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Isometries and Computability Structures

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Abstract: We investigate the relationship between computable metric spaces (X, d, α) and (X, d, β), where (X, d) is a given metric space. In the case of Euclidean space, α and β are equivalent up to isometry, which does not hold in general. We introduce the notion of effectively dispersed metric space and we use it in the proof of the following result: if (X, d, α) is effectively totally bounded, then (X, d, β) is also effectively totally bounded. This means that the property that a computable metric space is effectively totally bounded (and in particular effectively compact) depends only on the underlying metric space. In the final section of this paper we examine compact metric spaces (X, d) such that there are only finitely many isometries $X \to X$. We prove that in this case a stronger result holds than the previous one: if (X, d, α) is effectively totally bounded, then α and β are equivalent. Hence if (X, d, α) is effectively totally bounded, then (X, d) has a unique computability structure.
Key Words: computable metric space, computability structure, effective total boundedness, effective dispersion, effective compactness, isometry
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1 Introduction

Let $k \in \mathbf{N}$, $k \geq 1$. We say that a function $f : \mathbf{N}^k \to \mathbf{Q}$ is **recursive** if there exist recursive functions $a, b, c : \mathbf{N}^k \to \mathbf{N}$ such that $f(x) = (-1)^{c(x)} \frac{a(x)}{b(x)+1}$, $\forall x \in \mathbf{N}^k$. A function $f : \mathbf{N}^k \to \mathbf{R}$ is said to be **recursive** if there exists a recursive function $F : \mathbf{N}^{k+1} \to \mathbf{Q}$ such that $|f(x) - F(x, i)| < 2^{-i}$, $\forall x \in \mathbf{N}^k$, $\forall i \in \mathbf{N}$.

A tuple (X, d, α) is said to be a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbf{N} \to X$ is a sequence dense in (X, d) (i.e. a sequence which range is dense in (X, d)) such that the function $\mathbf{N}^2 \to \mathbf{R}$, $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is recursive (we use notation $\alpha = (\alpha_i)$). We say that α is an **effective separating sequence** in (X, d) (cf. [Yasugi, Mori and Tsujji 1999]). If (X, d, α) is a computable metric space, then a sequence (x_i) in X is said to be **recursive** in (X, d, α) if there exists a recursive function $F : \mathbf{N}^2 \to \mathbf{N}$ such that $d(x_i, \alpha_{F(i,k)}) < 2^{-k}$, $\forall i, k \in \mathbf{N}$ and a point $a \in X$ is said to be **recursive** in (X, d, α) if the constant sequence $a, a, \ldots$ is recursive. For example, if $q : \mathbf{N} \to \mathbf{Q}$ is a recursive surjection, then $(\mathbf{R}, d, q)$ is a computable metric space, where d is the Euclidean metric on $\mathbf{R}$. A sequence (x_i) is recursive in this computable metric space if and only if (x_i) is a recursive sequence of real numbers and $a \in \mathbf{R}$ is a recursive point in this space if and only if a is a recursive number.

Let (X, d) be a metric space and let $\mathcal{S}$ be a nonempty set whose elements are sequences in X. We say that $\mathcal{S}$ is a **computability structure** on (X, d) (cf.

[Yasugi, Mori and Tsujji 1999]) if the following four properties hold:

(i) if $(x_i), (y_j) \in \mathcal{S}$, then the function $\mathbf{N}^2 \to \mathbf{R}$, $(i, j) \mapsto d(x_i, y_j)$ is recursive;

(ii) if $(x_i)_{i\in\mathbf{N}} \in \mathcal{S}$, then $(x_{f(i)})_{i\in\mathbf{N}} \in \mathcal{S}$ for any recursive function $f : \mathbf{N} \to \mathbf{N}$;

(iii) if (y_i) is a sequence in X such that $d(y_i, x_{F(i,k)}) < 2^{-k}$, $\forall i, k \in \mathbf{N}$, where $F : \mathbf{N}^2 \to \mathbf{N}$ is a recursive function and $(x_i) \in \mathcal{S}$, then $(y_i) \in \mathcal{S}$;

(iv) there exists $(x_i) \in \mathcal{S}$ such that (x_i) is dense in (X, d).

Let (X, d) be a metric space. If α is an effective separating sequence in (X, d), then the set $\mathcal{S}_\alpha$ of all recursive sequences in (X, d, α) is a computability structure on (X, d). Suppose now that α and β are effective separating sequences in (X, d). We say that α is **equivalent** to β, $\alpha \sim \beta$, if α is a recursive sequence in (X, d, β). It follows easily that $\alpha \sim \beta$ if and only if $\mathcal{S}_\alpha = \mathcal{S}_\beta$.

A closed subset S of a computable metric space (X, d, α) is said to be **recursively enumerable** if $\{i \in \mathbf{N} \mid I_i \cap S \neq \emptyset\}$ is an r.e. set, where (I_i) is some effective enumeration of all open rational balls in (X, d, α) (by an open rational ball we mean an open ball with rational radius and with center α_i, for some $i \in \mathbf{N}$), **co-recursively enumerable** if $X \setminus S = \bigcup_{i\in\mathbf{N}} I_{f(i)}$, where $f : \mathbf{N} \to \mathbf{N}$ is a recursive function and **recursive** if it is both r.e. and co-r.e. ([Brattka and Presser 2003]). It is not hard to see that if $\alpha \sim \beta$, then S is r.e. (co-r.e.) in (X, d, α) if and only if S is r.e. (co-r.e.) in (X, d, β) and consequently S is recursive in (X, d, α) if and only if S is recursive in (X, d, β). Hence the notions of recursive enumerability, co-recursive enumerability and recursiveness of a set are examples of notions which depend only on the induced computability structure and not on particular α which induces that structure.

If α and β are effective separating sequences in a metric space (X, d), then α and β need not be equivalent. For example, if $c \in \mathbf{R}$ is a nonrecursive number and (α_i) a recursive sequence of real numbers dense in $(\mathbf{R}, d)$, where d is the Euclidean metric, then $(\alpha_i + c)$ is an effective separating sequence in $(\mathbf{R}, d)$, c is a recursive point in $(\mathbf{R}, d, (\alpha_i + c))$ and c is not recursive in $(\mathbf{R}, d, (\alpha_i))$. Hence (α_i) and $(\alpha_i + c)$ are not equivalent.

Let $(X, d, (\alpha_i))$ be a computable metric space and f an isometry of (X, d). By an isometry of (X, d) we mean a surjective map $f : X \to X$ such that $d(f(x), f(y)) = d(x, y)$, $\forall x, y \in X$. Then $(X, d, (f(\alpha_i)))$ is also a computable metric space and in general the sequences (α_i) and $(f(\alpha_i))$ are not equivalent by the previous example. Note that f "maps" the computability structure induced by (α_i) on the computability structure induced by $(f(\alpha_i))$, i.e.

$$\mathcal{S}_{(f(\alpha_i))} = \{(f(x_i)) \mid (x_i) \in \mathcal{S}_{(\alpha_i)}\}.$$

In particular, if A is the set of all recursive points in $(X, d, (\alpha_i))$ and B the set of all recursive points in $(X, d, (f(\alpha_i)))$, then $f(A) = B$.

We say that effective separating sequences (α_i) and (β_i) in a metric space (X, d) are **equivalent up to isometry** if $(\alpha_i) \sim (f(\beta_i))$ for some isometry f of (X, d). It is easy to see that this relation is an equivalence relation on the set of all effective separating sequences in (X, d).

A metric space (X, d) is said to be **totally bounded** if for each $\varepsilon > 0$ there exist finitely many points $y_0, \ldots, y_m \in X$ such that $X = \bigcup_{0 \leq i \leq m} B(y_i, \varepsilon)$. Here $B(x, r)$ for $x \in X$ and $r > 0$ denotes the open ball of radius r centered at x. If (X, d, α) is a computable metric space, then the sequence α is dense in (X, d) and we have the following conclusion: the metric space (X, d) is totally bounded if and only if for each $k \in \mathbf{N}$ there exists $m \in \mathbf{N}$ such that $X = \bigcup_{0 \leq i \leq m} B(\alpha_i, 2^{-k})$. We say that a computable metric space (X, d, α) is **effectively totally bounded** if there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that

$$X = \bigcup_{i=0}^{f(k)} B(\alpha_i, 2^{-k}),$$

$\forall k \in \mathbf{N}$ ([Yasugi, Mori and Tsujji 1999]).

Example 1. If S is a recursive nonempty compact subset of $\mathbf{R}^n$, then there exists a recursive sequence (x_i) in S and a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $S \subseteq \bigcup_{0 \leq i \leq f(k)} B(x_i, 2^{-k})$, $\forall k \in \mathbf{N}$ ([Zhou 1996, Weihrauch 2000]) and therefore $(S, d, (x_i))$ is an effectively totally bounded computable metric space, where d is the Euclidean metric on S.

Example 2. Let $\omega : \mathbf{N} \to \mathbf{Q}$ be a recursive sequence which converges to a nonrecursive number $\gamma \in \mathbf{R}$ and such that $\omega(0) = 0$, $\omega(i) < \omega(i+1)$, $\forall i \in \mathbf{N}$. It is easy to construct a recursive sequence of rational numbers α which is dense in $[0, \gamma]$. Then the tuple $([0, \gamma], d, \alpha)$ is a computable metric space, where d is the Euclidean metric on $[0, \gamma]$. Suppose that $([0, \gamma], d, \alpha)$ is effectively totally bounded. Then $[0, \gamma] = \bigcup_{0 \leq i \leq f(k)} B(\alpha_i, 2^{-k})$, $\forall k \in \mathbf{N}$, for some recursive function $f : \mathbf{N} \to \mathbf{N}$. If $h : \mathbf{N} \to \mathbf{Q}$ is defined by $h(k) = \max\{\alpha_i \mid 0 \leq i \leq f(k)\}$, $k \in \mathbf{N}$, then h is a recursive function and $|\gamma - h(k)| < 2^{-k}$, $\forall k \in \mathbf{N}$ which contradicts the fact that γ is a nonrecursive number. Hence the computable metric space $([0, \gamma], d, \alpha)$ is not effectively totally bounded, although the metric space $([0, \gamma], d)$ is totally bounded.

