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On Completeness of Pseudosimple Sets

Vadim Bulitko
(Odessa State University, Ukraine
ul. Perekopskoj divizii 67-40
Odessa 270062, UKRAINE
bulitko.odessa@REX.iasnet.com)

Abstract: The paper contains completeness criterions for pseudosimple sets. Those criterions are constructed using effectivization of the definitions as well as extensionally bounded functions.

Key Words: Completeness, Pseudosimple set, Effectivization, Extensionally bounded function

Category: G

A recursively enumerable (r.e.) set is called complete (Turing-complete) if any r.e. set can be reduced to that set using Turing reducibility. Optionally complete sets can have some interesting properties, for instance they can be simple, pseudosimple, etc. Problems of constructing completeness criterions for simple sets by an effectivization of definitions have been investigated by McLaughlin, Smullyan, Lachlan, Arslanov, etc. History and systematical description of that can be found in [Arslanov 87].

Simple sets fall into class $\mathcal{E}_1$ of r.e. set classification by Uspenskij and Dekker-Myhill [Rogers 67]. For creative sets ($\mathcal{E}_4$) the criterions have been constructed even before. However we do not know any specific criterion for pseudosimple ($\mathcal{E}_2$) and pseudocreative ($\mathcal{E}_3$) sets. In this paper we propose some completeness criterions for pseudosimple sets.

Most of our notation follows [Rogers 67] also *r.* means *recursive*, *r.e.* means *recursively enumerable* and *i.r.e.* means *infinite recursively enumerable*. We use $[a,b]$ for $\{x \in N \mid a \le x \le b\}$ and $\rightleftharpoons$ as "$\Leftrightarrow$ under definition". Sometimes we use "number of" as "index of". Let us recall the following definitions from [Rogers 67]:

Definition 1. A is *pseudosimple* $\rightleftharpoons$ (1) A is r.e. but not r.; (2) $\left(\exists i.r.e. C \subset \overline{A}\right)$ $\forall W_z \left[W_z \subset \left(\overline{A} \backslash C\right) \Rightarrow |W_z| < \infty\right]$. Such C we will denote C^A.

Definition 2. A is *pseudosimple with center* C $\rightleftharpoons$ (1) A is pseudosimple; (2) $\left(\exists i.r.e. C \subset \overline{A}\right) \forall W_z \left[W_z \subset \overline{A} \Rightarrow |W_z \backslash C| < \infty\right]$.

Above definitions allow the following effectivization:

Definition 1′. A is *weakly effectively pseudosimple* $\rightleftharpoons$ (1) A is pseudosimple; (2) $(\exists \text{ total } f \le_T A) \forall W_z \left[W_z \subset \left(\overline{A} \backslash C^A\right) \Rightarrow |W_z| \le f(z)\right]$.

Definition 2′. A is *weakly effectively pseudosimple with center* C $\rightleftharpoons$ (1) A is pseudosimple with center C; (2) $(\exists \text{ total } f \le_T A) \forall W_z [W_z \subset \overline{A} \Rightarrow |W_z \backslash C| \le f(z)]$.

First prove two complementary lemmas.

Lemma 1. For any complete set A and any r.e. set M the following function is total and recursive in A:

$$f(z) = \begin{cases} 1, W_z \cap M \ne \emptyset, \\ 0, W_z \subset \overline{M}. \end{cases}$$

Proof. $\{x \mid W_x \cap M \neq \emptyset\} \leq_T A$ because $\{x \mid W_x \cap M \neq \emptyset\}$ is an r.e. set. Therefore f is computable with oracle A.□

Lemma 2. Let A be complete and g be partial recursive in A. Then the following function is partial recursive in A:

$$\phi(z) = \begin{cases} \left|W_{g(z)}\right|, g(z)\downarrow \; and \left|W_{g(z)}\right| < \infty, \\ \uparrow, otherwise. \end{cases}$$

Proof. Let us make computation of ϕ as follows. Given z we start computation of $g(z)$ and if it ends, we enumerate numbers of r.e. sets $W_{g(z)} \cap [0, \infty]$, $W_{g(z)} \cap [1, \infty]$, ..., $W_{g(z)} \cap [i, \infty]$, ... until some of them fall into $\{x \mid W_x = \emptyset\}$ which is recursive in A. Let it happen when $i = n$ so we set $\phi(z)$ equal to $\left|W_{g(z)} \cap [0, n \Leftrightarrow 1]\right|$. Note we use uniform recursiveness of W_i in respect to complete A.□

Theorem 1. (A is pseudosimple and complete) $\Leftrightarrow$ (A is weakly effectively pseudosimple and $C^A \leq_T A$).

