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The Separation of Relativized Versions of P and DNP for the Ring of the Reals

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Abstract: We consider the uniform BSS model of computation where the machines can perform additions, multiplications, and tests of the form $x \geq 0$. The oracle machines can also check whether a tuple of real numbers belongs to a given oracle set or not. We present oracle sets containing positive integers and pairs of numbers, respectively, such that the classes P and DNP relative to these oracles are not equal. The first set is constructed by diagonalization techniques and the second one is derived from the Knapsack Problem.

Key Words: BSS model, oracle machine, relativizations, binary non-determinism, digital non-determinism
Category: F.1, F.1.1, F.1.2, F.1.3

1 Introduction

We consider the uniform BSS model of computation (cp. [Blum et al. 1989]), where the non-deterministic machines are able to guess arbitrary real numbers in one step and the digital non-deterministic machines are restricted to be machines using only zeros and ones as guesses (cp. also [Poizat 1995]). For the corresponding polynomial time complexity classes we do not know whether one of the inclusions in $\mathrm{P}_{\mathbb{R}} \subseteq \mathrm{DNP}_{\mathbb{R}} \subseteq \mathrm{NP}_{\mathbb{R}}$ is strict. However, there are oracles $\mathcal{O} \subseteq \mathbb{R}^\infty$ such that the classes $\mathrm{P}_{\mathbb{R}}^{\mathcal{O}}$ and $\mathrm{NP}_{\mathbb{R}}^{\mathcal{O}}$ are equal or are not (cp. [Emerson 1994]). Emerson's proof technique can also be used to separate relativized versions of $\mathrm{DNP}_{\mathbb{R}}$ and $\mathrm{NP}_{\mathbb{R}}$. Since, by assigning a positive integer i to any digital non-deterministic oracle machine working in polynomial time such that the machine corresponding to i does not use any $r > i$ in a query on its own code, we get a sequence of sets of codes of machines, $(K_i)_{i\geq 1}$, which allows to define a suitable oracle $\mathcal{Q}_1 = \bigcup_{i\geq 1} W_i$ by diagonalization. If we define $W_j = \bigcup_{i<j} V_i$ by $V_0 = \emptyset$ and $V_i = \{(\mathrm{code}(\mathcal{N}^{W_i}), i+1) \in K_i \times \mathbb{N} \mid \mathcal{N}^{W_i} \text{ does not accept } \mathrm{code}(\mathcal{N}^{W_i})\}$, then the problem $L_1 =_{\mathrm{df}} \{\mathbf{y} \mid (\exists n \in \mathbb{N}^+)((\mathbf{y}, n) \in \mathcal{Q}_1)\}$ cannot be in $\mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_1}$. On the other hand, since the non-deterministic BSS machines are able to guess any integer in one step, we get $L_1 \in \mathrm{NP}_{\mathbb{R}}^{\mathcal{Q}_1}$ and, consequently, $\mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_1} \neq \mathrm{NP}_{\mathbb{R}}^{\mathcal{Q}_1}$. Moreover, it is possible to show $\mathrm{DNP}_{\mathbb{R}}^{\mathbb{Z}} \neq \mathrm{NP}_{\mathbb{R}}^{\mathbb{Z}}$ and $\mathrm{DNP}_{\mathbb{R}}^{\mathbb{Q}} \neq \mathrm{NP}_{\mathbb{R}}^{\mathbb{Q}}$ by analogy with [Gaßner, 2009 (1)].

The discussion on corresponding relativizations for several types of groups in [Gaßner 2008] also shows which known constructions can be transferred to which type of structures. Whereas the separation of $\mathrm{DNP}_{\Sigma}^{\mathcal{O}}$ and $\mathrm{NP}_{\Sigma}^{\mathcal{O}}$ by constructing an oracle $\mathcal{O}$ as above is possible for any infinite structure Σ, the separation of $\mathrm{P}_{\Sigma}^{\mathcal{O}}$ and $\mathrm{DNP}_{\Sigma}^{\mathcal{O}}$ by the method introduced in [Baker et al. 1975] can be successfully used only for structures Σ of countable signature.

