The Riesz Representation Operator on the Dual of $C[0; 1]$ is Computable

Tahereh Jafarikhah
(University of Tarbiat Modares, Tehran, Iran
t.jafarikhah@modares.ac.ir)

Klaus Weihrauch
(University of Hagen, Hagen, Germany
Klaus.Weihrauch@FernUni-Hagen.de)

Abstract: By the Riesz representation theorem, for every linear functional $F : C[0; 1] \to \mathbb{R}$ there is a function $g : [0; 1] \to \mathbb{R}$ of bounded variation such that

$$F(h) = \int h \, dg \quad (h \in C[0; 1]).$$

A computable version is proved in [Lu and Weihrauch(2007)]: a function $g$ can be computed from $F$ and its norm, and $F$ can be computed from $g$ and an upper bound of its total variation. In this article we present a much more transparent proof. We first give a new proof of the classical theorem from which we then can derive the computable version easily. As in [Lu and Weihrauch(2007)] we use the framework of TTE, the representation approach for computable analysis, which allows to define natural concepts of computability for the operators under consideration.

Key Words: computable analysis, Riesz representation theorem
Category: F.0, F.1.1

1 Introduction

The Riesz representation theorem for continuous functionals on $C[0; 1]$, the Banach space of continuous functions $h : [0; 1] \to \mathbb{R}$ endowed with the supremum norm, can be stated as follows [Goffman and Pedrick(1965), Heuser(2006)]:

**Theorem 1 (Riesz representation theorem).** For every continuous linear operator $F : C[0; 1] \to \mathbb{R}$ there is a function $g : [0; 1] \to \mathbb{R}$ of bounded variation such that

$$F(h) = \int h \, dg \quad (h \in C[0; 1])$$

and

$$V(g) = \|F\|.$$

Here, $\int h \, dg$ is the Riemann-Stieltjes integral [Schechter(1997)]. The reversal of this theorem is almost trivial: the operator $h \mapsto \int h \, dg$ is continuous and linear.
A computable version of the Riesz representation theorem has been proved in [Lu and Weihrauch(2007)]: a function $g$ can be computed from $F$ and its norm, and $F$ can be computed from $g$ and an upper bound of its total variation. This proof, however, is complicated and partly intransparent. In this article we present a simpler and much more transparent proof which starts with a new proof of the classical theorem from which the computable version can be derived easily.

The classical Riesz representation theorem can be proved as follows [Goffman and Pedrick(1965), Heuser(2006)]: By the Hahn-Banach theorem, the operator $F$ has a continuous extension $F$ to the Banach space $B[0;1]$ of bounded functions $h : [0;1] \to \mathbb{R}$ such that $\|F\| = \|F\|$. Then define $g$ by $g(x) := F(\chi_{[0;x])}$, where $\chi_{[0;x]}$ is the characteristic function of $[0;x]$. In our proof, from $F$ and $\|F\|$ we define a dense set of points $x$ in which $g$ will be continuous. For these points $x$ we can compute $F$ to $\chi_{[0;x]}$, then we define $g(x) := F(\chi_{[0;x])}$.

In Section 2 we extend the definition of the Variation and the Riemann-Stieltjes integral to partial functions $g : \subseteq [0;1] \to \mathbb{R}$ the domains of which are dense in the unit interval. We observe that $\int h dg$ can be defined already from any restriction of $g$ to a countable dense subset of it domain.

In Section 3 we introduce the set $PC_F$ of the points $x$ which do not contribute to $\|F\|$ and define $F(\chi_{[0;x]}$) as the limit of $F(h_i)$ where $(h_i)_i$ is a sequence of continuous functions "converging" to $\chi_{[0;x]}$. We prove that $g_F$ is continuous with no continuous proper extension, and that its total variation is $\|F\|$. Furthermore, $F(h) = \int h dg_F$ for all continuous functions $f : [0;1] \to \mathbb{R}$.

In Section 4 we shortly summarize the computability concepts used in the following. In particular we define our representation of the functions with countable dense domain and finite variation.

Finally, in Section 5 we prove that from $F$ and $\|F\|$ a restriction $g$ of $g_F$ can be computed (a function of bounded variation representing $F$), and that $F$ can be computed from $g$ and a upper bound of $\text{Var}(g)$.

## 2 The Riemann-Stieltjes integral

We recall the definition of the Riemann-Stieltjes integral. We study only the special case of functions on the unit interval $[0;1]$. Results for arbitrary intervals $[a;b]$ can be derived easily from the special case. In our context it seems to be appropriate to generalize the definitions to partial functions $g : \subseteq [0;1] \to \mathbb{R}$ of bounded variation.

A partition of the real interval $[0;1]$ is a sequence $Z = (x_0, x_1, \ldots, x_n)$, $n \geq 1$, of real numbers such that $0 = x_0 < x_1 < \ldots < x_n = 1$. The partition $Z$ has precision $k$, if $x_i - x_{i-1} < 2^{-k}$ for $1 \leq i \leq n$. A partition $Z' = (x'_0, x'_1, \ldots, x'_m)$, is finer than $Z$, if $\{x_0, x_1, \ldots, x_n\} \subseteq \{x'_0, x'_1, \ldots, x'_m\}$. $Z$ is a partition for $g : \subseteq [0;1] \to \mathbb{R}$ if $\{x_0, x_1, \ldots, x_n\} \subseteq \text{dom}(g)$. For a partition $Z$ for $g$ define
\[ S(g, Z) := \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|. \] (1)

The \textit{variation} of \( g \) is defined by
\[ V(g) := \sup \{ S(g, Z) | Z \text{ is a partition for } g \}. \] (2)

The function \( g \) is of \textit{bounded variation} if its variation \( V(g) \) is finite.

\textbf{Definition 2.} Let \( BV \) be the set of (partial) functions \( g : \subseteq [0;1] \rightarrow \mathbb{R} \) of bounded variation such that \( \{0,1\} \subseteq \text{dom}(g) \) and \( \text{dom}(g) \) is dense in \([0;1] \).

The relation to the usual definitions with total functions \( g \) is given by the following lemma.

\textbf{Lemma 3.}  
\begin{enumerate}
  \item Let \( g, g' \in BV \) such that \( g \) is a restriction of \( g' \). Then \( V(g) \leq V(g') \).
  \item For every function \( g \in BV \) the extension \( \overline{g} : [0;1] \rightarrow \mathbb{R} \) defined by
    \[ \overline{g}(x) := \lim_{y \rightarrow x} g(y) \text{ for } x \notin \text{dom}(g) \] (3)
    is of bounded variation such that \( V(g) = V(\overline{g}) \).
\end{enumerate}

\textbf{Proof:} (1) Obvious.

(2) Suppose this limit from below does not exist. Then there is an increasing sequence \( (y_i)_i \) converging to \( x \) such that the sequence \( (g(y_i))_i \) does not converge, hence there is some \( \varepsilon > 0 \) such that \( \forall i \exists j > i \) \( |g(y_i) - g(y_j)| > \varepsilon \). Therefore, for every \( n \) there is some partition \( Z_n = (0, y_{i_0}, y_{i_1}, \ldots, y_{i_n}, 1) \) for \( g \) such that \( S(g, Z_n) > n \cdot \varepsilon \). But \( g \) is of bounded variation, hence \( \overline{g}(x) \) exists.

Since \( \text{dom}(g) \subseteq \text{dom}(\overline{g}) \), \( V(g) \leq V(\overline{g}) \). On the other hand suppose \( X := (0 = x_1, x_2, \ldots, x_n = 1) \) is a partition for \( \overline{g} \) and let \( \varepsilon > 0 \). For \( 1 \leq i \leq n \) there are \( y_i \in \text{dom}(g) \) such that \( x_{i-1} < y_i < x_i \) and \( |g(y_i) - \overline{g}(x_i)| < \varepsilon/(2n) \), hence for \( Y := (0, y_1, y_2, \ldots, y_n, 1) \), \( |S(\overline{g}, X) - S(g, Y)| < \varepsilon \). Therefore, \( V(\overline{g}) \leq V(g) \). \( \square \)

On the space \( C[0;1] \) of continuous functions \( h : [0;1] \rightarrow \mathbb{R} \) the norm is defined by \( ||h|| := \sup_{x \in [0;1]} |h(x)| \). On the space \( C'[0;1] \) of the linear continuous operators \( F : C[0;1] \rightarrow \mathbb{R} \) the norm is defined by \( ||F|| := \sup_{||h|| \leq 1} |F(h)| \).