It is not hard to check that if α and β are equivalent effective separating sequences in a metric space (X, d), then (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded. Furthermore, if f is an isometry of (X, d) and (α_i) an effective separating sequence, then $(X, d, (\alpha_i))$ is effectively totally bounded if and only if $(X, d, (f(\alpha_i)))$ is effectively totally bounded. This follows immediately from the fact that $f(B(x, r)) = B(f(x), r)$,

$\forall x \in X$, $\forall r > 0$. Therefore, if α and β are effective separating sequences equivalent up to isometry, then (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded.

There exist totally bounded metric spaces with effective separating sequences nonequivalent up to isometry (Section 2). Nevertheless, the equivalence

$$(X, d, \alpha) \text{ effectively totally bounded } \Leftrightarrow (X, d, \beta) \text{ effectively totally bounded} \tag{1}$$

holds in general and that is a result which will be proved in Section 3 where we introduce the notion of effectively dispersed metric space. In Section 2 we also prove that each two effective separating sequence in Euclidean space $\mathbf{R}^n$ are equivalent up to isometry.

In Section 4 we examine compact metric spaces (X, d) such that there are only finitely many isometries of (X, d). We prove that in this case a stronger result holds than (1): if α and β are effective separating sequences in (X, d) such that (X, d, α) is effectively totally bounded, then $\alpha \sim \beta$. This implies the following: if there exists an effective separating sequence α in (X, d) such that (X, d, α) is effectively totally bounded, then (X, d) has a unique computability structure.

1.1 Basic techniques

Let $k, n \in \mathbf{N}$, $k, n \geq 1$. By a recursive function $f : \mathbf{N}^k \to \mathbf{N}^n$ we mean a function whose component functions $f_1, \dots, f_n : \mathbf{N}^k \to \mathbf{N}$ are recursive. In the following proposition we state some elementary facts.

Proposition 1. *(i) Let $T \subseteq \mathbf{N}^{k+n}$ be a recursively enumerable set. Then the set $S = \{x \in \mathbf{N}^k \mid \exists y \in \mathbf{N}^n \ (x, y) \in T\}$ is recursively enumerable.*

(ii) Let $S \subseteq \mathbf{N}^{k+n}$ be a recursively enumerable set such that for each $x \in \mathbf{N}^k$ there exists $y \in \mathbf{N}^n$ such that $(x, y) \in S$. Then there exists a recursive function $f : \mathbf{N}^k \to \mathbf{N}^n$ such that $(x, f(x)) \in S$, $\forall x \in \mathbf{N}^k$.

In the following proposition we state some elementary facts about recursive functions $\mathbf{N}^k \to \mathbf{R}$.

Proposition 2. *(i) If $f, g : \mathbf{N}^k \to \mathbf{R}$ are recursive, then $f + g, f - g : \mathbf{N}^k \to \mathbf{R}$ are recursive.*

(ii) If $f : \mathbf{N}^k \to \mathbf{R}$ and $F : \mathbf{N}^{k+1} \to \mathbf{R}$ are functions such that F is recursive and $|f(x) - F(x, i)| < 2^{-i}$, $\forall x \in \mathbf{N}^k$, $\forall i \in \mathbf{N}$, then f is recursive.

(iii) If $f : \mathbf{N}^{k+1} \to \mathbf{R}$ and $\varphi : \mathbf{N}^k \to \mathbf{N}$ are recursive functions, then the functions $g, h : \mathbf{N}^k \to \mathbf{R}$ defined by $g(x) = \max_{0 \leq i \leq \varphi(x)} f(i, x)$, $h(x) = \min_{0 \leq i \leq \varphi(x)} f(i, x)$, $x \in \mathbf{N}^k$, are recursive.

(iv) If $f, g : \mathbf{N}^k \to \mathbf{R}$ is a recursive function, then the set $\{x \in \mathbf{N}^k \mid f(x) < g(x)\}$ is r.e.

We say that a function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is **recursive** if the function $\overline{\Phi} : \mathbf{N}^{k+n} \to \mathbf{N}$ defined by

$$\overline{\Phi}(x, y) = \chi_{\Phi(x)}(y),$$

$x \in \mathbf{N}^k$, $y \in \mathbf{N}^n$ is recursive. Here $\mathcal{P}(\mathbf{N}^n)$ denotes the set of all subsets of $\mathbf{N}^n$, and $\chi_S : \mathbf{N}^n \to \mathbf{N}$ denotes the characteristic function of $S \subseteq \mathbf{N}^n$. A function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is said to be **recursively bounded** if there exists a recursive function $\varphi : \mathbf{N}^k \to \mathbf{N}$ such that $\Phi(x) \subseteq \{0, \dots, \varphi(x)\}^n$, $\forall x \in \mathbf{N}^k$, where $\{0, \dots, \varphi(x)\}^n$ equals the set of all $(y_1, \dots, y_n) \in \mathbf{N}^n$ such that $\{y_1, \dots, y_n\} \subseteq \{0, \dots, \varphi(x)\}$.

We say that a function $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ is **r.r.b.** if Φ is recursive and recursively bounded. The proof of the following proposition is straightforward.

Proposition 3. *If $\Phi, \Psi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ are r.r.b. functions, then the sets $\{x \in \mathbf{N}^k \mid \Phi(x) = \Psi(x)\}$, $\{x \in \mathbf{N}^k \mid \Phi(x) \subseteq \Psi(x)\}$, $\{x \in \mathbf{N}^k \mid \Phi(x) = \emptyset\}$ are recursive.*

It is not hard to prove the following proposition.

Proposition 4. *Let $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ and $\Psi : \mathbf{N}^{n+k} \to \mathcal{P}(\mathbf{N}^m)$ be r.r.b. functions. Let $\Lambda : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^m)$ be defined by*

$$\Lambda(x) = \bigcup_{z \in \Phi(x)} \Psi(z, x),$$

$x \in \mathbf{N}^k$. Then Λ is an r.r.b. function.

Example 3. If $\alpha, \beta : \mathbf{N}^k \to \mathbf{N}$ and $f : \mathbf{N}^{k+1} \to \mathbf{N}^n$ are recursive functions, then the function $\mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$, $x \mapsto \{f(i, x) \mid \alpha(x) \leq i \leq \beta(x)\}$ is r.r.b.

It is not hard to prove the following lemma.

Lemma 5. *Let $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^k)$ be r.r.b. and let $T \subseteq \mathbf{N}^n$ be r.e. Then the set $S = \{x \in \mathbf{N}^k \mid \Phi(x) \subseteq T\}$ is r.e.*

Let $\sigma : \mathbf{N}^2 \to \mathbf{N}$ and $\eta : \mathbf{N} \to \mathbf{N}$ be some fixed recursive functions with the following property: $\{(\sigma(i, 0), \dots, \sigma(i, \eta(i))) \mid i \in \mathbf{N}\}$ is the set of all nonempty finite sequences in $\mathbf{N}$, i.e. the set $\{(a_0, \dots, a_n) \mid n \in \mathbf{N},\ a_0, \dots, a_n \in \mathbf{N}\}$. Such functions, for instance, can be defined using the Cantor pairing function. We are going to use the following notation: $(i)_j$ instead of $\sigma(i, j)$ and $\overline{i}$ instead of $\eta(i)$. Hence

$$\{((i)_0, \dots, (i)_{\overline{i}}) \mid i \in \mathbf{N}\}$$

is the set of all nonempty finite sequences in $\mathbf{N}$.

Lemma 6. *Let $\Phi : \mathbf{N}^k \to \mathcal{P}(\mathbf{N}^n)$ be an r.r.b. function and let $f : \mathbf{N}^n \to \mathbf{R}$ be a recursive function. Then there exist recursive functions $\varphi, \psi : \mathbf{N}^k \to \mathbf{R}$ such that*

$$\varphi(x) = \min_{y \in \Phi(x)} f(y),\ \psi(x) = \max_{y \in \Phi(x)} f(y)$$

for each $x \in \mathbf{N}^k$ *such that* $\Phi(x) \neq \emptyset$.

Proof. Let $\alpha : \mathbf{N} \to \mathbf{N}^n$ be some recursive surjection. Let $\Gamma : \mathbf{N} \to \mathcal{P}(\mathbf{N}^n)$ be defined by

$$\Gamma(i) = \{\alpha((i)_j) \mid 0 \leq j \leq \bar{i}\}.$$

Then Γ is r.r.b. (Example 3). Note that each nonempty subset of $\mathbf{N}^n$ equals $\Gamma(i)$ for some $i \in \mathbf{N}$. Therefore, for each $x \in \mathbf{N}^k$ there exists $i \in \mathbf{N}$ such that ($\Phi(x) = \Gamma(i)$ or $\Phi(x) = \emptyset$.) By Proposition 3 and Proposition 1(ii) there exists a recursive function $\lambda : \mathbf{N}^k \to \mathbf{N}$ such that $\Phi(x) = \Gamma(\lambda(x))$ for each $x \in \mathbf{N}^k$ such that $\Phi(x) \neq \emptyset$. Now we define $\varphi : \mathbf{N}^k \to \mathbf{R}$ by

$$\varphi(x) = \min_{0 \leq j \leq \overline{\lambda(x)}} f(\alpha((\lambda(x))_j)),\ \psi(x) = \max_{0 \leq j \leq \overline{\lambda(x)}} f(\alpha((\lambda(x))_j)),$$

$x \in \mathbf{N}^k$. Then φ and ψ have the desired property (recursiveness of these functions follows from Proposition 2(iii)). $\square$

Lemma 7. *There exists a recursive function* $\zeta : \mathbf{N}^2 \to \mathbf{N}$ *such that for all* $m, p \in \mathbf{N}$ *each finite sequence* $x_0, \dots, x_p$ *in* $\{0, \dots, m\}$ *is equal to* $(i)_0, \dots, (i)_{\bar{i}}$ *for some* $i \in \mathbf{N}$ *such that* $i \leq \zeta(m, p)$.

2 Computability structures on Euclidean space

Let $n \geq 1$ and let d be the Euclidean metric on $\mathbf{R}^n$. The main step in proving that every two effective separating sequences in $(\mathbf{R}^n, d)$ are equivalent up to isometry is the following proposition.