For infinite structures of uncountable signature Σ, we do not know any general method for defining an oracle $\mathcal{O}$ satisfying $\mathrm{P}_{\Sigma}^{\mathcal{O}} \neq \mathrm{DNP}_{\Sigma}^{\mathcal{O}}$ that is only based on logical techniques. However, if we consider the *BSS machines* over the reals which can perform additions, multiplications, and tests of the form $x \geq 0$, we can make use of special integers for characterizing the behavior of the deterministic oracle machines on selected inputs and the resulting sequence of these integers allows us to construct an oracle set $\mathcal{Q}_2 \subseteq \mathbb{N}$ satisfying $\mathrm{P}_{\mathbb{R}}^{\mathcal{Q}_2} \neq \mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_2}$ by means of diagonalization techniques. Moreover, we can derive an oracle $\mathcal{Q}_3 \subseteq \mathbb{N} \times \mathbb{R}$ satisfying $\mathrm{P}_{\mathbb{R}}^{\mathcal{Q}_3} \neq \mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_3}$ from the Real Knapsack Problem such that we get a more natural decision problem as the oracle. Our constructions are based on constructions of oracles $\mathcal{Q} \subseteq \mathbb{R}^{\infty}$ presented in [Gaßner, 2009 (2)]. Here we will show that it is even possible to fix the arity of the tuples in the oracle sets.

Remark. L. Blum, M. Shub, and S. Smale introduced a uniform model of computation over the ring and over the field of the reals, respectively. To simplify matters, we want to consider only the ring operations.

For other models of computation over algebraic structures, a summary of papers where diagonalization techniques have been applied is given, for instance, in [Bürgisser 1999].

2 The Separation by Diagonalization Techniques

Every BSS machine using an oracle $\mathcal{B} \subseteq \mathbb{R}^{\infty}$ is determined by its machine constants and a program. Since the characters of the programs including all indices can be encoded by a finite sequence in $\{0,1\}^{\infty}$ independently of the used constants and the oracle $\mathcal{B}$, we are able to consider a sequence of sets of oracle BSS machines where the i^{th} set contains each $\mathcal{N}_i^{\mathcal{B},\mathbf{c}}$ deciding a problem in $\mathrm{P}_{\mathbb{R}}^{\mathcal{B}}$ by means of a program P_i and its own constants in $\mathbf{c} = (c_1, \ldots, c_{k_i})$ in a time bounded by a polynomial p_i. Moreover, for any $\mathcal{B} \subseteq \mathbb{N}$, the behavior of any $\mathcal{N}_i^{\mathcal{B},\mathbf{c}}$ on inputs $(0, \ldots, 0, m) \in \mathbb{N}^{n_i}$ for some n_i and for large positive integers m can be characterized by an integer $L_{\mathrm{char}}(i, \mathbf{c})$ that we will define in order to generate a sequence of integers, $(N_j)_{j \geq 1}$, and a sequence of sets of machines, $(\mathcal{K}_j^{\mathcal{B}})_{j \geq 1}$. Our aim is that all machines in any $\mathcal{K}_j^{\mathcal{B}} =_{\mathrm{df}} \{\mathcal{N}_i^{\mathcal{B},\mathbf{c}} \mid N_j = c(i, L_{\mathrm{char}}(i, \mathbf{c}))\}$ have the same behavior on inputs $(0, \ldots, 0, m) \in \mathbb{N}^{n_i}$ for large m. Note, that we use the Cantor numbers $c(x_1, x_2) = \frac{1}{2}((x_1 + x_2)^2 + 3x_1 + x_2)$ and $c(x_1, \ldots, x_n) = c(c(x_1, \ldots, x_{n-1}), x_n)$.