In the following let \( h : [0;1] \rightarrow \mathbb{R} \) be a continuous function and let \( g \in BV \). For any partition \( Z = (x_0, x_1, \ldots, x_n) \) of \([0;1]\) for \( g \) define
\[ S(g, h, Z) := \sum_{i=1}^{n} h(x_i)(g(x_i) - g(x_{i-1})). \] (4)
Since \( h \) is continuous and its domain is compact, it has a (uniform) modulus of continuity, i.e., a function \( m : \mathbb{N} \rightarrow \mathbb{N} \) such that \( |h(x) - h(y)| \leq 2^{-k} \) if \( |x - y| \leq 2^{-m(k)} \). We may assume that the function \( m \) is non-decreasing.

**Lemma 4 [Lu and Weihrauch(2007)].** Let \( h : [0;1] \rightarrow \mathbb{R} \) be a continuous function with modulus of continuity \( m : \mathbb{N} \rightarrow \mathbb{N} \) and let \( g \in \text{BV} \). Then there is a unique number \( I \in \mathbb{R} \) such that
\[
|I - S(g, h, Z)| \leq 2^{-k}V(g)
\]
for every partition \( Z \) for \( g \) with precision \( m(k+1) \).

A proof is given in [Lu and Weihrauch(2007)]. A revised proof is given in the appendix.

**Definition 5.** The number \( I \) from Lemma 4 is called the Riemann-Stieltjes integral and is denoted by \( \int h \, dg \).

Notice that by Lemma 4 the integral \( \int f \, dg \) is determined already by the values of the function \( g \) on an arbitrary set \( X \) that is dense in \( \text{dom}(g) \), since there are partitions of arbitrary precision that contain of points only from the set \( X \).

**Corollary 6.** Let \( g, g' \in \text{BV} \). Suppose \( A \subseteq \text{dom}(g) \cap \text{dom}(g') \) is dense in \( [0;1] \) such that \( \{0,1\} \subseteq A \) and \( (\forall x \in A)g(x) = g'(x) \). Then \( \int h \, dg = h \, dg' \) for every \( h \in C[0;1] \).

**Proof:** Obvious. \( \square \)

### 3 Another proof of the classical theorem

In this section we present a proof of the (non-computable) Riesz representation theorem which we will effectivize in Section 5. Let \( P_{g} \) be the (countable) set of polygon functions \( h : [0;1] \rightarrow \mathbb{R} \) with rational vertices and let \( R_{I} := \{(a;b) \mid a, b \in \mathbb{Q}, 0 \leq a < b \leq 1 \} \) be the set of open rational subintervals of \( (0;1) \). By the Weierstraß approximation theorem \( P_{g} \) is dense in \( C[0;1] \). In the following let \( F : C[0;1] \rightarrow \mathbb{R} \) be a linear continuous functional.

**Definition 7.** For \( h \in C[0;1], Y \subseteq [0;1] \), and \( x \in (0;1) \) define \( \text{NZ}(h), \|F\|_{Y} \) and \( \text{PC}_{F} \subseteq (0;1) \) as follows:
\[
\text{NZ}(h) := \{x \in [0;1] \mid h(x) \neq 0\}, \quad \|F\|_{Y} := \sup\{\|F(h)\| \mid h \in C[0;1], \|h\| \leq 1, \text{NZ}(h) \subseteq Y\}, \quad x \in \text{PC}_{F} : \iff \inf\{\|F\|_{J} \mid x \in J \in R_{I}\} = 0.
\]
Lemma 8. 1. \( \|F\|_Y \leq \|F\|_Z \) if \( Y \subseteq Z \),
2. \( \|F\|_{J_1 + \ldots + J_n} \leq \|F\| \) if the \( J_i \) are pairwise disjoint.
3. \( |F(h_1)| + \ldots + |F(h_n)| \leq \|F\| \) if \( |h_i| \leq 1 \) for \( i = 1, \ldots, n \) and the sets \( \text{NZ}(h_i) \) are pairwise disjoint.

Proof: (1) Obvious.
(2) Let \( \varepsilon > 0 \). For \( i = 1, \ldots, n \) there is a continuous function \( h_i \) such that \( |h_i| \leq 1 \), \( \text{NZ}(h_i) \subseteq J_i \) and \( |F(h_i)| \geq \|F\|_{J_i} - \varepsilon \). We may assume \( F(h_i) \geq 0 \) (if \( F(h_i) < 0 \), replace \( h_i \) by \( -h_i \)). Since the sets \( \text{NZ}(h_i) \) are pairwise disjoint, \( \sum_i |h_i| \leq 1 \). We obtain
\[
\sum_i |F|_{J_i} \leq n\varepsilon + \sum_i |F(h_i)| = n\varepsilon + \sum_i F(h_i) = n\varepsilon + F(\sum_i h_i) \leq n\varepsilon + \|F\|.
\]
This is true for all \( \varepsilon > 0 \), hence \( \sum_i \|F|_{J_i} \leq \|F\| \).
(3) This follows from (2). \( \square \)

At most countably many points can have a positive contribution to \( \|F\| \).

Lemma 9. The complement \((0;1) \setminus \text{PC}_F \) of \( \text{PC}_F \) is at most countable.

Proof: For \( n \in \mathbb{N} \) let \( T_n \) be the set of all \( x \in (0;1) \) such that \( \inf\{\|F\|_{J} \mid x \in J\} > 2^{-n} \). Suppose, \( \text{card}(T_n) \geq N > 2^n \cdot \|F\| \). Then there are \( N \) points \( x_1, \ldots, x_N \in T_n \) and pairwise disjoint intervals \( J_1, \ldots, J_N \) such that \( x_i \in J_i \). Since \( \|F\|_{J_i} > 2^{-n} \) for all \( i \), \( \sum_i \|F\|_{J_i} > N \cdot 2^{-n} > \|F\| \). But this is false by Lemma 8. Therefore, \( T_n \) is finite for every \( n \) and \( (0;1) \setminus \text{PC}_F = \bigcup_n T_n \) is at most countable. \( \square \)

We define slanted step functions (Figure 2) as approximations of characteristic functions \( \chi_{[0;1]} \).

Definition 10. For \( I = (a; b) \in \text{RI} \) let \( s_I \in \text{Pg} \), the slanted step function at \( I \), be the polygon function whose graph has the vertices \((0,1), (a,1), (b,0), \) and \((1,0)\).

Suppose \( J, K \subseteq L \). Then \( \text{NZ}(s_J - s_K) \subseteq L \) and \( \|s_J - s_K\| \leq 1 \), hence \( |F(s_J) - F(s_K)| = |F(s_J - s_K)| \leq \|F\|_L \), therefore
\[
|F(s_J) - F(s_K)| \leq \|F\|_L \text{ if } J, K \subseteq L.
\] (8)
In the classical proof (Section 1) \( g(x) \) can be defined as \( \mathcal{T}(\chi_{[0;1]}) \), where \( \mathcal{T} \) is the Hahn-Banach extension of \( F \) to the bounded real functions. We replace this definition as follows considering only points of continuity:

**Definition 11.** Define a function \( g_F : \subseteq \mathbb{R} \to \mathbb{R} \) as follows: \( \text{dom}(g_F) := \{0, 1\} \cup \text{PC}_F \), \( g(0) := 0, \ g(1) := F(1) \). For \( x \in \text{PC}_F \) let \( (J_n)_{n \in \mathbb{N}} \) be a sequence of rational intervals such that \( x \in J_{n+1} \subseteq J_n \) and \( \lim_{n \to \infty} \text{length}(J_n) = 0 \). Then let \( g_F(x) := \lim_{n \to \infty} F(s_{J_n}) \).

Since \( x \in \text{PC}_F \), \( \lim_{n \to \infty} \|F\|_{J_n} = 0 \) by monotonicity in \( J \) of \( \|F\|_J \). We show that \( g_F(x) \) exists and does not depend on the specific sequence \( (J_n)_{n \in \mathbb{N}} \).