Proposition 8. *Let* $a_0, \dots, a_n$ *be recursive points in* $\mathbf{R}^n$ *which are geometrically independent (i.e.* $a_1 - a_0, \dots, a_n - a_0$ *are linearly independent vectors) and let* (x_i) *be a sequence in* $\mathbf{R}^n$ *such that* $(d(x_i, a_k))_{i \in \mathbf{N}}$ *is a recursive sequence of real numbers for each* $k \in \{0, \dots, n\}$. *Then* (x_i) *is a recursive sequence.*

Proof. For $k \in \{0, \dots, n\}$ let $v_k : \mathbf{N} \to \mathbf{R}$ be the function defined by

$$v_k(i) = d(x_i, a_k),\ i \in \mathbf{N}.$$

Let $i \in \mathbf{N}$. For $k \in \{0, \dots, n\}$ we have

$$\langle x_i - a_k | x_i - a_k \rangle = v_k(i)^2, \tag{2}$$

where $(x, y) \mapsto \langle x | y \rangle$, $x, y \in \mathbf{R}^n$, is the inner product. It follows from (2) that for each $k \in \{1, \dots, n\}$ we have

$$\langle x_i - a_k | x_i - a_k \rangle - \langle x_i - a_0 | x_i - a_0 \rangle = v_k(i)^2 - v_0(i)^2$$

which implies

$$\langle x_i| - 2a_k + 2a_0\rangle = v_k(i)^2 - v_0(i)^2 - \langle a_k|a_k\rangle + \langle a_0|a_0\rangle.$$

Hence there exist recursive functions $s_1, \ldots, s_n : \mathbf{N} \to \mathbf{R}$ such that

$$\langle x_i|a_k - a_0\rangle = s_k(i), \tag{3}$$

$\forall i \in \mathbf{N}$, $\forall k \in \{1, \ldots, n\}$. For $i \in \mathbf{N}$ let $x_i^1, \ldots, x_i^n \in \mathbf{R}$ be numbers such that $x_i = (x_i^1, \ldots, x_i^n)$. Let A be the $n \times n$ matrix whose $k-th$ row is the $n-$tuple $a_k - a_0$, i.e. $A = \begin{pmatrix} a_1 - a_0 \\ \vdots \\ a_n - a_0 \end{pmatrix}$. It follows from (3) that $A \begin{pmatrix} x_i^1 \\ \vdots \\ x_i^n \end{pmatrix} = \begin{pmatrix} s_1(i) \\ \vdots \\ s_n(i) \end{pmatrix}$. The rank of the matrix A is clearly n, hence A is invertible and we have

$$\begin{pmatrix} x_i^1 \\ \vdots \\ x_i^n \end{pmatrix} = A^{-1} \begin{pmatrix} s_1(i) \\ \vdots \\ s_n(i) \end{pmatrix}. \tag{4}$$

In general, if B is an invertible matrix, then each element of B^{-1} can be written as the quotient of the determinants of matrices which consist of certain elements of B. Therefore each element of A^{-1} is a recursive number and it follows from (4) that $(x_i^1)_{i\in\mathbf{N}}, \ldots, (x_i^n)_{i\in\mathbf{N}}$ are recursive sequences. Hence $(x_i)_{i\in\mathbf{N}}$ is a recursive sequence. □

Proposition 8 is essentially a consequence of the fact that we can compute each component of x_i by certain formula which involves addition, subtraction, multiplication and division of numbers $d(x_i, a_0), \ldots, d(x_i, a_n)$ and components of the points $a_0, \ldots, a_n$. It follows from Proposition 8 that for geometrically independent recursive points $a_0, \ldots, a_n$ in $\mathbf{R}^n$ and $x \in \mathbf{N}$ the following implication holds:

$$\text{the numbers } d(x, a_0), \ldots, d(x, a_n) \text{ are recursive } \Rightarrow \text{ the point } x \text{ is recursive.} \tag{5}$$

However, in a general computable metric space it is not possible to find $n \in \mathbf{N}$ and recursive points $a_0, \ldots, a_n$ such that the implication (5) holds. This shows the following example.

Example 4. Let p be the metric on $\mathbf{R}^2$ given by $p((x_1, y_1), (x_2, y_2)) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$. If (α_i) is a recursive dense sequence in $\mathbf{R}^2$, then $(\mathbf{R}^2, p, (\alpha_i))$ is a computable metric space and the induced computability structure coincides with the usual computability structure on $\mathbf{R}^2$. Suppose $(x_0, y_0), \ldots, (x_k, y_k)$ are any recursive points in $\mathbf{R}^2$. Let $M > 0$ be some upper bound of the set $\{|x_0|, |y_0|, \ldots, |x_k|, |y_k|\}$. Let $a, b \in \mathbf{R}$ be such that $a > 3M$, $|b| < M$ and such that a is a recursive, and b a nonrecursive number. Then $p((a, b), (x_0, y_0)), \ldots$ $p((a, b), (x_k, y_k))$ are recursive numbers, but (a, b) is a nonrecursive point.

The following corollary is an immediate consequence of Proposition 8.

Corollary 9. *Suppose* $(\mathbf{R}^n, d, \alpha)$ *is a computable metric space,* $f : \mathbf{R}^n \to \mathbf{R}^n$ *an isometry and* $a_0, \dots, a_n$ *recursive points in* $(\mathbf{R}^n, d, \alpha)$ *which are geometrically independent and such that* $f(a_0), \dots, f(a_n)$ *are recursive points in* $\mathbf{R}^n$ *in the usual sense. Then* $f \circ \alpha$ *is a recursive sequence in the usual sense.*

The next step in proving that every two effective separating sequences in $(\mathbf{R}^n, d)$ are equivalent up to isometry is the following lemma.

Lemma 10. *Let* $a_0, \dots, a_n$ *be geometrically independent points in* $\mathbf{R}^n$ *such that* $d(a_i, a_j)$ *is a recursive number for all* $i, j \in \{0, \dots, n\}$*. Then there exists an isometry* $f : \mathbf{R}^n \to \mathbf{R}^n$ *such that* $f(a_0), \dots, f(a_n)$ *are recursive points.*

Proof. By the Gram-Schmidt orthogonalization process there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbf{R}^n$ such that the sets $\{a_1 - a_0, \dots, a_j - a_0\}$ and $\{e_1, \dots, e_j\}$ span the same linear subspace of $\mathbf{R}^n$ for each $j \in \{1, \dots, n\}$. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be the composition of the map $g : \mathbf{R}^n \to \mathbf{R}^n$, $x \mapsto x - a_0$ and the map $h : \mathbf{R}^n \to \mathbf{R}^n$, $h(t_1 e_1 + \dots + t_n e_n) = (t_1, \dots, t_n)$, $t_1, \dots, t_n \in \mathbf{R}$. Then f is an isometry and $f(a_0) = (0, \dots, 0)$,

$$f(a_k) \in \{(t_1, \dots, t_k, 0, \dots, 0) \mid t_1, \dots, t_k \in \mathbf{R},\ t_k \neq 0\},$$

$\forall k \in \{1, \dots, n\}$. We prove now that $f(a_k)$ is a recursive point for each $k \in \{0, \dots, n\}$. This is clearly true for $k = 0$. For $k \in \{1, \dots, n\}$ let $b_1^k, \dots, b_k^k \in \mathbf{R}$ be such that

$$f(a_k) = (b_1^k, \dots, b_k^k, 0, \dots, 0).$$

Suppose that $f(a_0), \dots, f(a_{k-1})$ are recursive points for some $k \in \{1, \dots, n\}$. Let us prove that $f(a_k)$ is recursive. For $l \in \{0, \dots, k-1\}$ let

$$r_l = d(f(a_k), f(a_l)). \tag{6}$$

Note that the numbers $r_0, \dots, r_{k-1}$ are recursive. It follows from (6) for $l = 0$ that

$$(b_1^k)^2 + (b_2^k)^2 + \dots + (b_k^k)^2 = r_0^2 \tag{7}$$

and for $l = 1$ that

$$(b_1^k - b_1^1)^2 + (b_2^k)^2 + \dots + (b_k^k)^2 = r_1^2. \tag{8}$$

Subtracting (8) from (7) we get that b_1^k is a recursive number. We get from (6) for $l = 2$ that

$$(b_1^k - b_1^2)^2 + (b_2^k - b_2^2)^2 + \dots + (b_k^k)^2 = r_2^2$$

which, together with (7), now implies that b_2^k is recursive. Repeating this argument for $l = 3, \dots, k-1$ we obtain that $b_3^k, \dots, b_{k-1}^k$ are recursive. Now (7) implies that b_k^k is recursive and therefore $f(a_k)$ is a recursive point. We conclude that $f(a_0), \dots, f(a_n)$ are recursive points. $\square$

Proposition 11. *Let (α_i) be an effective separating sequence in $\mathbf{R}^n$. Then there exists an isometry $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $(f(\alpha_i))$ is a recursive sequence in $\mathbf{R}^n$.*

Proof. Let $i_0, \dots, i_n \in \mathbf{N}$ be such that $\alpha_{i_0}, \dots, \alpha_{i_n}$ are geometrically independent points. By Lemma 10 there exists an isometry $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(a_{i_0}), \dots, f(a_{i_n})$ are recursive points. The claim of the theorem now follows from Corollary 9. □

Note the following: if (x_i) and (y_i) are recursive dense sequences in $\mathbf{R}^n$, then (x_i) and (y_i) are equivalent as effective separating sequences. This and Proposition 11 imply the following.

Theorem 12. *If α and β are effective separating sequences in $(\mathbf{R}^n, d)$, then α and β are equivalent up to isometry.*

Euclidean space $\mathbf{R}^n$ is not totally bounded, but each open (or closed) ball in $\mathbf{R}^n$ is totally bounded. We say that a computable metric space (X, d, α) **can be exhausted effectively by totally bounded balls** if there exists $\tilde{x} \in X$ and a recursive function $F : \mathbf{N}^2 \to \mathbf{N}$ such that

$$B(\tilde{x}, m) \subseteq \bigcup_{i=0}^{F(k,m)} B(\alpha_i, 2^{-k}),$$

$\forall k, m \in \mathbf{N}$. It is clear that if such a function F exists for one $\tilde{x} \in X$, then it exists for each $\tilde{x} \in X$. It is obvious that each effectively totally bounded computable metric space can be exhausted effectively by totally bounded balls. Furthermore, if α is some recursive dense sequence in $\mathbf{R}^n$, then $(\mathbf{R}^n, d, \alpha)$ can be exhausted effectively by totally bounded balls. It is easy to conclude from this and Theorem 12 that any computable metric space of the form $(\mathbf{R}^n, d, \alpha)$ can be exhausted effectively by totally bounded balls.

In the contrast to the fact that the equivalence (1) holds in general, which will be proved later, the equivalence

$$\begin{gathered}(X, d, \alpha) \text{ can be exhausted effectively by totally bounded balls} \\ \Updownarrow \\ (X, d, \beta) \text{ can be exhausted effectively by totally bounded balls}\end{gathered} \tag{9}$$

does not hold in general, as the following example shows.