Proof. $\Rightarrow$. Let A be pseudosimple and complete then obviously $C^A \leq_T A$. Define total f as follows:

$$f(z) = \begin{cases} 0, W_z \cap \left(C^A \cup A\right) \neq \emptyset, \\ \left|W_z\right|, W_z \subset \left(\overline{A} \backslash C^A\right). \end{cases}$$

f is computable with oracle A by lemma 1,2, and because $A \cup C^A$ is an r.e. set, and $\overline{A} \backslash C^A$ is immune. Thus A is weakly effectively pseudosimple.

$\Leftarrow$. Using f, which is recursive in A, and possibility to write out elements of $\overline{A} \backslash C^A$ in increasing order with oracle A, construct initial segment $\left\{\alpha_0, \alpha_1, \ldots, \alpha_{f(z)}\right\} = X(z)$ of enumeration of $\overline{A} \backslash C^A$ in increasing order. An r.e. index of $X(z)$ can be computed with oracle A because $X(z)$ is finite and recursive in A. Let $g(z)$ be that r.e. index. So $g \leq_T A$ and $\forall z \left[W_z \neq W_{g(z)}\right]$. Thus A is complete by the Arslanov theorem [Arslanov 87]. (The Arslanov theorem used here and below states that for any r.e.s. A [(A is complete) $\Leftrightarrow$ ($\exists$ total $f \leq_T A$ $\forall z[W_z \neq W_{f(z)}]$)]).□

Another completeness criterion can be obtained introducing concept of special function.

Definition 3. Let A be weakly effectively pseudosimple. We call total and recursive in A function f *special* for A if

$$f(z) = \begin{cases} \left|W_z\right| + t, W_z \subset (\overline{A} \backslash C^A), t \geq 1, \\ 0, otherwise. \end{cases}$$

Theorem 2. (A is pseudosimple and complete) $\Leftrightarrow$ (A is weakly effectively pseudosimple and there is f special for A).

Proof. $\Rightarrow$. Define f as follows:

$$f(z) = \begin{cases} 0, W_z \cap \left(C^A \cup A\right) \neq \emptyset, \\ \left|W_z\right| + 1, W_z \subset \left(\overline{A} \backslash C^A\right). \end{cases}$$

f can be computed with oracle A by lemmas 1,2, and because $A \cup C^A$ is an r.e. set, and $\overline{A} \backslash C^A$ is immune. So A is weakly effectively pseudosimple with special function f.

$\Leftarrow$. Let g be defined as follows:

$$W_{g(z)} = \begin{cases} \{a\}\,, \text{where } a \in \left(\overline{A} \backslash C^A\right) \text{ if } f(z) = 0, \\ \{b\}\,, \text{where } b \in A \text{ if } f(z) \neq 0. \end{cases}$$

g can be computed with oracle A because f is recursive in A. Also g is total and obviously does not have fixed points so we can apply the Arslanov theorem that causes A is complete.□

There is also another approach to effectivization of the definitions. The approach is based on use of functions evaluating complexity characteristics of set other than the number of elements. For instance, Lachlan in [Lachlan 68] used the number of value changes of characteristic function for a given r.e. set on a cohesive set building sufficient conditions for complete maximal sets. Completeness criterion for maximal sets [Bulitko 92] constructed using Kolmogorov complexity of set initial segments is another example. Next step in that direction is use of extensionally bounded functions to construct completeness criterions for special classes of simple sets. That approach has been posed by V.K.Bulitko in [Bulitko 95]. In present paper we use that approach to describe pseudosimple and complete sets.

Definition 4. Total f is *extensionally bounded (e.b.)* $\rightleftharpoons \forall W_x \exists c \forall W_z$ $[W_z = W_x \Rightarrow f(z) \leq c]$.