**Lemma 12.** The function \( g_F \) is well-defined.

**Proof:** For every \( \varepsilon > 0 \) there is some \( n \) such that \( \|F\|_{J_n} < \varepsilon \). By (8) for \( k > n \), \( |F(s_{J_k}) - F(s_{J_n})| \leq \|F\|_{J_n} < \varepsilon \), hence \( \lim_{n \to \infty} F(s_{J_n}) \) exists.

Let \( (L_n)_{n \in \mathbb{N}} \) be another sequence of rational intervals such that \( x \in L_{n+1} \subseteq L_n \) and \( \lim_{n \to \infty} \|F\|_{J_n} = 0 \). Then \( \lim_{n \to \infty} F(s_{L_n}) \) exists accordingly. Let \( K_n := J_n \cap L_n \). By (8), \( |F(s_{J_n}) - F(s_{K_n})| \leq \|F\|_{J_n} \) and \( |F(s_{L_n}) - F(s_{K_n})| \leq \|F\|_{L_n} \), hence \( |F(s_{J_n}) - F(s_{L_n})| \leq \|F\|_{J_n} + \|F\|_{L_n} \). Therefore, \( \lim_n |F(s_{J_n}) - F(s_{L_n})| = 0 \) and finally \( \lim_n F(s_{J_n}) = \lim_n F(s_{L_n}) \). \( \square \)

**Lemma 13.** Suppose \( J, K, L \in \text{RI} \), \( J, K \subseteq L \) and \( x, y \in \text{PC}_F \cap L \). Then

\[
|F(s_J) - F(s_K)| \leq \|F\|_L, \quad (9)
\]
\[
|F(s_J) - g_F(y)| \leq \|F\|_L, \quad (10)
\]
\[
|g_F(x) - g_F(y)| \leq \|F\|_L. \quad (11)
\]

**Proof:**

(9): By (8).

(10): For every \( \varepsilon > 0 \) there is some \( K \subseteq L \) such that \( y \in K \) and \( |F(s_K) - g_F(y)| \leq \varepsilon \). Then by (9), \( |F(s_J) - g_F(y)| \leq |F(s_J) - F(s_K)| + |F(s_K) - g_F(y)| \leq \|F\|_L + \varepsilon \). Therefore \( |F(s_J) - g_F(y)| \leq \|F\|_L \).

(11): For every \( \varepsilon > 0 \) there is some \( J \subseteq L \) such that \( x \in J \) and \( |F(s_J) - g_F(x)| \leq \varepsilon \). Then by (10), \( |g_F(x) - g_F(y)| \leq |g_F(x) - F(s_J)| + |F(s_J) - g_F(y)| \leq \|F\|_L + \varepsilon \). Therefore \( |g_F(x) - g_F(y)| \leq \|F\|_L \). \( \square \)

We will prove some further properties of the function \( g_F \). In the following, \( \lim_{y \searrow x} g_F(y) \) abbreviates \( \lim_{y \in \text{dom}(g_F)} \), \( y \searrow x \) \( g_F(y) \) and \( \lim_{y \searrow x} g_F(y) \) abbreviates \( \lim_{y \in \text{dom}(g_F)} \), \( y \searrow x \) \( g_F(y) \).

**Lemma 14.** For all \( x \in (0; 1) \\
1. \lim_{y \searrow x} g_F(y) \) and \( \lim_{y \searrow x} g_F(y) \) exist,
2. \( |\lim_{y \searrow x} g_F(y) - \lim_{y \searrow x} g_F(y)| = \inf_{x \in J} \|F\|_J \).
Proof:

(1) Suppose that \( \lim_{y \to x} g_F(y) \) does not exist. Then there is an increasing sequence \((y_i)_i\) from \(\text{PC}_F\) converging to \(x\) such that the sequence \((g_F(y_i))_i\) does not converge, hence there is some \(\varepsilon > 0\) such that \((\forall N) (\exists i, j > N) |g_F(y_i) - g_F(y_j)| > \varepsilon\). Therefore, for every \(N\) we can find \(y_{i_0} < \ldots < y_{i_{2N}}\) from the sequence \((y_i)_i\) such that \(|g_F(y_{i_{2k}}) - g_F(y_{i_{2k}-1})| > \varepsilon\), for \(1 \leq k \leq N\). Hence there are pairwise disjoint rational intervals \(J_1, J_2, \ldots, J_N\) such that \(y_{i_{2k}-1}, y_{i_{2k}} \in J_k\) for \(1 \leq k \leq N\). Then by (11), \(|F||J_k| > \varepsilon\) for each \(1 \leq k \leq N\). By Lemma 8, \(|F||J| > N \varepsilon\). Since this is true for all numbers \(N\), \(|F||\) is unbounded. Contradiction.

(2) Let \(a = \inf_{x \in J} \|F\|_J\) and \(\delta > 0\). There is some \(J \in \text{RI}\) such that
\[
x \in J \quad \text{and} \quad |\|F\|_J - a| < \delta.
\]

“\(\leq\)”: By (11) and (12) for \(y_1, y_2 \in J \cap \text{PC}_F\), \(|g_F(y_1) - g_F(y_2)| \leq |\|F\|_J - a| + \delta\), hence \(|\lim_{y \to x} g_F(y) - \lim_{y \to x} g_F(y)| \leq a + \delta\). Since this is true for all \(\delta > 0\), “\(\leq\)” is true.

“\(\geq\)”: An example of the functions, intervals etc. defined in the following is shown in Figure 1. There is a rational polygon \(h\) such that
\[
\text{NZ}(h) \subseteq J, \quad \|h\| \leq 1 \quad \text{and} \quad |F(h) - \|F\|_J| < \delta.
\]

The function \(h\) can be chosen such that
\[
K \subseteq J; \quad x \in K \quad \text{and} \quad (\forall y \in K) h(y) = c
\]
for some \(K \in \text{RI}\) and some \(c\) such that \(0 < |c| \leq 1\). We may assume \(0 < c \leq 1\) if \(c < 0\) replace \(h\) by \(-h\) and \(y \in K \cap \text{PC}_F\), \(y_0 < x < y_\delta\) such that
\[
|\lim_{y \to x} g_F(y) - g_F(y_0)| < \delta \quad \text{and} \quad |\lim_{y \to x} g_F(y) - g_F(y_\delta)| < \delta.
\]

There are \(L, R \in \text{RI}\) such that \(L, R \subseteq K\), \(L < x < R\), \(y_0 \in L\), \(y_\delta \in R\) and
\[
|\|F\|_L| < \delta \quad \text{and} \quad |\|F\|_R| < \delta.
\]

Let \(m_L\) and \(m_R\) be the center of \(L\) and \(R\) respectively. Let \(t_L : [0;1] \to \mathbb{R}\) be the rational polygon whose graph has the vertices \((0,0), (\inf L, 0), (m_L, c), (\sup L, 0), (1,0)\) and let \(t_R : [0;1] \to \mathbb{R}\) be the rational polygon whose graph has the vertices \((0,0), (\inf R, 0), (m_R, c), (\sup R, 0), (1,0)\). Then \(|F(t_L)| \leq |\|F\|_L| < \delta\) and \(|F(t_R)| \leq |\|F\|_R| < \delta\).

Let \(h' := h - t_L - t_R\). Then
\[
|F(h') - F(h)| = |F(t_L) + F(t_R)| \leq 2\delta.
\]

Let \(N\) be the interval \((m_L, m_R)\). Let \(h_0\) be the polygon function whose graph has the vertices \((0,0), (m_L, 0), (\sup L, c), (\inf R, c), (m_R, 0), (1,0)\). Let \(h := h' - h_0\).
Therefore $F(a) \leq +6h_\beta$ is small. From the above estimations, 

$$|F(h') - F(h_0)| = |F(\tilde{h})| \leq a + \delta - |F(h_0')| \leq a + \delta - \|F\|_N + \delta \leq 2\delta.$$ 

Therefore $F(\tilde{h})$ is small. From the above estimations, 

$$|a| \leq |a - \|F\|_J + \|F\|_J - F(h)| + |F(h) - F(h')| + |F(h') - F(h_0)| + |F(h_0)|,$$

hence $a \leq \delta + \delta + 2\delta + 2\delta + |F(h_0)|$, that is,

$$a \leq 6\delta + |F(h_0)|.$$ 

Therefore, $|F(h_0)|$ is big. By construction, $0 < c = \|h_0\| \leq 1$. Let $\tilde{h} := h_0/c$. Then $a \leq 6\delta + |F(\tilde{h})|$. 