Example 5. Let the number γ be as in Example 2. It is easy to construct a recursive sequence of rational numbers α' which is dense in $\langle -\infty, \gamma]$. Let d be the Euclidean metric on $\langle -\infty, 0]$ and let (x_i) be some recursive sequence of real numbers which is dense in $\langle -\infty, 0]$. Then the computable metric space

$(\langle -\infty, 0], d, (x_i))$ can be exhausted effectively by totally bounded balls. On the other hand, if $\alpha : \mathbf{N} \to \langle -\infty, 0]$ is defined by $\alpha(i) = \alpha'(i) - \gamma$, then α is an effective separating sequence in $(\langle -\infty, 0], d)$ and the computable metric space $(\langle -\infty, 0], d, \alpha)$ cannot be exhausted effectively by totally bounded balls which can be deduced from the fact that 0 is not a recursive point in this space.

The previous example also shows that effective separating sequences in a metric space (X, d) need not be equivalent up to isometry; namely, it is easy to see that the equivalence (9) holds when α and β are equivalent up to isometry. The following two examples show that effective separating sequences in (X, d) need not be equivalent up to isometry even when (X, d) is totally bounded.

Example 6. Let $([0, \gamma], d, \alpha)$ be the computable metric space of Example 2. Let $\alpha' : \mathbf{N} \to \mathbf{R}$ be defined by $\alpha'(2i) = \frac{\alpha(i)}{2}$, $\alpha'(2i+1) = -\frac{\alpha(i)}{2}$, $i \in \mathbf{N}$ and let $\alpha'' : \mathbf{N} \to [0, \gamma]$ be defined by $\alpha''(i) = \alpha'(i) + \frac{\gamma}{2}$. Then α'' is an effective separating sequence in $([0, \gamma], d)$. Since the point $\frac{\gamma}{2}$ is recursive in $([0, \gamma], d, \alpha'')$, but not in $([0, \gamma], d, \alpha)$, and since $\frac{\gamma}{2}$ is a fixed point of each isometry of $([0, \gamma], d)$ (namely the only isometries are the identity and the map $t \mapsto \gamma - t$, $t \in [0, \gamma]$), we conclude that effective separating sequences α and α'' are not equivalent.

Example 7. Let S be the unit circle in $\mathbf{R}^2$ and let d be the Euclidean metric on S. Since S is a recursive set, there exists a recursive sequence (x_i) in S such that $(S, d, (x_i))$ is an effectively totally bounded computable metric space (Example 1). Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a rotation with the center $(0, 0)$ such that $f(1, 0)$ is a nonrecursive point. Then $(f(x_i))$ is an effective separating sequence in (S, d) nonequivalent to (x_i). Let $A = \{x_i \mid i \in \mathbf{N}\} \cup \{f(x_i) \mid i \in \mathbf{N}\}$, let $T = \{(x, y) \in S \mid x \leq 0 \text{ or } (x, y) \in A\}$ and let d' be the Euclidean metric on T. Then (x_i) and $(f(x_i))$ are effective separating sequences in (T, d') and it follows easily that they are not equivalent up to isometry in this metric space.

3 Effective total boundedness and effective dispersion

Let (X, d) be a metric space. A nonempty subset S of X is said to be $r-$**dense** in (X, d), where $r \in \mathbf{R}$, $r > 0$, if $X = \bigcup_{s \in S} B(s, r)$. Note that a set S is dense in (X, d) if and only if S is $r-$dense in (X, d) for all $r > 0$. We say that a finite sequence $x_0, \ldots, x_n$ of points in X is $r-$dense in (X, d) if the set $\{x_0, \ldots, x_n\}$ is $r-$dense in (X, d). Hence (X, d) is totally bounded if and only if for each $\varepsilon > 0$ there exists a finite sequence of points in X which is $\varepsilon-$dense in (X, d).

Let $s \in \mathbf{R}$. A nonempty subset S of X is said to be $s-$**dispersed** in (X, d) if $d(x, y) > s$, $\forall x, y \in S$, $x \neq y$. A finite sequence $x_0, \ldots, x_n$ of points in X is said to be $s-$**dispersed** in (X, d) if $d(x_i, x_j) > s$, $\forall i, j \in \{0, \ldots, n\}$, $i \neq j$. Note that if $x_0, \ldots, x_n$ is an s-dispersed finite sequence, then $\{x_0, \ldots, x_n\}$ is an $s-$dispersed set, while the converse does not hold in general.

Proposition 13. *Let (X, d) be a totally bounded metric space and let $s > 0$. Then the set $A = \{k \in \mathbf{N} \mid$ there exists a finite sequence $x_1, \dots, x_k$ which is $s-$dispersed in $(X, d)\}$ is finite.*

Proof. Let $y_0, \dots, y_p$ be an $\frac{s}{2}-$dense finite sequence in (X, d). Suppose that a finite sequence $x_1, \dots, x_k$ is $s-$dispersed. For each $i \in \{1, \dots, k\}$ let $j_i \in \{0, \dots, p\}$ be such that $x_i \in B(y_{j_i}, \frac{s}{2})$. If $i, i' \in \{1, \dots, k\}$, $i \neq i'$, then $j_i \neq j_{i'}$ since $d(x_i, x_{i'}) > s$. Therefore we have an injection $\{1, \dots, k\} \to \{0, \dots, p\}$, hence $k < p$. This shows that A is finite. □

Let (X, d) be a totally bounded metric space. If $S \subseteq X$, $S \neq \emptyset$, and $s > 0$, then, by Proposition 13, the set $\{k \in \mathbf{N} \mid$ there exists a finite sequence $x_1, \dots, x_k$ of points in S which is $s-$dispersed in $(X, d)\}$ is finite. We denote the maximum of this set by $\Lambda(S, s)$. If $x_0, \dots, x_n$ is a finite sequence in X, then we will write $\Lambda(x_0, \dots, x_n; s)$ instead of $\Lambda(\{x_0, \dots, x_n\}, s)$.

Example 8. With the Euclidean metric on $[0, 3]$ we have $\Lambda([0, 1], s) = 1$ if $s \geq 1$, $\Lambda([0, 1], s) = 2$ if $s \in \left[\frac{1}{2}, 1\right\rangle$ and $\Lambda(0, 1, 3; s) = \begin{cases} 1, \ 3 \leq s, \\ 2, \ 1 \leq s < 3, \\ 3, \ 0 < s < 1. \end{cases}$

Lemma 14. *Suppose (X, d) is a totally bounded metric space, $s > 0$ and $n = \Lambda(X, s)$. Let $x_0, \dots, x_{n-1}$ be a finite sequence which is $s-$dispersed in (X, d). Then $x_0, \dots, x_{n-1}$ is $2s-$dense.*

Proof. Let $a \in X$. Then the finite sequence $a, x_0, \dots, x_{n-1}$ is not $s-$dispersed and since $x_0, \dots, x_{n-1}$ is $s-$dispersed, there exists $i \in \{0, \dots, n-1\}$ such that $d(a, x_i) \leq s$. Hence the finite sequence $x_0, \dots, x_{n-1}$ is $2s-$dense. □

Now, let α and β be effective separating sequences in (X, d) such that the computable metric space (X, d, α) is effectively totally bounded. In order to prove that (X, d, β) is also effectively totally bounded, it would be enough to prove that for each $k \in \mathbf{N}$ we can effectively find the number $\Lambda(X, 2^{-k})$. Namely, in that case for any $k \in \mathbf{N}$ we can effectively find $i_1, \dots, i_n \in \mathbf{N}$ such that the finite sequence $\beta_{i_1}, \dots, \beta_{i_n}$ is $2^{-(k+1)}-$dispersed, where $n = \Lambda(X, 2^{-(k+1)})$ and then this finite sequence of points (and consequently the finite sequence $\beta_0, \dots, \beta_{\max\{i_1, \dots, i_n\}}$) must be $2^{-k}-$dense. However, the number $\Lambda(X, 2^{-k})$ cannot be found effectively in general, as the following example shows.

Example 9. Let (λ_i) be a recursive sequence of real numbers such that $\lambda_i \geq 0$, $\forall i \in \mathbf{N}$ and such that the set $\{i \in \mathbf{N} \mid \lambda_i = 0\}$ is not recursive ([Pour-El and Richards 1989]). We may assume $\lambda_i < 4^{-i}$, $\forall i \in \mathbf{N}$. Let $t_i = 4^{-i} + \lambda_i$, $i \in \mathbf{N}$, $X = \{t_i \mid i \in \mathbf{N}\} \cup \{0\}$ and let d be the Euclidean metric on X. Then $(X, d, (t_i))$ is an effectively totally bounded computable metric space. Let $i \in \mathbf{N}$. It is straightforward to check that $\Lambda(X,4^{-i}) = i+1$ if $\lambda_i = 0$ and $\Lambda(X,4^{-i}) = i+2$ if $\lambda_i > 0$. Therefore the function $\mathbf{N} \to \mathbf{N}$, $i \mapsto \Lambda(X,2^{-i})$ is not recursive.

A totally bounded metric space (X,d) is said to be **effectively dispersed** if there exists a recursive function $s : \mathbf{N} \to \mathbf{Q}$ such that $s_i \in \langle 0, 2^{-i}\rangle$, $\forall i \in \mathbf{N}$ and such that the function $\mathbf{N} \to \mathbf{N}$, $i \mapsto \Lambda(X,s_i)$ is recursive.

If X is a set and $p \in \mathbf{N}$ let $\mathcal{F}^p(X)$ denotes the set of all functions $x : \{0,\dots,p\} \to X$ (hence $\mathcal{F}^p(X)$ is the set of all finite sequences in X of the form $x_0,\dots,x_p$). Of course, for $x \in \mathcal{F}^p(X)$ and $i \in \{0,\dots,p\}$ we will denote $x(i)$ by x_i. If $x \in \mathcal{F}^p(X)$, then we say that the finite sequence x has length p and we write $p = \operatorname{length}(x)$.