Since the number of polynomials in $\mathbb{R}[x]$ which can be computed by a BSS machine in bounded time is finite, it is enough to consider a finite sequence of polynomials $f_1, f_2, \ldots, f_{s_i} \in \mathbb{R}[x]$ for characterizing the behavior of $\mathcal{N}_i^{\mathcal{B},\mathbf{c}}$ on $(0,\ldots,0,x) \in \mathbb{N}^{n_i}$. The number $L_{\mathrm{char}}(i,\mathbf{c}) =_{\mathrm{df}} c(i,\mu_1,\ldots,\mu_{s_i},\mu)$ results from the characterization of these polynomials. For any $k \in \{1,\ldots,s_i\}$, let μ_k be given by $\mu_k = \mathrm{code}(f_k) \in \mathbb{N} \setminus \{0,1,2\}$ if $f_k \in \mathbb{Q}[x]$ and $\mathrm{degree}(f_k) \le 1$, and by $\mu_k = \lim_{x\to\infty} \mathrm{sgn}(f_k(x)) + 1$ otherwise. Moreover, let μ be an integer such that the following conditions are satisfied for all $m \ge \mu$ and for all $k \in \{1,\ldots,s_i\}$.

$$\begin{array}{cl} f_k(m) < 2^m & \text{if degree}(f_k) = 0, \\ 0 < f_k(m) < 2^m & \text{if degree}(f_k) = 1 \text{ and } \lim_{x\to\infty} f_k(x) = \infty, \\ f_k(m) < 0 & \text{if degree}(f_k) \ge 1 \text{ and } \lim_{x\to\infty} f_k(x) = -\infty, \\ m < f_k(m) < 2^m & \text{if degree}(f_k) > 1 \text{ and } \lim_{x\to\infty} f_k(x) = \infty, \\ f_k(m) \notin \mathbb{N} & \text{if degree}(f_k) = 1 \text{ and } f_k \notin \mathbb{Q}[x]. \end{array}$$

Let $L_{i,1}, L_{i,2}, \ldots$ be an enumeration of $\{L_{\mathrm{char}}(i,\mathbf{c}) \mid \mathbf{c} \in \mathbb{R}^{k_i}\}$ and $N_1, N_2, \ldots$ be an enumeration of $\bigcup_{i\ge1}\{c(i,L_{i,1}), c(i,L_{i,2}), \ldots\}$ such that $N_j < N_{j+1}$. These numbers allow us to construct $\mathcal{Q}_2 = \bigcup_{i\ge1} V_i \subseteq \mathbb{N}$ recursively. Let $m_0 = 0$, $V_0 = \emptyset$, and $C_1 = N_1$. Stage $j \ge 1$: Let $i = i_j$, r, and n_i be integers such that $N_j = c(i, L_{i,r})$ and $p_i(n_i) < 2^{n_i}$ and let

$$\begin{array}{ll} Z_j & = \{c(1,C_j), \ldots, c(2^{n_i}, C_j)\}, \\ W_j & = \bigcup_{k<j} V_k, \\ V_j & = \{x \in Z_j \mid \\ & \quad \& \ (\exists \mathbf{c} \in \mathbb{R}^{k_i})(\mathcal{N}_i^{W_j,\mathbf{c}} \in \mathcal{K}_j^{W_j} \ \& \ \mathcal{N}_i^{W_j,\mathbf{c}} \text{ rejects } (0,\ldots,0,C_j) \in \mathbb{N}^{n_i} \\ & \quad \& \ \mathcal{N}_i^{W_j,\mathbf{c}} \text{ does not use } x \text{ in a query on input } (0,\ldots,0,C_j) \in \mathbb{N}^{n_i})\}, \\ C_{j+1} & = \max\{2^{C_j}, N_{j+1}, c(2^{n_i}, C_j)\}. \end{array}$$

We have $L_2 = \bigcup_{j\ge1}\{(0,\ldots,0,C_j) \in \mathbb{N}^{n_{i_j}} \mid V_j \cap Z_j \ne \emptyset\} \in \mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_2} \setminus \mathrm{P}_{\mathbb{R}}^{\mathcal{Q}_2}$ and thus $\mathrm{P}_{\mathbb{R}}^{\mathcal{Q}_2} \ne \mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_2}$. The proof can be done similarly as in [Gaßner, 2009 (2)].