**Figure 1:** The functions $h, h_0$ and $h'$
Theorem 15.

Proof: Since this is true for all $x$, let $S = (m, \sup L)$ and $T = (\inf R, m_R)$. By Lemma 13,

$$|g_F(y_\prec) - F(s_S)| \leq \|F\|_K$$

and

$$|g_F(y_\succ) - F(s_T)| \leq \|F\|_K,$$

hence by Lemma 13,

$$a \leq 6\delta + |F(\hat{h})|$$

$$= 6\delta + |F(s_T) - F(s_S)|$$

$$\leq 6\delta + |F(s_T) - g_F(y_\succ)| + |g_F(y_\succ) - \lim_{y_\succ x} g_F(y)|$$

$$+ |\lim_{y_\succ x} g_F(y) - g_F(y_\succ)| + |\lim_{y_\prec x} g_F(y) - g_F(y_\prec)| + |g_F(y_\prec) - F(s_S)|$$

$$\leq 6\delta + \|F\|_R + \delta + |\lim_{y_\prec x} g_F(y) - \lim_{y_\succ x} g_F(y)| = \|F\|_K$$

$$\leq |\lim_{y_\prec x} g_F(y) - \lim_{y_\succ x} g_F(y)| + 10\delta$$

Since this is true for all $\delta > 0$, “$\Rightarrow$” has been proved. $\square$

Theorem 15.

1. $g_F$ is continuous on $(0;1) \cap \text{dom}(g_F) = \text{PC}_F$,

2. no proper extension $g$ of $g_F$ is continuous on $(0;1) \cap \text{dom}(g)$,

3. $\text{Var}(g) = \|F\|$ for every restriction $g \in \text{BV}$ of $g_F$,

4. $\text{Var}(g_F) = \|F\|.$

Proof: 1. If $x \in \text{PC}_F$ then $\lim_{y_\prec x} g_F(y) = \lim_{y_\succ x} g_F(y)$ by Lemma 14. Therefore $g_F$ is continuous in $x$.

2. Let $g$ be an extension of $g_F$ and let $g$ be continuous in $x \in \text{dom}(g)$. Then

$$\lim_{y_\prec x} g_F(y) = \lim_{y_\succ x} g_F(y),$$

hence $\inf_{x \in J} \|F\|_J = 0$ by Lemma 14, that is, $x \in \text{PC}_F$.

3. $\text{Var}(g) \leq \|F\|$: Let $X := (x_0, x_1, \ldots, x_n)$ be a partition for $g$. Let $\varepsilon > 0$. By the definition of $g_F$ for every $0 < i < n$ there is an interval $K_i \in \text{RI}$ such that $x_i \in K_i$, $\sup K_i < \inf K_{i+1}$, $\|F\|_K_1 < \varepsilon$. Furthermore, for $0 < i < n$ there are intervals $L_i, R_i \in \text{RI}$ such that $L_i, R_i \subseteq K_i$ and $\sup L_i < x_i < \inf R_i$, Figure 2 shows the intervals and some corresponding slanted step functions. By Lemma 8 and Lemma 13,

$$S(g, X) = |g(x_1)| + \sum_{i=2}^{n-1} |g(x_i) - g(x_{i-1})| + |g(1) - g(x_{n-1})|$$

$$\leq |F(s_{L_1})| + \varepsilon + \sum_{i=2}^{n-1} (|F(s_{L_i} - s_{R_{i-1}})| + 2\varepsilon)$$

$$+ |F(1 - s_{R_{n-1}})| + \varepsilon$$

$$\leq 2n\varepsilon + \|F\|.$$
Figure 3 shows an example of the left end of the unit interval with the function \( h \). Let \( K \) and the following sequence of vertices:

\[ x_1, x_2, \ldots, x_{n-1}, x_n = 1 \]

Then \( x_i - x_{i-1} \) and \( \|F\|_{K_i} \) are sufficiently small for all \( 1 < i \leq n \). By Lemma 13 \( F(h_2) \) can be related to \( S(g, X) \) (and to \( S(g, h_0, X) \) in the proof of Theorem 16).

Let \( h_0 \in \text{Pg} \) and \( k \in \mathbb{N} \). Let \( m : \mathbb{N} \to \mathbb{N} \) be a modulus of continuity of \( h_0 \). Let \( n := 2^{m(k)+1} + 1 \). Since \( \text{dom}(g) \) is dense, there is a partition \( X = (0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1) \) for \( g \) such that \( x_i - x_{i-1} < 2^{-m(k)-1} \). Since all the \( x_i \in \text{PC}_F \), for every \( 0 < i < n \) there are rational intervals \( K_i, L_i, R_i \) such that

\[
\begin{align*}
&x_i \in K_i, \quad 0 < \inf K_i, \quad \sup K_i < \inf K_{i+1}, \quad \sup K_{i-1} < 1, \\
&\|F\|_{K_i} < 2^{-k}/n, \\
&\inf L_i = \inf K_i, \quad \sup L_i < x_i < \inf R_i \quad \sup R_i = \sup K_i.
\end{align*}
\]

Figure 3 shows an example of the left end of the unit interval with the function \( h_0 \) and the intervals.

For \( 1 \leq i \leq n \) define

\[
c_i := \max\{h_0(x) \mid \sup R_{i-1} \leq x \leq \inf L_i\},
\]

(where \( \sup R_0 := 0 \) and \( \inf L_n := 1 \)). Define a rational polygon function \( h_1 \) by the following sequence of vertices:

\( (\sup R_0, c_1), (\inf L_1, c_1), (\sup R_1, c_2), (\inf L_2, c_2), \ldots, (\sup R_{n-1}, c_n), (\inf L_n, c_n) \)

(see Figure 3, notice that \( c_i \) may be negative).

Suppose \( 1 \leq i \leq n \) and \( \sup R_{i-1} \leq x \leq \inf L_i \). Then \( x_{i-1} \leq x \leq x_i \) and \( h_1(x) = c_i = h_0(y) \) for some \( y \) with \( x_{i-1} \leq y \leq x_i \). Then \( |x - y| < 2^{-m(k)} \), hence \( |h_1(x) - h_0(x)| = |h_0(y) - h_0(x)| < 2^{-k} \).
Suppose $0 < i < n$ and $x \in K_i$. Then $h_1(x) = h_0(y)$ for some $y$ such that $x_{i-1} < y < x_{i+1}$. Since $x_{i-1} < x < x_{i+1}$, $|x - y| < 2^{-m(k)}$ and hence $|h_1(x) - h_0(x)| = |h_0(y) - h_0(x)| < 2^{-k}$.

Therefore, $\|h_1 - h_0\| < 2^{-k}$ and hence $|F(h_1) - F(h_0)| \leq \|F\| \cdot 2^{-k}$.

Let $1 \leq i \leq n$. Then $c_i = h_0(y)$ for some $x_{i-1} \leq y \leq x_i$. Since $|x_i - y| < 2^{-m(k)}$, $|h_0(x_i) - c_i| = |h_0(x_i) - h_0(y)| \leq 2^{-k}$.

From $h_1$ we construct a third function $h_2$ by replacing for every $0 < i < n$ the line segment from $(\inf L_i, c_i)$ to $(\sup R_i, c_i+1)$ in the graph of $h_1$ by the polygon $(\inf L_i, c_i), (\sup L_i, 0), (\inf R_i, 0), (\sup R_i, c_i+1)$ (see Figure 3). Then by Definition 10,

$$h_2 = c_1 s_{L_1} + \sum_{i=2}^{n-1} c_i (s_{L_i} - s_{R_{i-1}}) + c_n (1 - s_{R_{n-1}}).$$

For $0 < i < n$ let $d_i$ be the polygon function defined by the sequence of vertices $(0, 0), (\inf L_i, 0), (\sup L_i, h_1(\sup L_i)), (\inf R_i, h_1(\inf R_i)), (\sup R_i, 0), (1, 0)$.