If (X,d) is a metric space and $x \in \mathcal{F}^p(X)$, $p \geq 1$, let $\rho(x)$ denotes the real number defined by

$$\rho(x) = \min\{d(x_i,x_j) \mid i,j \in \{0,\dots,p\},\ i \neq j\}.$$

Let (X,d) be a metric space and let A be a nonempty bounded set in this space. For each $n \in \mathbf{N}$ we define the real number $C_n(A)$ (see [Kreinovich 1977]) by

$$C_n(A) = \sup\{\varepsilon \in \mathbf{R} \mid \exists x \in \mathcal{F}^{n+1}(A) \text{ such that } x \text{ is } \varepsilon-\text{dispersed}\}.$$

Note that

$$C_n(A) = \sup\{\rho(x) \mid x \in \mathcal{F}^{n+1}(A)\}.$$

Lemma 15. *Let (X,d) be a metric space, let A and B be nonempty bounded sets in this space and let $\varepsilon > 0$ be such that for each $a \in A$ there exists $b \in B$ such that $d(a,b) < \varepsilon$ and for each $b \in B$ there exists $a \in A$ such that $d(b,a) < \varepsilon$. Then for each $n \in \mathbf{N}$*

$$|C_n(A) - C_n(B)| \leq 2\varepsilon.$$

Lemma 15 can be proved easily using the following lemma, which is an immediate consequence of the triangle inequality in a metric space.

Lemma 16. *If (X,d) is a metric space, $a,b,a',b' \in X$ and $\varepsilon, r > 0$ such that $d(a,b) > r$, $d(a,a') < \varepsilon$ and $d(b,b') < \varepsilon$, then $d(a',b') > r - 2\varepsilon$.*

Lemma 17. *Let (X,d,α) be a computable metric space. For $l \in \mathbf{N}$ let $\alpha[l]$ denotes the finite sequence $\alpha_{(l)_0}, \dots, \alpha_{(l)_{\overline{l}}}$. Then there exists a recursive function $f : \mathbf{N} \to \mathbf{R}$ such that*

$$f(l) = \rho(\alpha[l])$$

for each $l \in \mathbf{N}$ such that $\operatorname{length}(\alpha[l]) \geq 1$ *(i.e. $\overline{l} \geq 1$).*

Proof. Since $\mathbf{N} \to \mathcal{P}(\mathbf{N}^2)$, $l \mapsto \{(i,j) \in \mathbf{N}^2 \mid i \neq j,\ 0 \leq i,j \leq \bar{l}\}$, is clearly an r.r.b. function and $\mathbf{N}^3 \to \mathbf{N}^2$, $(l,i,j) \mapsto ((l)_i,(l)_j)$, is a recursive function, Proposition 4 implies that the function $\mathbf{N} \to \mathcal{P}(\mathbf{N}^2)$,

$$l \mapsto \{((l)_i,(l)_j) \mid i \neq j,\ 0 \leq i,j \leq \bar{l}\}$$

is r.r.b. If we apply Lemma 6 to this function and the function $\mathbf{N}^2 \to \mathbf{R}$, $(i,j) \mapsto d(\alpha_i,\alpha_j)$, we get the claim of the lemma. □

Corollary 18. *Let (X,d,α) be a computable metric space and let $(s_k)_{k\in\mathbf{N}}$ be a recursive sequence of real numbers. With notation of the previous lemma we have that the set*

$$D = \{(l,k) \in \mathbf{N}^2 \mid \alpha[l] \textit{ is } s_k \textit{ dispersed}\}$$

is recursively enumerable.

Proof. For all $x \in \mathcal{F}^p(X)$, $p \geq 1$, and $r > 0$ we have that x is $r-$dispersed if and only if $\rho(x) > r$. Therefore,

$$(l,k) \in D \text{ if and only if } \rho(\alpha[l]) > s_k \text{ or } \bar{l} = 0.$$

The claim of the corollary now follows from Lemma 17 and Proposition 2(iv). □

Proposition 19. *Let (X,d,α) be a computable metric space. For $m \in \mathbf{N}$ let $A_m = \{\alpha_0,\ldots,\alpha_m\}$. Then the function $\mathbf{N}^2 \to \mathbf{R}$,*

$$(n,m) \mapsto C_n(A_m),$$

is recursive.

Proof. For $i \in \mathbf{N}$ let us denote by $\alpha[i]$ the finite sequence $\alpha_{(i)_0},\ \ldots,\ \alpha_{(i)_{\bar{i}}}$.

For all $n,m \in \mathbf{N}$ we have

$$C_n(A_m) = \max_{x\in\mathcal{F}^{n+1}(A_m)} \rho(x). \tag{10}$$

Let ζ be the function of Lemma 7. Then each element of $\mathcal{F}^{n+1}(\{0,\ldots,m\})$ is of the form $(i)_0,\ldots,(i)_{\bar{i}}$ for some $i \leq \zeta(m,n+1)$.

Let $\Phi : \mathbf{N}^2 \to \mathcal{P}(\mathbf{N})$ be defined by

$$\Phi(n,m) = \{i \leq \zeta(m,n+1) \mid \bar{i} = n+1 \text{ and } (i)_j \leq m,\ \forall j \in \{0,\ldots,\bar{i}\}\}.$$

Clearly, Φ is r.r.b. Let $n,m \in \mathbf{N}$. We have that the set of all finite sequences $(i)_0,\ldots,(i)_{\bar{i}}$ for $i \in \Phi(n,m)$ equals $\mathcal{F}^{n+1}(\{0,\ldots,m\})$. Therefore

$$\{\alpha[i] \mid i \in \Phi(n,m)\} = \mathcal{F}^{n+1}(A_m)$$

and, by (10),

$$C_n(A_m) = \max_{i\in\Phi(n,m)} \rho(\alpha[i]).$$

The claim of the proposition now follows from Lemma 17 and Lemma 6. □

Theorem 20. *Let (X, d) be a totally bounded metric space. Let α be an effective separating sequence in (X, d). Then the following statements are equivalent.*

(i) *the computable metric space (X, d, α) is effectively totally bounded;*

(ii) *the function $\mathbf{N} \to \mathbf{R}$, $n \mapsto C_n(X)$, is recursive;*

(iii) *the metric space (X, d) is effectively dispersed.*

Proof. Suppose that (i) holds. For $m \in \mathbf{N}$ let $A_m = \{\alpha_0, \ldots, \alpha_m\}$. Let $\varphi : \mathbf{N} \to \mathbf{N}$ be a recursive function such that $X = \bigcup_{i=0}^{\varphi(k)} B(\alpha_i, 2^{-k})$, $\forall k \in \mathbf{N}$. Then, by Lemma 15,

$$|C_n(X) - C_n(A_{\varphi(k)})| \leq 2 \cdot 2^{-k},$$

for all $n, k \in \mathbf{N}$. Therefore (ii) holds (Proposition 2(ii)).

Suppose now that (ii) holds and let us prove (iii). If X is finite, then (iii) clearly holds. Suppose X is infinite. Then $0 < C_{n+1}(X) \leq C_n(X)$, $\forall n \in \mathbf{N}$. We also have $\lim_{n\to\infty} C_n(X) = 0$, otherwise there exists $s > 0$ such that $C_n(X) > s$ for each $n \in \mathbf{N}$ which contradicts Proposition 13.

Let $r : \mathbf{N} \to \mathbf{Q}$ be a recursive function whose image is dense in $\mathbf{R}$. Now, for each $k \in \mathbf{N}$ there exists $i, n \in \mathbf{N}$ such that

$$r_i < 2^{-k} \text{ and } C_{n+1}(X) < r_i < C_n(X).$$

By Proposition 2(iv) and Proposition 1(ii) there exist recursive functions $\varphi, \psi : \mathbf{N} \to \mathbf{N}$ such that $r_{\varphi(k)} < 2^{-k}$ and $C_{\psi(k)+1}(X) < r_{\varphi(k)} < C_{\psi(k)}(X)$, $\forall k \in \mathbf{N}$. This, by definition of the numbers $C_n(X)$, $n \in \mathbf{N}$, implies

$$\Lambda(X, r_{\varphi(k)}) = \psi(k) + 2,$$

$\forall k \in \mathbf{N}$. Therefore (X, d) is effectively dispersed.

Finally, let us prove that (iii) implies (i). Let $s : \mathbf{N} \to \mathbf{Q}$ be a recursive function such that $0 < s_k < 2^{-k}$, $\forall k \in \mathbf{N}$ and such that

$$k \mapsto \Lambda(X, s_k), \ k \in \mathbf{N}, \tag{11}$$

is a recursive function. Let $k \in \mathbf{N}$. Then there exist a finite sequence $x_1, \ldots, x_p$ which is s_k-dispersed in (X, d), where $p = \Lambda(X, s_k)$. Since the sequence α is dense in (X, d), we easily conclude that there exist $i_1, \ldots, i_p$ such that the sequence $\alpha_{i_1}, \ldots, \alpha_{i_p}$ is s_k-dispersed.

For $l \in \mathbf{N}$ let us denote by $\alpha[l]$ the finite sequence $\alpha_{(l)_0}, \ldots, \alpha_{(l)_{\overline{l}}}$. Hence for each $k \in \mathbf{N}$ there exists $l \in \mathbf{N}$ such that

$$\alpha[l] \text{ is } s_k-\text{dispersed and } \overline{l} + 1 = \Lambda(X, s_k). \tag{12}$$

The fact that (11) is a recursive function, Lemma 18 and Proposition 1(ii) imply that there exists a recursive function $\lambda : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$ (12)

holds when $l = \lambda(k)$. Now Lemma 14 implies that $\alpha[\lambda(k)]$ is $2s_k$ dense for each $k \in \mathbf{N}$.

Let $f : \mathbf{N} \to \mathbf{N}$ be defined by

$$f(k) = \max\{(\lambda(k+1))_i \mid 0 \leq i \leq \overline{\lambda(k+1)}\}.$$

Clearly, f is recursive. It is obvious that the sequence $\alpha_0, \dots, \alpha_{f(k)}$ is $2 \cdot s_{k+1}-$ dense in (X, d) and since $2 \cdot s_{k+1} < 2 \cdot 2^{-(k+1)} = 2^{-k}$, this sequence is also $2^{-k}-$dense. Therefore (X, d, α) is effectively totally bounded. $\square$

Let (X, d, α) be a computable metric space. By Theorem 20

(X, d, α) is effectively totally bounded $\Leftrightarrow$ (X, d) is effectively dispersed.

Corollary 21. *Let α and β be effective separating sequences in a metric space (X, d). Then (X, d, α) is effectively totally bounded if and only if (X, d, β) is effectively totally bounded.*

A computable metric space (X, d, α) is said to be **effectively compact** (cf. [Yasugi, Mori and Tsujji 1999]) if (X, d, α) is effectively totally bounded and (X, d) is complete. If α and β are effective separating sequences in a metric space (X, d), then, by Corollary 21, (X, d, α) is effectively compact if and only if (X, d, β) is effectively compact.