3 An Oracle Derived from the Knapsack Problem

In [Koiran 1994] and [Meer 1992], the relationships $\mathrm{P}_{\mathrm{add}}^{=} \ne \mathrm{DNP}_{\mathrm{add}}^{=}$ and $\mathrm{P}_{\mathrm{lin}}^{=} \ne \mathrm{DNP}_{\mathrm{lin}}^{=}$ were proved for the additive BSS model over $(\mathbb{R}; \mathbb{R}; +, -; =)$ and the linear BSS model over $(\mathbb{R}; 1; +, -, \{\phi_r \mid r \in \mathbb{R}\}; =)$ where $\phi_r(x) = rx$. It was shown that the *Real Knapsack Problem*

$$\mathrm{KP}_{\mathbb{R}} = \bigcup_{n=1}^{\infty} \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid (\exists(\alpha_1,\ldots,\alpha_n) \in \{0,1\}^n)(\sum_{i=1}^{n} \alpha_i x_i = 1)\}$$

belongs to $\mathrm{DNP}_{\mathrm{add}}^{=}$ and does not belong to $\mathrm{P}_{\mathrm{lin}}^{=}$ (cp. also [Koiran 1993]). Since, for an input $(x_1,\ldots,x_n) \in \mathbb{R}^\infty$, a digital non-deterministic machine can guess any sequence $(\alpha_1,\ldots,\alpha_n) \in \{0,1\}^n$ and compute $\alpha_1 x_1 + \cdots + \alpha_n x_n$ in linear time, $\mathrm{KP}_{\mathbb{R}} \in \mathrm{DNP}_{\mathbb{R}}$ holds. It is not known whether $\mathrm{KP}_{\mathbb{R}} \in \mathrm{P}_{\mathbb{R}}$ holds. The proof techniques used in [Meer 1992] fail if the order test is permitted. However, the following oracle allows us to apply similar techniques as in [Meer 1992] in order to show the inequality of the resulting relativized versions of $\mathrm{P}_{\mathbb{R}}$ and $\mathrm{DNP}_{\mathbb{R}}$.

Let $E_0 = \mathbb{Q}$, let $\tau_1, \tau_2, \ldots$ be a sequence of transcendental numbers such that τ_{i+1} is transcendental over $E_i =_{\mathrm{df}} E_{i-1}(\tau_i)$, and let the following sets A_n for $n \geq 1$, the oracle $\mathcal{Q}_3$, and the decision problem L_3 be given.

$$\begin{aligned} A_n &= \{(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n \mid \alpha_n = 1\}. \\ \mathcal{Q}_3 &= \textstyle\bigcup_{n=1}^{\infty}\{(\sum_{i=1}^{n} \alpha_i 2^{i-1}, \sum_{i=1}^{n} \alpha_i v \tau_i) \,|\, (\alpha_1, \ldots, \alpha_n) \in A_n \ \&\ v \in \mathbb{Z} \setminus \{0\}\}. \\ L_3 &= \textstyle\bigcup_{n=1}^{\infty}\{(0, \ldots, 0, r) \in \mathbb{R}^{n+1} \mid (\exists z \in \mathbb{Z})((z, r) \in \mathcal{Q}_3)\}. \end{aligned}$$

In order to show that $L_3 \notin \mathrm{P}_{\mathbb{R}}^{\mathcal{Q}_3}$, we will consider any BSS machine $\mathcal{M}$ which uses $\mathcal{Q}_3$ as the oracle and only the constants $c_1, \ldots, c_k$.

Let $F_0 = \bigcup_{i=0}^{\infty} E_i$. For $i = 1, \ldots, k$, let $F_i = F_{i-1}$ and $d_i = 1$ if $c_i \in F_{i-1}$, let $F_i = F_{i-1}(c_i)$ and $d_i = \infty$ if c_i is not algebraic over F_{i-1}, and let $F_i = F_{i-1}[c_i]$ if there is an irreducible polynomial $p_i \in F_{i-1}[x]$ of degree $d_i \geq 2$ with $p_i(c_i) = 0$. In the latter setting, c_i is algebraic over F_{i-1}. We will use that there is some i_0 such that the coefficients of each of these polynomials p_i have the form $w\tau_1^{m_1} \cdots \tau_{i_0}^{m_{i_0}} c_1^{j_1} \cdots c_{i-1}^{j_{i-1}}$ for some $w \in \mathbb{Z}$.