Then $h_2 = h_1 - \sum_{i=1}^{n-1} d_i$. Since $NZ(d_i) \subseteq K_i$ and $\|d_i\| \leq \|h_0\|$, $|F(h_2) - F(h_1)| \leq \sum_{i=1}^{n-1} |F(d_i)| \leq \sum_{i=1}^{n-1} \|F\| \cdot \|h_0\| \leq \|h_0\| \cdot 2^{-k}$.

We prove $\|F\| \leq \text{Var}(g)$. There is some $h_0 \in \text{Pg}$ such that $\|h_0\| \leq 1$ and $\|F\| \leq |F(h_0)| + 2^{-k}$. Since $|c_i| \leq 1$ and by Lemma 13,
\[ \|F\| \leq |F(h_0 - h_1)| + |F(h_1 - h_2)| + |F(h_2)| + 2^{-k} \]
\[ \leq \|F\| \cdot 2^{-k} + \|h_0\| \cdot 2^{-k} + |F(h_2)| + 2^{-k} \]
\[ \leq |F(s_{L_1})| + \sum_{i=0}^{n-1} |F(s_{L_i} - s_{R_{i-1}})| + |F(1 - s_{R_{n-1}})| \]
\[ + (\|F\| + 2) \cdot 2^{-k} \]
\[ \leq |g(x_1)| + 2^{-k}/n + \sum_{i=2}^{n-1} (|g(x_i) - g(x_{i-1})| + 2 \cdot 2^{-k}/n) \]
\[ + |g(1) - g(x_{n-1})| + 2^{-k}/n + (\|F\| + 2) \cdot 2^{-k} \]
\[ \leq \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + 2 \cdot 2^{-k} + (\|F\| + 2) \cdot 2^{-k} \]
\[ = S(g, X) + (\|F\| + 4) \cdot 2^{-k} \]
\[ \leq \text{Var}(g) + (\|F\| + 4) \cdot 2^{-k}. \]

Since this is true for all \( k \), \( \|F\| \leq \text{Var}(g) \).

4. This follows from 3. \( \square \)

**Theorem 16.** Let \( g \in \text{BV} \) be a restriction of \( g_F \). Then for every \( h \in C[0;1] \), \( F(h) = \int h \, dg \).

**Proof:** Let \( h \in C[0;1] \) and \( k \in \mathbb{N} \). There is a function \( h_0 \in \text{P}g \) such that \( \|h - h_0\| \leq 2^{-k} \). Let \( m, n, X, K_i, L_i, R_i, c_i, h_1, h_2 \) be the objects introduced in the proof of Theorem 15.3. We prove that \( |F(h) - S(g, h, X)| \) is small. By the results that we have already shown,

\[ |F(h) - F(h_2)| \leq |F(h) - F(h_0)| + |F(h_0) - F(h_1)| + |F(h_1) - F(h_2)| \]
\[ \leq \|F\| \cdot 2^{-k} + \|F\| \cdot 2^{-k} + \|h_0\| \cdot 2^{-k} \]
\[ = (2\|F\| + \|h_0\|) \cdot 2^{-k} \]

Since \( |F(s_{R_i}) + B| \leq |g(x_i) + B| + \|F\| \cdot K_i \) etc. by Lemma 13, \( c_i \leq \|h_0\| \), and \( |h_0(x_i) - c_i| \leq 2^{-k} \),

\[ |F(h_2) - S(g, h_0, X)| \]
\[ \leq c_1 F(s_{L_1}) + \sum_{i=2}^{n-1} c_i (F(s_{L_i}) - F(s_{R_{i-1}})) + c_n (F(1) - F(s_{R_{n-1}})) \]
\[ - \sum_{i=1}^{n} h_0(x_i) (g(x_i) - g(x_{i-1})) \]
\[ \leq \left| c_1 g(x_1) + \sum_{i=2}^{n-1} c_i (g(x_i) - g(x_{i-1})) + c_n (g(1) - g(x_{n-1})) \right| \\
- \sum_{i=1}^n h_0(x_i)(g(x_i) - g(x_{i-1})) \right| \\
+ |c_1| \|F\|_{K_1} + \sum_{i=2}^{n-1} |c_i| (\|F\|_{K_i} + \|F\|_{K_{i-1}}) + |c_n| \|F\|_{K_{n-1}} \\
\leq \sum_{i=1}^n (c_i - h_0(x_i)) (g(x_i) - g(x_{i-1})) \right| + 2 \|h_0\| \cdot 2^{-k} \\
\leq \sum_{i=1}^n |c_i - h_0(x_i)| \cdot |g(x_i) - g(x_{i-1})| + \|h_0\| \cdot 2^{-k+1} \\
\leq 2^{-k} \cdot S(g, X) + \|h_0\| \cdot 2^{-k+1} \\
\leq 2^{-k} \cdot \text{Var}(g) + \|h_0\| \cdot 2^{-k+1} \\
= (\|F\| + 2 \|h_0\|) \cdot 2^{-k} \\
\]

Furthermore,

\[ |S(g, h_0, X) - S(g, h, X)| = \left| \sum_{i=1}^n (h_0(x_i) - h(x_i))(g(x_i) - g(x_{i-1})) \right| \\
\leq 2^{-k} \sum_{i=1}^n |g(x_i) - g(x_{i-1})| \\
= 2^{-k} \cdot S(g, X) \\
\leq 2^{-k} \cdot \text{Var}(g) \\
= 2^{-k} \cdot \|F\|. \]

Combining these results we obtain

\[ |F(h) - S(g, h, X)| \\
\leq |F(h) - F(h_2)| + |F(h_2) - S(g, h_0, X)| + |S(g, h_0, X) - S(g, h, X)| \\
\leq (2\|F\| + \|h_0\|) \cdot 2^{-k} + (\|F\| + 2 \|h_0\|) \cdot 2^{-k} + 2^{-k} \cdot \|F\| \\
\leq (\|F\| + \|h\| + 1) \cdot 2^{-k+2} \]

Since \(X\) has precision \(m(k)\), \(\|h\| = \int f \, dg\) by Lemma 4. Therefore, \(|F(h) - \int f \, dg| \leq (3\|F\| + 2\|h\| + 2) \cdot 2^{-k+1}\). Since this is true for all \(k\), \(F(h) = \int f \, dg\). \(\square\)

4 Concepts from Computable Analysis

For studying computability we use the representation approach (TTE) for Computable Analysis [Weihrauch(2000), Brattka et al.(2008)]. Let \(\Sigma\) be a finite al-
phabet. Computable functions on $\Sigma^*$ (the set of finite sequences over $\Sigma$) and $\Sigma^\omega$ (the set of infinite sequences over $\Sigma$) are defined by Turing machines which map sequences to sequences (finite or infinite). On $\Sigma^*$ and $\Sigma^\omega$ finite or countable tupling will be denoted by $\langle \rangle$ [Weihrauch(2000)]. The tupling functions and the projections of their inverses are computable.

In TTE, sequences from $\Sigma^*$ or $\Sigma^\omega$ are used as “names” of abstract objects such as rational numbers, real numbers, real functions or points of a metric space. We consider computability of multi-functions w.r.t. multi-representations [Weihrauch(2000)], [Brattka et al.(2008)], [Weihrauch(2008), Sections 3,6,8,9].

A representation of a set $X$ is a function $\delta : C \to X$ where $C = \Sigma^*$ or $C = \Sigma^\omega$. If $\delta(p) = x$ we call $p$ a $\delta$-name of $x$. If $f : X \Rightarrow Y$ is a multi-function (on represented sets) then $f(x)$ is the set of $y \in Y$ which are accepted as a result of $f$ applied to $x$. (Example: $f : \mathbb{R} \Rightarrow \mathbb{Q}$, $f(x) := \{a \in \mathbb{Q} \mid x < a\}$, we may say: “the multi-function $f$ finds some rational upper bound of $x$”.)