We will say that a metric space (X, d) is **effectively compact** if there exists α such that (X, d, α) is an effectively compact computable metric space. Corollary 21 says that (X, d, β) is an effectively totally bounded computable metric for every effective separating sequence β in an effectively compact metric space (X, d).

Note that a compact metric space (X, d) is effectively compact if and only if it is effectively dispersed and it has at least one effective separating sequence.

4 Isometries and effective compactness

We have seen in Section 2 that each two effective separating sequences in $\mathbf{R}^n$ with the Euclidean metric are equivalent up to isometry. Examples 5, 6 and 7 show that this property does not hold in general. Note, however, that metric spaces constructed in these examples are not effectively compact. In contrast to Example 6, every two effective separating sequences in $[0, 1]$ with the Euclidean metric are equivalent up to isometry, moreover they are equivalent as the following example shows.

Example 10. Let (α_i) be a recursive sequence of rational numbers such that $\{\alpha_i \mid i \in \mathbf{N}\} = \mathbf{Q} \cap [0, 1]$. Let d be the Euclidean metric on $[0, 1]$. Then (α_i)

is an effective separating sequence in $([0,1],d)$. Let β be an effective separating sequence in $([0,1],d)$. We claim that $\beta \sim \alpha$.

Choose $i_0 \in \mathbf{N}$ so that $\beta_{i_0} < \frac{1}{4}$. For each $k \in \mathbf{N}$ there exist $i,j \in \mathbf{N}$ such that $d(\beta_i,\beta_j) > 1-2^{-(k+2)}$, $d(\beta_i,\beta_{i_0}) < \frac{1}{4}$. Therefore there exist recursive functions $\varphi,\psi : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$ these two inequalities hold when $i=\varphi(k)$, $j=\psi(k)$. So for each $k \in \mathbf{N}$ we have

$$|\beta_{\varphi(k)}-\beta_{\psi(k)}| > 1-2^{-(k+2)},\ |\beta_{\varphi(k)}-\beta_{i_0}| < \frac{1}{4},$$

from which we easily conclude that $\beta_{\varphi(k)} < 2^{-(k+2)}$. Hence $d(0,\beta_{\varphi(k)}) < 2^{-(k+2)}$, $\forall k \in \mathbf{N}$, which means that 0 is a recursive point in the computable metric space $([0,1],d,\beta)$.

In general, it is easy to see that if (X,q,γ) is a computable metric space and x a recursive point in this space, then $\mathbf{N} \to \mathbf{R}$, $i \mapsto q(x,\gamma_i)$, is a recursive function.

Therefore, the function $i \mapsto d(0,\beta_i)$, $i \in \mathbf{N}$, is recursive, i.e. (β_i) is a recursive sequence in $\mathbf{R}$ and $(\alpha_i) \sim (\beta_i)$.

Example 10 says that $[0,1]$ with the Euclidean metric has a unique computability structure. On the other hand, the unit circle S^1 in $\mathbf{R}^2$ with the Euclidean metric is effectively compact, but it has nonequivalent effective separating sequences (Example 7), hence it has more than one computability structure. One obvious difference between these metric spaces is that there are infinitely many isometries $S^1 \to S^1$, but only two isometries $[0,1] \to [0,1]$.

As we will see in this section, the property of an effectively compact metric space (X,d) that there are only finitely many isometries $X \to X$ implies that (X,d) has a unique computability structure, not just for those (X,d) which are metric subspaces of Euclidean space, but in general.

The idea how to prove this is in certain sense similar to the idea used in Example 10. As we noticed in that example, if $x,y \in [0,1]$ are such that $d(x,y)$ is close to 1, then x and y are close respectively to 0 and 1 or to 1 and 0. Let us now observe this situation in the case of a compact metric space (X,d) and let us, for simplicity, take that (X,d) is such that there exists exactly one isometry $X \to X$ (the identity). The question is this: if $x_0,\dots,x_p$ and $y_0,\dots,y_p$ are finite sequences in X and if the number $d(x_i,x_j)$ is close to the number $d(y_i,y_j)$ for all i,j, what can be said about the distances between the points $x_0,\dots,x_p$ and $y_0,\dots,y_p$ respectively? As we will see in Proposition 26, under certain conditions the point x_i must be close to the point y_i for each i. Using this fact, it will be possible to prove that effective separating sequences α and β in (X,d) are equivalent (under assumption that (X,d) is effectively compact): we can effectively find numbers $v_0,\dots,v_p$ and $w_0,\dots,w_p$ such that $d(\alpha_{v_i},\alpha_{v_j})$ is close to $d(\beta_{w_i},\beta_{w_j})$ for all $i,j \in \{0,\dots,p\}$ and then it follows that β_{w_i} is an approximation of α_{v_i} for each i; if we ensure that each α_k is close to some α_{v_i}, then each α_k can be effectively approximated by some $\beta_{k'}$ and this means that the sequences α and β are equivalent.

As we will see, the described idea can be generalized to the case when (X, d) has more that one isometry onto itself (but finitely many) and this will give the desired result.

First, we need some facts about finite sequences in a metric space.

Let X be a set. Let $\mathcal{G}(X)$ be the set of all sequences $(v^k)_{k\in\mathbf{N}}$ in $\bigcup_{p=0}^{\infty}\mathcal{F}^p(X)$ such that

$$\mathrm{length}(v^k) < \mathrm{length}(v^{k+1}), \forall k \in \mathbf{N}.$$

If $v = (v^k)_{k\in\mathbf{N}} \in \mathcal{G}(X)$, then clearly any subsequence of v is also an element of $\mathcal{G}(X)$.

If $(v^k)_{k\in\mathbf{N}}$ is a sequence in $\bigcup_{p=0}^{\infty}\mathcal{F}^p(X)$, then for $k, i \in \mathbf{N}$, $i \leq \mathrm{length}(v^k)$, we denote $(v^k)_i$ by v_i^k.

Let (X, d) be a metric space. We say that $(v^k)_{k\in\mathbf{N}} \in \mathcal{G}(X)$ is $l-$**convergent** in (X, d), $l \in \mathbf{N}$, if for each $l' \in \{0, \dots, l\}$ the sequence $k \mapsto v_{l'}^{l'+k}, k \in \mathbf{N}$, converges in (X, d) (note that the fact $(v^k)_{k\in\mathbf{N}} \in \mathcal{G}(X)$ implies $\mathrm{length}(v^k) \geq k, \forall k \in \mathbf{N}$).

Lemma 22. *Let (X, d) be a metric space and $x_0 \in X$. If $(v^k)_{k\in\mathbf{N}} \in \mathcal{G}(X)$ and $l \in \mathbf{N}$ are such that the sequence $k \mapsto v_l^{k+l}, k \in \mathbf{N}$, converges to x_0, then for each subsequence $(w^k)_{k\in\mathbf{N}}$ of $(v^k)_{k\in\mathbf{N}}$ the sequence $k \mapsto w_l^{k+l}, k \in \mathbf{N}$, converges to x_0.*

Proof. If $(w^k)_{k\in\mathbf{N}}$ is a subsequence of $(v^k)_{k\in\mathbf{N}}$, then $w^k = v^{\varphi(k)}, \forall k \in \mathbf{N}$, where $\varphi : \mathbf{N} \to \mathbf{N}$ is some increasing function (i.e. $\varphi(i) < \varphi(i+1), \forall i \in \mathbf{N}$). Therefore for each $k \in \mathbf{N}$ we have

$$w_l^{k+l} = v_l^{\varphi(k+l)} = v_l^{(\varphi(k+l)-l)+l}$$

from which we conclude that $k \mapsto w_l^{k+l}, k \in \mathbf{N}$, is a subsequence of $k \mapsto v_l^{k+l}, k \in \mathbf{N}$, and the claim of the lemma follows. $\square$

Lemma 23. *Let (X, d) be a compact metric space and suppose $v = (v^k)_{k\in\mathbf{N}} \in \mathcal{G}(X)$ is $l-$convergent in (X, d) for some $l \in \mathbf{N}$. Then there exists a subsequence of v which is $(l+1)-$convergent in (X, d).*

Proof. Let us observe the sequence $k \mapsto v_{l+1}^{k+(l+1)}, k \in \mathbf{N}$. Since (X, d) is compact, there is an increasing function $\varphi : \mathbf{N} \to \mathbf{N}$ such that $k \mapsto v_{l+1}^{\varphi(k)+(l+1)}, k \in \mathbf{N}$, is a convergent sequence. Now, let $w = (w^k)_{k\in\mathbf{N}}$ be a sequence defined by

$$w^k = v^{\varphi(k)+l+1},$$

$k \in \mathbf{N}$. Then w is a subsequence of v, $w \in \mathcal{G}(X)$ and for each $k \in \mathbf{N}$ we have $w_{l+1}^{k+l+1} = v_{l+1}^{\varphi(k+l+1)+l+1}$, hence the sequence $k \mapsto w_{l+1}^{k+l+1}, k \in \mathbf{N}$, is a subsequence of $k \mapsto v_{l+1}^{\varphi(k)+l+1}, k \in \mathbf{N}$, and therefore is convergent. For $l' \in \{0, \ldots, l\}$ the sequence $k \mapsto w_{l'}^{k+l'}, k \in \mathbf{N}$, is convergent by Lemma 22. Hence w is $(l+1)-$convergent. $\square$

Let (X, d) be a metric space. If $x, y \in \mathcal{F}^p(X)$, then we denote the number $\max_{0 \leq i \leq p} |x_i - y_i|$ by $d(x, y)$. The function $X \times X \to \mathbf{R}$, $(x, y) \mapsto d(x, y)$, is a metric on $\mathcal{F}^p(X)$.

Let $x, y \in \mathcal{F}^p(X)$. We say that x and y are **isometrically equivalent** and we denote that by $x \sim_{\text{iso}} y$ if $d(x_i, x_j) = d(y_i, y_j), \forall i, j \in \{0, \ldots, p\}$. Similarly, sequences (x_i) and (y_i) in X are said to be isometrically equivalent, $(x_i) \sim_{\text{iso}} (y_i)$, if $d(x_i, x_j) = d(y_i, y_j), \forall i, j \in \mathbf{N}$.