Any value computed by $\mathcal{M}$ on input $(0, \ldots, 0, x) \in \mathbb{R}^{\infty}$ can be described by some term of the form $\sum_{j,j_1,\ldots,j_k=0}^{t} \alpha_{j_1,\ldots,j_k,j} c_1^{j_1} \cdots c_k^{j_k} x^j$ where each $\alpha_{j_1,\ldots,j_k,j}$ is an integer and, consequently, by some term of the form $\frac{1}{r_0}\sum_{j=0}^{t} r_{j+1} x^j$ where we have

$$r_j = \sum_{\substack{m_1,\ldots,m_{i_0} \leq m_0 \\ j_\mu < \min\{d_\mu, j_0\}}} z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,j} \tau_1^{m_1} \cdots \tau_{i_0}^{m_{i_0}} c_1^{j_1} \cdots c_k^{j_k}$$

for some m_0, j_0, and $z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,j} \in \mathbb{Z}$ and

$$z_{m'_1,\ldots,m'_{i_0},j'_1,\ldots,j'_k,0} \neq 0 \text{ for some } m'_1, \ldots, m'_{i_0}, j'_1, \ldots, j'_k. \tag{1}$$

Thus, for any input $(0, \ldots, 0, x) \in \mathbb{R}^{\infty}$, a non-trivial oracle query whether $(z, q(x)) \in \mathcal{Q}_3$ holds (where degree$(q) \geq 1$) can be answered yes only if an equation of the form

$$\sum_{j=0}^{t} r_{j+1} x^j = r_0 \sum_{i=1}^{n} v_i \tau_i \tag{2}$$

is satisfied for some $(v_1, \ldots, v_n) \in vA_n$ where v is in $\mathbb{Z} \setminus \{0\}$ and vA_n is defined by $vA_n = \{(v\alpha_1, \ldots, v\alpha_n) \in \{0, v\}^n \mid \alpha_n = 1\}$. For $n' > i_0$, $v' \in \mathbb{Z} \setminus \{0\}$, and $(0, \ldots, 0, v'_{i_0+1}, \ldots, v'_{n'}) \in v'A_{n'}$, $x = \sum_{i=i_0+1}^{n'} v'_i \tau_i$ satisfies (2) if and only if

$$\begin{aligned} &\sum_{\substack{m_1,\ldots,m_{i_0} \leq m_0 \\ j_\mu < \min\{d_\mu, j_0\}}} (\textstyle\sum_{j=0}^{t} x^j z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,j+1}) \tau_1^{m_1} \cdots \tau_{i_0}^{m_{i_0}} c_1^{j_1} \cdots c_k^{j_k} \\ = &\sum_{\substack{m_1,\ldots,m_{i_0} \leq m_0 \\ j_\mu < \min\{d_\mu, j_0\}}} (\textstyle\sum_{i=1}^{n} v_i \tau_i) z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,0} \tau_1^{m_1} \cdots \tau_{i_0}^{m_{i_0}} c_1^{j_1} \cdots c_k^{j_k} \end{aligned}$$

is satisfied and, consequently, only if the equations

$$\begin{aligned}
&\sum_{m_1,\ldots,m_{i_0}\le m_0} z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,1}\tau_1^{m_1}\cdots\tau_{i_0}^{m_{i_0}}\\
&= (\sum_{m_1,\ldots,m_{i_0}\le m_0} z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,0}\tau_1^{m_1}\cdots\tau_{i_0}^{m_{i_0}})(\textstyle\sum_{i=1}^{i_0} v_i\tau_i),\\
&(\sum_{m_1,\ldots,m_{i_0}\le m_0} z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,2}\tau_1^{m_1}\cdots\tau_{i_0}^{m_{i_0}})(\textstyle\sum_{i=i_0+1}^{n'} v'_i\tau_i)\\
&= (\sum_{m_1,\ldots,m_{i_0}\le m_0} z_{m_1,\ldots,m_{i_0},j_1,\ldots,j_k,0}\tau_1^{m_1}\cdots\tau_{i_0}^{m_{i_0}})(\textstyle\sum_{i=i_0+1}^{n} v_i\tau_i)
\end{aligned}$$

are satisfied for any $j_1 < \min\{d_1, j_0\}, \ldots, j_k < \min\{d_k, j_0\}$.