For representations $\gamma : \subseteq Y \to M$ and $\gamma_0 : \subseteq Y_0 \to M_0$, a function $h : \subseteq Y \to Y_0$ is a $(\gamma,\gamma_0)$-realization of a multi-function $f : \subseteq M \Rightarrow M_0$, iff for all $p \in Y$ and $x \in M$,

$$\gamma(p) = x \in \text{dom}(f) \implies \gamma_0 \circ h(p) \in f(x).$$

(18)

Fig. 4 illustrates the definition.

![Figure 4](image.jpg)

**Figure 4:** $h(p)$ is a name of some $y \in f(x)$, if $p$ is a name of $x \in \text{dom}(f)$.

The multi-function $f$ is called $(\gamma,\gamma_0)$-computable, if it has a computable $(\gamma,\gamma_0)$-realization and $(\gamma,\gamma_0)$-continuous if it has a continuous realization. The definitions can be generalized straightforwardly to multi-functions $f : M_1 \times \ldots \times M_n \Rightarrow M_0$ for represented sets $M_i$. 
For two representations \( \delta_i : \Sigma^\omega \to M_i \) \( i = 1, 2 \) the canonical representation \([\delta_1, \delta_2]\) of the product \( M_1 \times M_2 \) is defined by

\[
[\delta_1, \delta_2](p_1, p_2) = (\delta_1(p_1), \delta_2(p_2)).
\]

(19)

For two representations \( \delta \subseteq \Sigma^\omega \to M \) \( i = 1, 2 \), \( \delta_1 \leq \delta_2 \) \( (\delta_1 \text{ is reducible to } \delta_2) \) iff there is a computable function \( h : \subseteq \Sigma^\omega \to \Sigma^\omega \) such that \( (\forall p \in \text{dom}(\delta_1)) \delta_1(p) = \delta_2 h(p) \). (If \( p \) is a \( \delta_1 \)-name of \( x \) then \( h(p) \) is a \( \delta_2 \)-name of \( x \).)

We use various canonical notations \( \nu : \subseteq \Sigma^* \to X \): \( \nu_\mathbb{N} \) for the natural numbers, \( \nu_\mathbb{Q} \) for the polygon functions on \([0;1]\) whose graphs have rational vertices, and \( \nu_I \) for the set \( \text{RI} \) open subintervals \((a;b)\subseteq(0;1)\) with rational endpoints. For functions \( m : \mathbb{N} \to \mathbb{N} \) we use the canonical representation \( \delta_\mathbb{N} : \subseteq \Sigma^\omega \to \mathbb{N} = \{ m \mid m : \mathbb{N} \to \mathbb{N} \} \) defined by \( \delta_\mathbb{N}(p) = m \) if \( p = 1^{m(0)}1^{m(1)}01^{m(2)}0\ldots \). For the real numbers we use the Cauchy representation \( \rho : \subseteq \Sigma^\omega \to \mathbb{R} \), \( \rho(p) = x \) if \( p \) is (encodes) a sequence \((a_i)_{i\in\mathbb{N}}\) of rational numbers such that for all \( i \), \( |x-a_i| \leq 2^{-i} \). By the Weierstraß approximation theorem the countable set of \( \text{Pg} \) of polygon functions with rational vertices is dense in \( C[0;1] \). Therefore, \( C[0;1] \) with notation \( \nu_{\mathbb{Q}} \) of the set \( \text{Pg} \) is a computable metric space [Weihrauch(2000)] for which we use the canonical representation \( \delta_{\mathbb{Q}} \) defined as follows: \( \delta_{\mathbb{Q}}(p) = h \) if \( p \) is (encodes) a sequence \((h_i)_{i\in\mathbb{N}}\) of polygons \( h_i \in \text{Pg} \) such that for all \( i \), \( \|h-h_i\| \leq 2^{-i} \) [Weihrauch(2000)]. For the space \( C(C[0;1],\mathbb{R}) \) of the continuous (not necessarily linear) functions \( F : C[0;1] \to \mathbb{R} \) we use the canonical representation \( [\delta_{\mathbb{Q}} \to \rho] \) [Weihrauch(2000), Weihrauch and Grubba(2009)]. It is determined uniquely up to equivalence by (U) and (S):

(U) the function \( \text{APPLY} : (F,h) \to F(h) \) is \( ([\delta_{\mathbb{Q}} \to \rho],\delta_{\mathbb{Q}},\rho) \)-computable,

(S) if for some representation \( \delta \) of a subset of \( C(C[0;1],\mathbb{R}) \), \( \text{APPLY} \) is \( (\delta,\delta_{\mathbb{Q}},\rho) \)-computable then \( \delta \leq [\delta_{\mathbb{Q}} \to \rho] \).

(U) corresponds to the “universal Turing machine theorem” and (S) to the “smn-theorem” from computability theory. Roughly speaking, \( [\delta_{\mathbb{Q}} \to \rho] \) is the “poorest” representation of the set \( C(C[0;1],\mathbb{R}) \) for which the \( \text{APPLY} \) function becomes computable.

For converting the classical proof mentioned in Section 2 we needed a representation of the set \( B[0;1] \) of bounded functions \( g : [0;1] \to \mathbb{R} \). Since it has a cardinality bigger than that of \( \Sigma^\omega \), it has no representation. To overcome this difficulty it would suffice to extend \( F \) to the Banach space \( B_1[0;1] \) generated by the continuous functions and all the characteristic function \( \chi_{[0;x]} \), \( 0 \leq x \leq 1 \). However, since this space is not separable we do not know any reasonable representation of it. We solve the problem by (implicitly) extending \( F \) only to functions \( \chi_{[0;x]} \) from a countable dense set of points \( x \) in which \( g \) is continuous and for which we can compute \( g(x) := \overline{F}(\chi_{[0;x]}) \) from \( F \) and \( \|F\| \). Remember
that every function of bounded variation has at most countably many points of discontinuity.

Finally, for formulating a computable version of the Riesz representation theorem we need a representation for functions of bounded variation. In our context the only application of a function \( g \) of bounded variation is to compute the Riemann-Stieltjes integral \( \int h \, dg \) for continuous functions \( h \). By Corollary 6, it suffices to know \( g \) on a countable dense set containing 0 and 1. Therefore it will suffice to consider only functions from BV with countable domain.

**Definition 17.** Let \( BVC := \{ g \in BV \mid \text{dom}(g) \text{ is countable} \} \). Define a representation \( \delta_{BVC} : \subseteq \Sigma^\omega \to BVC \) as follows: \( \delta_{BVC}(p) = g \iff \) there are sequences \( p_0, q_0, p_1, q_1, \ldots \in \Sigma^\omega \) such that \( p = \langle p_0, q_0 \rangle, \langle p_1, q_1 \rangle, \ldots \), \( \rho(p_0) = 0, \rho(p_1) = 1 \) and \( \text{graph}(g) = \{ (\rho(p_i), \rho(q_i)) \mid i \in \mathbb{N} \} \).

Informally, a \( \delta_{BVC} \)-name of \( g \) is a list of its graph. For proving computability of multi-functions on represented sets we use “generalized Turing machines” (GTMs) [Tavana and Weihrauch(2011)]. We call a generalized Turing machine \( M \) on represented sets computable, if all multi-functions on the represented sets occurring in \( M \) are computable. We use the following result: the multi-function \( f_M \) computed by a computable GTM \( M \) on represented sets is computable.

For a representation \( \delta : \subseteq \Sigma^\omega \to Z \) a subset \( Y \subseteq Z \) is \( \delta \)-r.e., iff there is a Type-2 machine \( N \) such that for all \( p \in \text{dom}(\delta) \),

\[ N \text{ halts on input } p \iff \delta(p) \in Y. \]

And \( Y \subseteq Z \) is \( \delta \)-decidable, iff \( Y \) and \( Z \setminus Y \) are \( \delta \)-r.e. [Weihrauch(2000)]. As an example, \( x < y \) for real numbers is \([\rho, \rho]\)-r.e.

### 5 The computable Riesz representation theorem

In the following “computable”, “recursively enumerable” and “decidable” means computable, recursively enumerable and decidable, respectively, w.r.t. the notations and multi-representations mentioned in Section 4.