Theorem 3. (A is pseudosimple and complete) $\Leftrightarrow$ (A is r.e. but not r. & $\exists i.r.e.C^A \subset \overline{A}$ such that the following function f :

$$f(z) = \begin{cases} |W_z \cap [0,z]| + 1, W_z \subset (\overline{A} \backslash C^A), \\ 0, otherwise, \end{cases}$$

is e.b. and recursive in A).

Proof. $\Rightarrow$. Obviously A is r.e. but not r. f is computable with oracle A by lemma 1 and because $\overline{A} \backslash C^A$ is immune. For any number i of r.e. $X \subset \left(\overline{A} \backslash C^A\right)$ $f(i)$ is bounded by cardinality of X that is finite because A is pseudosimple. f is equal to 0 for a number of an r.e. set which is not a subset of $\overline{A} \backslash C^A$. Therefore f is e.b.

$\Leftarrow$. Assume $\exists W_z \subset \left(\overline{A} \backslash C^A\right) [|W_z| = \infty]$. Then taking in account infiniteness of W_z index set we get f is not bounded on W_z index set. That contradicts with the definition of e.b. function. Therefore our assumption is wrong and A is pseudosimple. Completeness of A can be proved in the same way as in theorem 2.□

Let us consider pseudosimple sets with center.

Theorem 4. (A is pseudosimple with recursive center C and complete) $\Leftrightarrow$ (A is weakly effectively pseudosimple with recursive center C).

Proof. $\Rightarrow$. Define f as follows:

$$f(z) = \begin{cases} 0, (W_z \cap A) \neq \emptyset, \\ |W_z \backslash C|\,, W_z \subset \overline{A}. \end{cases}$$

f is computable with oracle A by lemmas 1,2 and $\left(\forall W_z \subset \overline{A}\right) [|W_z \backslash C| < \infty]$. Therefore A is weakly effectively pseudosimple with recursive center.

$\Leftarrow$. We can get total function $g \leq_T A$ that does not have fixed points doing in the same way as in the first theorem. Using the Arslanov theorem we obtain completeness of A.□

Situation regarding case of non-recursive (in general) center is described by the following theorems and remarkable enough. Trying to get the criterion we succeeded in way of using e.b functions *only*.

Theorem 5. (A is weakly effectively pseudosimple with center C recursive in A) $\Rightarrow$ (A is pseudosimple with center C and complete).

Proof. Follows from the first theorem.□

Definition 5. Let A be weakly effectively pseudosimple with center C. Call total and computable with oracle A function f bounding for A if

$$f(z) = \begin{cases} |W_z \backslash C| + t, W_z \subset \overline{A}, t \geq 1, \\ 0, otherwise. \end{cases}$$

Theorem 6. (A is weakly effectively pseudosimple with center C and there is bounding for A function f) $\Rightarrow$ (A is pseudosimple with center C and complete).

Proof. Define g as follows:

$$W_{g(z)} = \begin{cases} \{a\}, \text{where } a \in \overline{A} \text{ if } f(z) = 0, \\ \{b\}, \text{where } b \in A \text{ if } f(z) \neq 0. \end{cases}$$

g can be computed with oracle A because f is recursive in A. Also g is total and does not have fixed points. Thus A is complete by the Arslanov theorem.□

Theorem 7. (A is pseudosimple with center C and complete) $\Leftrightarrow$ (A is r.e. but not r.; $\exists i.r.e.C \subset \overline{A}$; the following f is e.b. and computable with oracle A):

$$f(z) = \begin{cases} |W_z \backslash C \cap [0, z]| + 1, W_z \subset \overline{A}, \\ 0, otherwise. \end{cases}$$

Proof. $\Rightarrow$. Obviously A is r.e. but not r. f is computable with oracle A by lemma 1 and because having oracle A we are able to write out all of W_z elements not belonging to C and situated in $[0, z]$. f is e.b. because f is bounded on numbers of r.e. $X \subset \overline{A}$ by $|X \backslash C|$, which is finite because A is pseudosimple with center, and f is equal to 0 on numbers of an r.e. set which is not a subset of $\overline{A}$.

$\Leftarrow$. A is pseudosimple with center C because f is e.b. and therefore $(\forall W_z \subset \overline{A})$ $[|W_z \backslash C| < \infty]$ (otherwise we would get unbounded f taking in account that any r.e. set has an infinite number of its indexes). Completeness of A can be proved in the same way as in theorem 6.□

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