Since (1) holds, the latter equations can be satisfied only if we have $v_i = 0 \Leftrightarrow v'_i = 0$ for all $i = i_0+1, \ldots, \min\{n, n'\}$, $v_i = 0$ for $i = n'+1, \ldots, n$ (if $n' < n$), and $v'_i = 0$ for $i = n+1, \ldots, n'$ (if $n < n'$). Thus, we have shown the following.

Lemma 1. *Let $n > i_0$, $(0, \ldots, 0, v_{i_0+1}, \ldots, v_n) \in vA_n$, and $x = \sum_{i=i_0+1}^{n} v_i\tau_i$. A non-trivial oracle query whether $(z, q(x)) \in \mathcal{Q}_3$ holds executed by $\mathcal{M}$ can be answered yes on inputs of the form $(0, \ldots, 0, x)$ only if $z = \sum_{i=1}^{i_0} \alpha_i \cdot 2^{i-1} + \mathrm{sgn}(|v_{i_0+1}|) \cdot 2^{i_0} + \cdots + \mathrm{sgn}(|v_n|) \cdot 2^{n-1}$ holds for some $\alpha_1, \ldots, \alpha_{i_0} \in \{0, 1\}$.*

Lemma 2. $L_3 \notin \mathrm{P}_{\mathbb{R}}^{\mathcal{Q}_3}$.

Proof. Let us assume that L_3 is decidable by a machine $\mathcal{M}$ described above in a time bounded by some polynomial p and let n_0 be an integer such that $n_0 > i_0$ and $p(n_0+1) < 2^{n_0-i_0-1}$. We want to consider the computation path P of $\mathcal{M}$ on inputs $(0, \ldots, 0, x) \in \mathbb{R}^{n_0+1}$ which can be uniquely described by conditions of the form $(g_j(x), h_j(x)) \notin \mathcal{Q}_3$ and $f_j(x) > 0$ ($j \le t$, $t \le p(n_0+1)$) where g_j and h_j are polynomials which satisfy, for any j, degree$(g_j) > 0$ or degree$(h_j) > 0$ and each f_j is defined by some equation of the form $f_j(x) = x^{n_j} + a_{n_j-1}x^{n_j-1} + \cdots + a_1x + a_0$.

Let $\tau > 0$ be transcendental over F_k and greater than all zeros of $f_1, \ldots, f_t$. Then, the path P is traversed by $\mathcal{M}$ on $(0, \ldots, 0, \tau) \in \mathbb{R}^{n_0+1} \setminus L_3$. If $g_j(\tau)$ is in $\mathbb{Z}$, then the polynomial g_j is constant. Since, for the set $G = \{g_j(\tau) \mid j \le t \;\&\; g_j \text{ is a constant function}\}$, we have $|G| \le p(n_0+1) < 2^{n_0-i_0-1}$, there is some $(0, \ldots, 0, x_0) \in \mathbb{R}^{n_0+1}$ with $x_0 = \sum_{i=i_0+1}^{n_0} w\alpha_i\tau_i$ satisfying a), b), and c).

a) $(0, \ldots, 0, \alpha_{i_0+1}, \ldots, \alpha_{n_0}) \in A_{n_0}$ and $\alpha_{n_0} \ne 0$ and $w \in \mathbb{Z} \setminus \{0\}$,

b) $x_0 > \tau$,

c) $z + \sum_{i=i_0+1}^{n_0} \alpha_i \cdot 2^{i-1} \notin G$ for any $z \in \{0, \ldots, 2^{i_0} - 1\}$.

By b) we have $f_j(x_0) > 0$. If $g_j(x_0)$ is an integer, then we can show, by analogy with the derivation of Lemma 1, that g_j is constant and, consequently, $g_j(x_0) \in G$ holds. Thus, by c) and by Lemma 1, P is also traversed by $\mathcal{M}$ on $(0, \ldots, 0, x_0) \in \mathbb{R}^{n_0+1}$. But this contradicts a) by which $(0, \ldots, 0, x_0)$ belongs to L_3. □

Because of $L_3 \in \mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}_3}$, we get the following.

Proposition 3. *There is an oracle $\mathcal{Q}$ derived from the Real Knapsack Problem such that $\mathrm{P}_{\mathbb{R}}^{\mathcal{Q}} \ne \mathrm{DNP}_{\mathbb{R}}^{\mathcal{Q}}$.*

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