First, from \( F \) and \( \|F\| \) we will compute some \( g \in BVC \) such that \( F(h) = \int h \, dg \). By the next lemma for every rational interval \( I \) we can compute subintervals \( J \) with arbitrarily small \( \|F\|_J \).

**Lemma 18.** There is a computable multi-function

\[ e : (F, z, I, n) \mapsto J \]

that maps every continuous linear functional \( F : C[0;1] \to \mathbb{R} \), its norm \( z \), every open rational interval \( I = (a; b) \subseteq [0;1] \) and every \( n \in \mathbb{N} \) to some open rational interval \( J \subseteq I \) such that \( J \subseteq I \), \( \text{length}(J) \leq 2^{-n} \) and \( \|F\|_J \leq 2^{-n} \).
Precisely speaking, the multi-function \( e \) is \( ([\delta_C \to \rho], \rho, \nu_1, \nu_2) \)-computable.

**Proof:** By Lemma 9 there is some \( x \in I \) such that \( x \in PC_F \). By Definition 7 there is some \( J, x \in J \in RI \), such that \( J \subseteq I \), length\( (J) \leq 2^{-n} \) and \( \| F \|_J \leq 2^{-n} \). We show that the multi-function \( e \) is \( ([\delta_C \to \rho], \rho, \nu_1, \nu_2) \)-computable.

For \( F, z = \| F \|, I = (a; b), n \in \mathbb{N}, J \in RI \) and \( \mathcal{J} \in Pg \) consider the conditions

\[
\begin{align*}
\mathcal{J} &\subseteq I, \quad \text{length}(J) \leq 2^{-n}, \\
\mathcal{J}(x) &= 0 \text{ for } x \in J, \\
\| \mathcal{J} \| &\leq 1, \\
|F(\mathcal{J})| &> \| F \| - 2^{-n}.
\end{align*}
\]

Conditions (20-22) are decidable (relative to their representations). Since \( x < y \) is \( [\rho, \rho] \)-r.e. and \( (F, \mathcal{J}) \rightarrow F(\mathcal{J}) \) is computable, (23) is r.e. Therefore, here is a Type-2-machine \( M \) that halts on input \((p_1, p_2, u_3, u_4, u_5, u_6)\) iff

\[
(F, \| F \|, I, n, J, \mathcal{J}) := ([\delta_C \to \rho], \rho, \nu_1, \nu_2, \nu_1, \nu_2)(p_1, p_2, u_3, u_4, u_5, u_6)
\]

satisfies (20-23). From \( M \) a Type-2 machine \( N \) can be constructed which on input \((p_1, p_2, u_3, u_4)\) (by the usual step counting technique) searches for \((u_5, u_6)\) such that \( M \) halts on input \((p_1, p_2, u_3, u_4, u_5, u_6)\).

First we show that \( J = \nu_1(u_5) \) and \( \mathcal{J} = \nu_2(u_6) \) exist.

Since \( Pg \) is dense in \( C[0; 1], \| F \| = \sup\{|F(h)| \mid h \in Pg, \| h \| \leq 1\} \). Therefore, there is a function \( h \in Pg \) with \( \| h \| \leq 1 \) such that \( |F(h)| > \| F \| - 2^{-n-1} \). As we have shown (replace above \( n \) by \( n + 1 \)) there is a rational interval \( L \subseteq I \) such that length\( (L) \leq 2^{-n} \) and \( \| F \|_L \leq 2^{-n-1} \). Let \( (a_2; b_2) \subseteq L \) such that \( h \) has no vertex in \((a_2; b_2)\). Let \( a_1 := a_2 + (b_2 - a_2)/3, b_1 := b_2 - (b_2 - a_2)/3 \) and \( J := (a_1; b_1) \).

Define a function \( f_0 \in Pg \) by its vertices as follows:

\[
(0, 0), (a_2, 0), (a_1, h(a_1)), (b_1, h(b_1)), (b_2, 0), (1, 0)
\]

and let \( \mathcal{J} := h - f_0 \). Then \( \| f_0 \| \leq 1 \) and \( \| F(f_0) \| \leq 2^{-n-1} \) since \( NZ(f_0) \subseteq L \). Since \( h \) and \( f_0 \) have no vertex in the interval \((a_2; a_1), |h(x) - f_0(x)| \leq |h(a_2)| \leq 1 \) for \( a_2 \leq x \leq a_1 \), correspondingly \( |h(x) - f_0(x)| \leq 1 \) for \( b_1 \leq x \leq b_2 \), and \( |h(x) - f_0(x)| = 0 \) for \( a_1 \leq x \leq b_1 \). We obtain \( \| \mathcal{J} \| \leq 1 \). Furthermore,

\[
|F(\mathcal{J})| = |F(h - f_0)| \geq |F(h)| - |F(f_0)| \geq \| F \| - 2^{-n}.
\]

Therefore, \( J \) and \( \mathcal{J} \) exist.

It remains to show that \( J \) has the properties requested in the lemma. Obviously, \( J \subseteq I \) and length\( (J) \leq 2^{-n} \). Suppose \( h \in C[0; 1], \| h \| \leq 1 \) and \( NZ(h) \subseteq J \). Since \( NZ(h) \) and \( NZ(\mathcal{J}) \) are disjoint and of norm \leq 1, by Lemma 8, \( |F(h)| + |F(\mathcal{J})| \leq \| F \| \) hence \( |F(h)| \leq \| F \| - |F(\mathcal{J})| < 2^{-n} \). Therefore, \( \| F \|_J \leq 2^{-n} \). \( \Box \)
By iterating the function \( e \) from Lemma 18 in every open rational interval we can find some point \( x \in PC_F \) and the value \( g_F(x) \).

**Lemma 19.** The multi-function \( G : (F, \|F\|, I) \models (x, g_F(x)) \) mapping \( F \), its norm and an interval \( I \in RI \) to \( (x, g_F(x)) \) for some \( x \in I \cap PC_F \) is computable.

**Proof:** Let \( J_{n-1} := I \). For every \( n \in \mathbb{N} \) let \( J_n \) be a result of applying the multi-function \( e \) from Lemma 18 to \( (F, \|F\|, J_{n-1}, n) \). Then \( (J_n)_{n \in \mathbb{N}} \) is a properly nested sequence of intervals with \( \text{length}(J_n) \leq 2^{-n} \). It converges to some point \( x \in I \). Since for all \( n \), \( x \in J_n \) and \( \|F\| \mid J_n \leq 2^{-n} \), \( x \in PC_F \). Furthermore, by Lemma 13, \( |g_F(x) - F(s_{J_n})| \leq 2^{-n} \). Therefore \( (s_{J_n})_{n \in \mathbb{N}} \) converges fast to \( g_F(x) \).

Let \( M_1 \) be a computable GTM computing the multi-function \( e \) from Lemma 18. From \( M_1 \) we can construct a computable GTM that on input \( (F, \|F\|, I, n) \) computes in turn some \( J_0, J_1, \ldots, J_n \) and then \( (J_n, F(s_{J_n})) \) as its result.

By [Weihrauch(2008), Theorem 35] the multi-function \( (F, \|F\|, I) \models (J_n, F(s_{J_n}))_{n \in \mathbb{N}} \) is computable (where the canonical representation considered for sequences [Weihrauch(2000)]). Since the limit operations for nested sequences of intervals converging to a point and for fast converging Cauchy sequences of real numbers are computable [Weihrauch(2000)], \( (x, g_F(x)) \) can be computed from \( (J_n, F(s_{J_n}))_{n \in \mathbb{N}} \). Therefore, the multi-function \( G \) is computable. \( \Box \)

We can now prove our computable version of the Riesz representation theorem.

**Theorem 20 (computable Riesz representation).**

The multi-function \( RRT : (F, \|F\|) \models g \) mapping every functional \( F : C[0; 1] \rightarrow \mathbb{R} \) and its norm to some function \( g \in BVC \) such that
- \( F(h) = \int hdg \) (for all \( h \in C[0; 1] \)),
- \( g \) is continuous on \( \text{dom}(g) \setminus \{0, 1\} \),
- \( g(0) = 0 \) and \( \|F\| = \text{Var}(g) \)

is \((\delta_C \rightarrow \rho), \rho, \delta_{BVC})\)-computable.