If $x, y \in \mathcal{F}^p(X)$ and $r \in \mathbf{R}$, then we say that x and y are $r-$**isometrically equivalent**, $x \sim_{\text{iso}}^{\leq r} y$, if

$$|d(x_i, x_j) - d(y_i, y_j)| \leq r, \forall i, j \in \{0, \ldots, p\},$$

and we say that x and y are **strictly** $r-$**isometrically equivalent**, $x \sim_{\text{iso}}^{< r} y$, if

$$|d(x_i, x_j) - d(y_i, y_j)| < r, \forall i, j \in \{0, \ldots, p\}.$$

If $\alpha = (\alpha_i)_{i \in \mathbf{N}}$ is a sequence in a set X and $p \in \mathbf{N}$, then we denote the finite sequence $\alpha_0, \ldots, \alpha_p$ by $\alpha_{\leq p}$.

Lemma 24. *Let (X, d) be a compact metric space and let $\alpha = (\alpha_i)_{i \in \mathbf{N}}$ be a dense sequence in this metric space. Suppose $v = (v^k)_{k \in \mathbf{N}} \in \mathcal{G}(X)$ is such that*

$$v^k \sim_{\text{iso}} \left(\alpha_{\leq \text{length}(v^k)}\right), \forall k \in \mathbf{N}.$$

Then there exists a sequence (γ_i) in X such that the following two properties are satisfied:

(i) $(\gamma_i) \sim_{\text{iso}} (\alpha_i)$;

(ii) *for each $\varepsilon > 0$ and each $q \in \mathbf{N}$ there exists $l \in \mathbf{N}$ such that* $\text{length}(v^l) \geq q$ *and*

$$d(\gamma_i, v_i^l) < \varepsilon, \forall i \in \{0, \ldots, q\}.$$

Proof. Observe the sequence in X defined by $k \mapsto v_0^k, k \in \mathbf{N}$. Compactness of (X, d) implies that there exists an increasing function $\varphi : \mathbf{N} \to \mathbf{N}$ such that $k \mapsto v_0^{\varphi(k)}, k \in \mathbf{N}$, is a convergent sequence. Let $a(0)$ be the subsequence $(v^{\varphi(k)})_{k \in \mathbf{N}}$ of v. Then $a(0)$ is a $0-$convergent element of $\mathcal{G}(X)$. By Lemma 23 there exists a subsequence $a(1)$ of $a(0)$ which is $1-$convergent. Repeating this argument, we obtain a sequence $a(0), a(1), \ldots, a(l), \ldots$ in $\mathcal{G}(X)$ such that $a(l)$ is $l-$convergent and $a(l+1)$ is a subsequence of $a(l)$ for each $l \in \mathbf{N}$.

For $l \in \mathbf{N}$ let γ_l be the limit of the sequence $k \mapsto a(l)_l^{k+l}, k \in \mathbf{N}$. We claim that $(\gamma_l)_{l \in \mathbf{N}}$ is the desired sequence. Note that, by Lemma 22, for all $l, l' \in \mathbf{N}$, $l' \geq l$, the sequence $k \mapsto a(l')_l^{k+l}, k \in \mathbf{N}$, converges to γ_l. If $l \in \mathbf{N}$, then, using notation $a(l) = (a(l)^k)_{k \in \mathbf{N}}$, we have that for each $k \in \mathbf{N}$ the finite sequence $a(l)^k$ is isometrically equivalent to $\alpha_{\leq m}$ for some $m \in \mathbf{N}$, namely $a(l)$ is a subsequence of v, hence $a(l)^k = v^{k'}$ for some $k' \in \mathbf{N}$.

Let $i, j \in \mathbf{N}$. Let $l \in \mathbf{N}$ be such that $l \geq i$, $l \geq j$. Then $\gamma_i = \lim_{k \to \infty} a(l)_i^{k+i}$ and since $k \mapsto a(l)_i^{k+l}, k \in \mathbf{N}$, is a subsequence of $k \mapsto a(l)_i^{k+i}, k \in \mathbf{N}$, we have $\gamma_i = \lim_{k \to \infty} a(l)_i^{k+l}$. In the same way we get $\gamma_j = \lim_{k \to \infty} a(l)_j^{k+l}$. Now

$$d\left(a(l)_i^{k+l}, a(l)_j^{k+l}\right) = d(\alpha_i, \alpha_j), \forall k \in \mathbf{N},$$

implies $d(\gamma_i, \gamma_j) = d(\alpha_i, \alpha_j)$. Hence $(\gamma_i) \sim_{\text{iso}} (\alpha_i)$.

Let $\varepsilon > 0$ and $q \in \mathbf{N}$. For each $i \in \{0, \ldots, q\}$ we have $\gamma_i = \lim_{k \to \infty} a(q)_i^{k+q}$. For $i \in \{0, \ldots, q\}$ let $k_i \in \mathbf{N}$ be such that $d(\gamma_i, a(q)_i^{k+q}) < \varepsilon, \forall k \geq k_i$. Let $k = \max\{k_0, \ldots, k_q\}$. Then $d(\gamma_i, a(q)_i^{k+q}) < \varepsilon, \forall i \in \{0, \ldots, q\}$. Let $l \in \mathbf{N}$ be such that $a(q)^{k+q} = v^l$. Then $d(\gamma_i, v_i^l) < \varepsilon, \forall i \in \{0, \ldots, q\}$. □

Lemma 25. *Let (X, d) be a compact metric space, $p \in \mathbf{N}$, $a \in \mathcal{F}^p(X)$ and $(v^N)_{N \in \mathbf{N}}$ a sequence in $\mathcal{F}^p(X)$ such that $v^N \sim_{\text{iso}}^{\leq 2^{-N}} a$, $\forall N \in \mathbf{N}$. Then there exists $w \in \mathcal{F}^p(X)$ such that $w \sim_{\text{iso}} a$ and such that $d(w, u) \geq r$ whenever $u \in \mathcal{F}^p(X)$ and $r > 0$ are such that $d(v^N, u) \geq r, \forall N \in \mathbf{N}$.*

Proof. Using the fact that (X, d) is compact, it is easy to conclude that there exists a subsequence $(v^{N_k})_{k \in \mathbf{N}}$ of $(v^N)_{N \in \mathbf{N}}$ such that $(v_i^{N_k})_{k \in \mathbf{N}}$ is a convergent sequence in (X, d) for each $i \in \{0, \ldots, p\}$. Let $w \in \mathcal{F}^p(X)$ be such that $w_i = \lim_{k \to \infty} v_i^{N_k}$, $\forall i \in \{0, \ldots, p\}$. For all $i, j \in \{0, \ldots, p\}$ we have

$$|d(v_i^{N_k}, v_j^{N_k}) - d(a_i, a_j)| \leq 2^{-N_k}, \forall k \in \mathbf{N},$$

and therefore $d(w_i, w_j) = d(a_i, a_j)$. Hence $w \sim_{\text{iso}} a$. Actually the sequence $(v^{N_k})_{k \in \mathbf{N}}$ converges to w in $\mathcal{F}^p(X)$ with respect to metric $(x, y) \mapsto d(x, y)$. So $d(w, u) < r$ for some $u \in \mathcal{F}^p(X)$ and $r > 0$ implies $d(v^{N_k}, u) < r$ for some $k \in \mathbf{N}$. □

Proposition 26. *Let (X, d) be a compact metric space such that there exist exactly n isometries $X \to X$ ($n \in \mathbf{N}, n \geq 1$). Let $\alpha = (\alpha_i)_{i \in \mathbf{N}}$ be a dense sequence in this metric space. Then for each $\varepsilon > 0$ and each $q \in \mathbf{N}$ there exist $N, p \in \mathbf{N}$, $p > q$, and $u_1, \ldots, u_n \in \mathcal{F}^p(X)$ such that $u_i \sim_{\text{iso}} \alpha_{\leq p}, \forall i \in \{1, \ldots, n\}$, and such that the following implication holds:*

$$v \in \mathcal{F}^p(X),\ v \sim_{\text{iso}}^{\leq 2^{-N}} \alpha_{\leq p} \ \Rightarrow\ d(v, u_i) < \varepsilon \text{ for some } i \in \{1, \ldots, n\}. \qquad (13)$$

Proof. Let $f_1, \dots, f_n$ be all isometries $X \to X$. Let $i, j \in \{1, \dots, n\}, i \neq j$. Since $f_i \neq f_j$ and α is dense in (X, d), there exists $k \in \mathbf{N}$ such that $f_i(\alpha_k) \neq f_j(\alpha_k)$. From this we conclude the following: there exists $p_0 \in \mathbf{N}$ and $\varepsilon_0 > 0$ such that

$$d((f_i \circ \alpha)_{\leq p_0}, (f_j \circ \alpha)_{\leq p_0}) > \varepsilon_0, \forall i, j \in \{1, \dots, n\}, i \neq j.$$

(Of course, $g \circ \alpha$ for $g : X \to X$ denotes the sequence $(g(\alpha_i))_{i \in \mathbf{N}}$.)

Let us suppose that the claim of the proposition is not true. Then there exist $\varepsilon > 0$ and $q \in \mathbf{N}$ such that there exist no N, p and $u_1, \dots, u_n$ with the stated property. Let $k_0 = \max\{p_0, q\} + 1$. Let $k \in \mathbf{N}$. For $i \in \{1, \dots, n\}$ let

$$u_i = (f_i \circ \alpha)_{\leq k+k_0}.$$

Then each u_i is isometrically equivalent to $\alpha_{\leq k+k_0}$. From this and the fact that $k + k_0 > q$ we conclude that for each $N \in \mathbf{N}$ the implication (13) does not hold (with $p = k + k_0$). Therefore for each $N \in \mathbf{N}$ there exists $v^N \in \mathcal{F}^{k+k_0}(X)$ such that $v^N \sim_{\text{iso}}^{\leq 2^{-N}} \alpha_{\leq k+k_0}$ and $d(v, u_i) \geq \varepsilon$ for each $i \in \{1, \dots, n\}$. It follows from Lemma 25 that there exists $w \in \mathcal{F}^{k+k_0}(X)$ such that $w \sim_{\text{iso}} \alpha_{\leq k+k_0}$ and $d(w, u_i) \geq \varepsilon$.

