mathcal{F})$ and some $(n+1)$-family even for $n = 0$.

**Theorem 10.** For a set $A$ and a 1-family $\mathcal{G}$ we have

$$\mathcal{J}(\mathcal{G}) \leq \Sigma J(\emptyset) \oplus \mathcal{E}(A) \implies \mathcal{J}(\mathcal{G}) \leq \Sigma J(\emptyset)$$

and, therefore, $\mathcal{J}(\mathcal{G}) \not\leq \Sigma J(\emptyset) \oplus \mathcal{E}(A)$. Thus, no 1-family can be a double jump inversion of $A$.

**Proof.** (Sketch) Let us look at the jump of $\mathcal{J}(\mathcal{G}) = \mathcal{J}(\mathcal{M}_5)$ for 1-families $\mathcal{G}$. Because of [Kalimullin and Puzarenko 2009], all $\Sigma$-predicates in $\mathcal{M}_5$ can be encoded in the sets

$$A_1 \oplus A_2 \oplus \cdots \oplus A_m \oplus E(\mathcal{G}),$$

where $A_i \in \mathcal{G}$ and the set $E(\mathcal{G}) = \{u : (\exists A \in \mathcal{G})[D_u \subseteq A]\}$ codes the $\exists$-theory of $\mathcal{M}_5$. But the family of enumeration jumps of these sets cannot fully represent the jump of the whole $\mathcal{G}$ since we need to keep the information when a jump for
a tuple $A_1, \ldots, A_m$ is an extension of the jump for a tuple $A_1, \ldots, A_m, A_{m+1}$. In fact, the jump $\mathcal{J}(\mathcal{S})$ (up to $\Sigma$-equivalence) can be viewed as a structure coding the jumps of the sets $A \in E(\mathcal{S}) \oplus \mathcal{S}$ extended by the similar coding of the jumps of elements of the $\oplus$-closure of $E(\mathcal{S}) \oplus \mathcal{S}$ with an additional binary operation which maps coding places of $J(X), J(Y)$ to the coding places of $J(X \oplus Y)$. Each coding instance should be generated by this binary operation from the instances coding jumps of the elements of $E(\mathcal{S}) \oplus \mathcal{S}$. The last instances should be marked by a special predicate. We omit technical details and a technical verification. Informally, such structure allows to compute all $\Sigma$-types in $\mathcal{M}_\mathcal{S}$, and, therefore, to build an isomorphic copy of the original $\mathcal{J}(\mathcal{S})$.

Suppose that $\mathcal{J}(\mathcal{S}) \leq \Sigma J(\emptyset) \oplus \mathcal{E}(A) = \{ J(\emptyset) \oplus \{ 2n, 2n + 1 \} : n \in \omega \} \cup \{ J(\emptyset) \oplus \{ 2n \} : n \in A \}$ as witnessed by some $\Sigma$-formula $\Phi$. For simplicity we assume that $\Phi$ has no parameters.

Note that the structure $\mathcal{M}_{J(\emptyset) \oplus \mathcal{E}(A)}$ is bi-embeddable with $\mathcal{M}_{J(\emptyset) \oplus \mathcal{H}_1} \leq \Sigma J(\emptyset)$, where

$$J(\emptyset) \oplus \mathcal{H}_1 = \{ J(\emptyset) \oplus \{ 2n, 2n + 1 \} : n \in \omega \}.$$  

Moreover, they are densely bi-embeddable in the sense that for every finite substructure $\mathcal{M}_0 \subseteq \mathcal{M}_{J(\emptyset) \oplus \mathcal{E}(A)}$ there is a substructure $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_{J(\emptyset) \oplus \mathcal{E}(A)}$ such that $\mathcal{M}_1 \cong \mathcal{M}_{J(\emptyset) \oplus \mathcal{H}_1}$, and vice versa. Considering the same formula $\Phi$ in $\mathcal{H}(\mathcal{M}_{J(\emptyset) \oplus \mathcal{H}_1})$ we get a structure $\mathcal{S} \leq \Sigma J(\emptyset)$ densely bi-embeddable with $\mathcal{J}(\mathcal{S})$. But $J(X) \subseteq J(Y)$ implies $J(X) = J(Y)$ so that this is possible only if $\mathcal{J}(\mathcal{S}) \equiv \mathcal{S}$. Hence, $\mathcal{J}(\mathcal{S}) \leq \Sigma J(\emptyset)$.

In the case when $\Phi$ has parameters we should change $\mathcal{H}_1$ by a 1-family in the form

$$\mathcal{H}_1 \cup \{ n_1 \} \cup \{ n_2 \} \cup \cdots \cup \{ n_k \}$$

for appropriate choice of $n_1, \ldots, n_k \in A$ (depending on the given parameters of $\Phi$) preserving the dense bi-embeddability property up to finitely many constants.

To prove the second part of the theorem suppose that $\mathcal{J}(\mathcal{S}) \equiv \Sigma J(\emptyset) \oplus \mathcal{E}(A)$. Then by the first part $\mathcal{J}(\mathcal{S}) \leq \Sigma \mathcal{J}(\emptyset)$. On the other hand, by Theorem 8

$$A \leq \Sigma \mathcal{J}(\mathcal{E}(A)) \leq \Sigma \mathcal{J}^2(\mathcal{S}) \leq \Sigma \mathcal{J}^2(\emptyset) \equiv \Sigma J^2(\emptyset),$$

so that $A \in \Sigma^0_3$.

Since $\mathcal{J}(\mathcal{E}^2(A)) \equiv \Sigma J(\emptyset) \oplus \mathcal{E}(A)$, by Theorem 8 we have also the following corollary:

**Corollary 11.** For a set a set $A \notin \Sigma^0_3$ there is no 1-family $\mathcal{S}$ such that $\mathcal{S} \equiv \Sigma \mathcal{E}^2(A)$, so that the least double jump inversion of a 0-family $A$ can not be replaced by a 1-family.
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