**Proof:** Let \( L_0, L_1, \ldots \) be a canonical numbering of the set \( RI \) of open rational intervals. By Lemma 19 there is a computable function \( G' \) mapping \( (F, \|F\|, n) \) to some \( (x_n, y_n) \in \mathbb{R}^2 \) where \( (x_0, y_0) = (0, 0), (x_1, y_1) = (1, F(1)) \) and \( (x_n, y_n) \in G(f, \|F\|, L_n) \) if \( n \geq 2 \). Since \( x_n \in PC_F \) and \( y_n = g_F(x_n) \) for all \( n \geq 2 \), \( \{(x_n, y_n) \mid n \in \mathbb{N}\} \) is the graph of a restriction \( g \) of \( g_F \). Since \( \{x_n \mid n \in \mathbb{N}\} \) is dense, \( g \in BVC \).

By Theorem 15, \( g \) is continuous on \( \text{dom}(g) \setminus \{0, 1\} \) and \( \text{Var}(g) = \|F\| \). Obviously, \( g(0) = 0 \). By Theorem 16, \( F(h) = \int hdg \) (for all \( h \in C[0; 1] \)).

By the type conversion theorem [Weihrauch(2008), Theorem 33], the multi-function \( (F, \|F\|) \models ((x_n, y_n))_{n \in \mathbb{N}} \) is \((\delta_C \rightarrow \rho), \rho, [\nu_N \rightarrow [\rho, \rho]]\)-computable. From a \([\nu_N \rightarrow [\rho, \rho]]\)-name of the sequence \( ((x_n, y_n))_{n \in \mathbb{N}} = ((x_0, y_0), (x_1, y_1), \ldots) \)
we can compute a $[\rho, \rho]^\omega$-name [Weihrauch(2000), Lemma 3.3.16] which is a $\delta_{\text{BVC}}$-name of $g$. \hfill \Box

Finally, we prove that a reverse of the Riesz representation theorem is computable.

**Theorem 21.** The operator $T : (g, l) \mapsto F$, mapping every $g \in \text{BVC}$ and every $l \in \mathbb{N}$ with $\text{Var}(g) \leq 2^l$ to the functional $F$ defined by $F(h) = \int h \, dg$ for all $h \in C[0; 1]$, is computable.

**Proof:** First we show that $(G, l, h) \mapsto \int h \, dg$ is computable. By Theorem 6.2.7 in [Weihrauch(2000)] a modulus $m : \mathbb{N} \to \mathbb{N}$ of continuity of $h$ can be computed from $h$. Let $\nu_{\text{fin}}$ be a canonical notation of the finite sequences of natural numbers. The set of all $(g, (i_1, \ldots, i_{n-1}), j)$ such that $(0, x_{i_1}, \ldots, x_{i_{n-1}}, 1)$ is a partition for $g$ of precision $j$ is $(\delta_{\text{BVC}}, \nu_{\text{fin}}, \nu_{\mathbb{N}})$-r.e. There is computable GTM on represented sets which on input $(g, j)$ finds a sequence $(i_1, \ldots, i_{n-1})$ such that $(0, x_{i_1}, \ldots, x_{i_{n-1}}, 1)$ is a partition for $g$ of precision $j$. Therefore from $(g, h, k, l)$ we can compute a sequence $(i_1, \ldots, i_{n-1})$ such that $X := (0, x_{i_1}, \ldots, x_{i_{n-1}}, 1)$ is a partition for $g$ of precision $m(k + l + 1)$. By Lemma 4, $| \int h \, dg - S(g, h, X) | \leq 2^{-l-k} V(g) \leq 2^{-k}$. The function $(g, h, X) \mapsto S(g, h, X)$ is computable (by a computable GTM). Therefore, from $(g, l, h, k)$ a number $y_k$ can be computed (multi-valued) such that $| \int h \, dg - y_k | \leq 2^{-k}$. By [Weihrauch(2008), Theorem 33] the multi-function $(g, l, h) \mapsto (y_k)_{k \in \mathbb{N}}$ is computable. By [Weihrauch(2000), Theorem 4.3.7], $(g, l, h) \mapsto \int h \, dg$ is $(\delta_{\text{BVC}}, \nu_{\mathbb{N}}, \delta_{\mathbb{C}}, \rho)$-computable. By [Weihrauch(2000), Theorem 3.3.15], $(g, l) \mapsto F$ such that $F(h) = \int h \, dg$ is $(\delta_{\text{BVC}}, \nu_{\mathbb{N}}, [\delta_{\mathbb{C}} \to \rho])$-computable. \hfill \Box

By Theorem 20, from $F$ and $\|F\|$ we can compute $g$ such that $\text{Var}(g) = \|F\|$, and by Theorem 21, from $g$ and an upper bound of $\text{Var}(g)$ we can compute $F$.

**References**


Appendix

Proof of Lemma 4

Since there are partitions for \( g \) of arbitrary precision, \( I \) is unique if it exists.

Next, we prove

\[
|S(g, h, Z_1) - S(g, h, Z_2)| \leq 2^{-k} V(g). \tag{24}
\]

for any two partitions \( Z_1, Z_2 \) for \( g \) with precision \( m(k + 1) \).

Let \( Z_1 = (x_0, x_1, \ldots, x_n) \) and let \( Z' \) be a refinement of \( Z_1 \). \( Z' \) can be written as

\[
x_0 = y_0^1, y_1^1, \ldots, y_{j_1}^1 = x_1 = y_0^2, y_1^2, \ldots, y_{j_2}^2 = x_2 \ldots \ldots = y_0^n, y_1^n, \ldots, y_{j_n}^n = x_n
\]

\((j_1, \ldots, j_n \geq 1)\). Then

\[
|S(g, h, Z_1) - S(g, h, Z')| = \left| \sum_{i=1}^{n} h(x_i) (g(x_i) - g(x_{i-1})) - \sum_{i=1}^{n} \sum_{l=1}^{j_i} h(y_i^l) (g(y_i^l) - g(y_{i-1}^l)) \right|
\]

\[
= \left| \sum_{i=1}^{n} h(x_i) \sum_{l=1}^{j_i} (g(y_i^l) - g(y_{i-1}^l)) - \sum_{i=1}^{n} \sum_{l=1}^{j_i} h(y_i^l) (g(y_i^l) - g(y_{i-1}^l)) \right|
\]

\[
= \left| \sum_{i=1}^{n} \sum_{l=1}^{j_i} (h(x_i) - h(y_i^l)) (g(y_i^l) - g(y_{i-1}^l)) \right|
\]

\[
\leq \sum_{i=1}^{n} \sum_{l=1}^{j_i} |h(x_i) - h(y_i^l)| |g(y_i^l) - g(y_{i-1}^l)|
\]

\[
\leq 2^{-k-1} \sum_{i=1}^{n} \sum_{l=1}^{j_i} |g(y_i^l) - g(y_{i-1}^l)| \quad \text{since } |x_i - y_i^l| \leq 2^{-m(k+1)}
\]

\[
\leq 2^{-k} V(g)
\]

Now let \( Z' \) be a common refinement of \( Z_1 \) and \( Z_2 \). Then

\[
|S(g, h, Z_1) - S(g, h, Z_2)| \leq |S(g, h, Z_1) - S(g, h, Z')| + |S(g, h, Z') - S(g, h, Z_2)| \leq 2^{-k} V(g).
\]
There is a sequence \((Z_k)_k\) of partitions for \(g\) such that \(Z_k\) has precision \(m(k+1)\). By (24) for \(j > k\), \(|S(g, h, Z_k) - S(g, h, Z_j)| \leq 2^{-k}V(g)\). Let \(I\) be the limit of the Cauchy sequence \((S(g, h, Z_k))_k\). Let \(Z\) be a partition of precision \(m(k+1)\). Then for every \(i > k\) by (24),

\[
|I - S(g, h, Z)| \leq |I - S(g, h, Z_i)| + |S(g, h, Z_i) - S(g, h, Z)| \\
\leq 2^{-i}V(g) + 2^{-k}V(g),
\]

hence \(|I - S(g, f, Z)| \leq 2^{-k}V(g)\).