We have the following conclusion. For each $k \in \mathbf{N}$ there exists $w^k \in \mathcal{F}^{k+k_0}(X)$ such that $w^k \sim_{\text{iso}} \alpha_{\leq k+k_0}$ and

$$d(w^k, (f_i \circ \alpha)_{\leq k+k_0}) \geq \varepsilon, \tag{14}$$

$\forall i \in \{1, \dots, n\}$. By Lemma 24 there exists a sequence $\gamma = (\gamma_i)$ in X such that $\gamma \sim_{\text{iso}} \alpha$ and such that for each $r > 0$ and each $q \in \mathbf{N}$ there exists $k \in \mathbf{N}$ such that $k+k_0 \geq q$ and $d(\gamma_i, w_i^k) < r, \forall i \in \{0, \dots, q\}$. Suppose that $(\gamma_i)_{i \in \mathbf{N}} = (f_j(\alpha_i))_{i \in \mathbf{N}}$ for some $j \in \{1, \dots, n\}$. Then the sequence (γ_i) is dense. Choose $r > 0$ so that $3r < \varepsilon$ and $q \in \mathbf{N}$ so that the finite sequence $\gamma_{\leq q}$ is $r-$dense. Let $k \in \mathbf{N}$ be such that $k + k_0 \geq q$ and

$$d(\gamma_i, w_i^k) < r, \tag{15}$$

$\forall i \in \{0, \dots, q\}$. Let $i' \in \{q+1, \dots, k+k_0\}$. Then there exists $i \in \{0, \dots, q\}$ such that $d(\gamma_i, w_{i'}^k) < r$. It follows

$$d(w_{i'}^k, w_i^k) \leq d(w_{i'}^k, \gamma_i) + d(\gamma_i, w_i^k) < r + r = 2r.$$

Now $d(\gamma_i, \gamma_{i'}) = d(\alpha_i, \alpha_{i'}) = d(w_i^k, w_{i'}^k) < 2r$ and so

$$d(w_{i'}^k, \gamma_{i'}) \leq d(w_{i'}^k, \gamma_i) + d(\gamma_i, \gamma_{i'}) < r + 2r = 3r < \varepsilon.$$

hence $d(w_{i'}^k, \gamma_{i'}) < \varepsilon$. This and (15) imply that $d(w_i^k, \gamma_i) < \varepsilon$ holds for each $i \in \{0, \dots, k+k_0\}$. But $\gamma_i = f_j(\alpha_i), \forall i \in \mathbf{N}$, therefore $d(w^k, (f_j \circ \alpha)_{\leq k+k_0}) < \varepsilon$. This is in contradiction with (14). Therefore

$$(\gamma_i)_{i \in \mathbf{N}} \neq (f_j(\alpha_i))_{i \in \mathbf{N}}, \tag{16}$$

$\forall j \in \{1, \dots, n\}$.

Now we define a map $g : X \to X$ in the following way. If $x \in X$, then $x = \lim_{i\to\infty} \alpha_{\varphi(i)}$, where $\varphi : \mathbf{N} \to \mathbf{N}$. The sequence $(\alpha_{\varphi(i)})_{i\in\mathbf{N}}$ is therefore Cauchy which, together with $\gamma \sim_{\text{iso}} \alpha$, implies that the sequence $(\gamma_{\varphi(i)})_{i\in\mathbf{N}}$ is Cauchy. We define $g(x)$ to be the limit of this sequence. (The metric space (X, d) is complete since it is compact.) This definition does not depend on the choice of the function φ : if $\psi : \mathbf{N} \to \mathbf{N}$ is such that $x = \lim_{i\to\infty} \alpha_{\psi(i)}$, then $\lim_{i\to\infty} d(\alpha_{\varphi(i)}, \alpha_{\psi(i)}) = 0$, therefore $\lim_{i\to\infty} d(\gamma_{\varphi(i)}, \gamma_{\psi(i)}) = 0$, which implies $\lim_{i\to\infty} \gamma_{\varphi(i)} = \lim_{i\to\infty} \gamma_{\psi(i)}$.

If $x, y \in X$ and $\varphi, \psi : \mathbf{N} \to \mathbf{N}$ are such that $x = \lim_{i\to\infty} \alpha_{\varphi(i)}$, $y = \lim_{i\to\infty} \alpha_{\psi(i)}$, then

$$d(x, y) = \lim_{i\to\infty} d(\alpha_{\varphi(i)}, \alpha_{\psi(i)}) = \lim_{i\to\infty} d(\gamma_{\varphi(i)}, \gamma_{\psi(i)}) = d(g(x), g(y)).$$

Hence g is an isometry (that g is surjective can be deduced from the compactness of (X, d), see [Sutherland 1975]). Note that $g(\alpha_i) = \gamma_i, \forall i \in \mathbf{N}$, hence $(\gamma_i)_{i\in\mathbf{N}} = (g(\alpha_i))_{i\in\mathbf{N}}$. It follows from (16) that $g \neq f_j, \forall j \in \{1, \dots, n\}$. But this contradicts the fact that $f_1, \dots, f_n$ are all isometries $X \to X$. $\square$

Let (X, d) be a metric space, $\alpha = (\alpha_i)$ a dense sequence in this space and $A \subseteq X$. Let $p \in \mathbf{N}$, $r, \varepsilon > 0$ and $u_1, \dots, u_n \in \mathcal{F}^p(A)$, where $n \in \mathbf{N}$, $n \geq 1$. We say that $u_1, \dots, u_n$ is a $(p, r, \varepsilon)-$**basis** for A in (X, d, α) if $u_i \sim_{\text{iso}}^{<r} \alpha_{\leq p}$ for each $i \in \{1, \dots, n\}$ and if the following holds: whenever $v \in \mathcal{F}^p(A)$ is such that $v \sim_{\text{iso}}^{\leq r} \alpha_{\leq p}$, then $d(v, u_i) < \varepsilon$ for some $i \in \{1, \dots, n\}$. A $(p, r, \varepsilon)-$basis $u_1, \dots, u_n$ for A in (X, d, α) is said to be a **proper** $(p, r, \varepsilon)-$**basis** if $u_i \sim_{\text{iso}} \alpha_{\leq p}$ for each $i \in \{1, \dots, n\}$. Note: if $u_1, \dots, u_n$ is a proper $(p, r, \varepsilon)-$basis for A, then $u_1, \dots, u_n$ is also a proper $(p, r', \varepsilon)-$basis for A for each $r' < r$.

Proposition 26 says that if (X, d) is a compact metric space such that there exist exactly n isometries $X \to X$, then for each $\varepsilon > 0$ and each $q \in \mathbf{N}$ there exist $p, N \in \mathbf{N}$, $p > q$, and a proper $(p, 2^{-N}, \varepsilon)-$basis $u_1, \dots, u_n$ for (X, d, α) (i.e. for X in (X, d, α)).

Suppose now that α is an effective separating sequence in (X, d). Is it possible, for given $k, q \in \mathbf{N}$, to find effectively numbers p, N, $p > q$, and numbers $i_0^1, \dots, i_p^1, \dots, i_0^n, \dots, i_p^n$ so that $u_1 = (\alpha_{i_0^1}, \dots, \alpha_{i_p^1}), \dots, u_n = (\alpha_{i_0^n}, \dots, \alpha_{i_p^n})$ is a $(p, 2^{-N}, 2^{-k})-$basis for (X, d, α)? We will see later that this is possible if the computable metric space (X, d, α) is effectively compact. The idea which will be used in the proof of this fact is to reduce the search for such a basis to a finite subset of X of the form $\{\alpha_0, \dots, \alpha_m\}$, $m \in \mathbf{N}$. In that sense, the following lemma and Lemma 29 will be useful.

Lemma 27. *Let $p \in \mathbf{N}$ and let $r, \varepsilon > 0$ be such that $\frac{r}{2} < \varepsilon$. If A is a $\frac{r}{4}-$dense set in (X, d) and $u_1, \dots, u_n$ is a $(p, r, \frac{\varepsilon}{2})-$basis for A in (X, d, α), then $u_1, \dots, u_n$ is a $(p, \frac{r}{2}, \varepsilon)-$basis for (X, d, α).*

Proof. Let $v \in \mathcal{F}^p(X)$ be such that $v \sim_{\mathrm{iso}}^{\leq \frac{r}{2}} \alpha_{\leq p}$. Since A is $\frac{r}{4}-$dense, there exists $a \in \mathcal{F}^p(A)$ such that $d(v,a) < \frac{r}{4}$. Let $i,j \in \{0,\dots,p\}$. Then

$$|d(v_i,v_j) - d(\alpha_i,\alpha_j)| \leq \frac{r}{2}. \tag{17}$$

Since $|d(v_i,v_j) - d(a_i,a_j)| \leq d(v_i,a_i) + d(v_j,a_j)$, we have $|d(v_i,v_j) - d(a_i,a_j)| < \frac{r}{2}$. This and (17) imply

$$|d(a_i,a_j) - d(\alpha_i,\alpha_j)| < r.$$

Hence $a \sim_{\mathrm{iso}}^{\leq r} \alpha_{\leq p}$ and therefore there exists $i \in \{1,\dots,n\}$ such that $d(a,u_i) < \frac{\varepsilon}{2}$. This, together with $d(a,v) < \frac{r}{4} < \frac{\varepsilon}{2}$, implies $d(v,u_i) < \varepsilon$. Hence $u_1,\dots,u_n$ is a $(p,\frac{r}{2},\varepsilon)-$basis for (X,d,α). □

Lemma 28. *If (X,d) is a metric space, $\delta > 0$ and $x,y,z \in \mathcal{F}^p(X)$ such that $d(x,y) < \delta$ and $y \sim_{\mathrm{iso}} z$, then $x \sim_{\mathrm{iso}}^{<2\delta} z$.*

Proof. Let $i,j \in \{0,\dots,p\}$. We have $d(y_i,x_i) < \delta$, $d(y_j,x_j) < \delta$, $|d(x_i,x_j) - d(y_i,y_j)| \leq d(x_i,y_i) + d(x_j,y_j)$ and therefore $|d(x_i,x_j) - d(z_i,z_j)| = |d(x_i,x_j) - d(y_i,y_j)| < 2\delta$. Hence $x \sim_{\mathrm{iso}}^{<2\delta} z$. □

Lemma 29. *Let $u_1,\dots,u_n$ be a proper $(p,r,\varepsilon)-$basis for (X,d,α), where $\frac{r}{2} < \varepsilon$. Suppose A is an $\frac{r}{2}-$dense set in (X,d). Then there exists a $(p,r,2\varepsilon)-$basis $u'_1,\dots,u'_n$ for A in (X,d,α).*

Proof. Since A is $\frac{r}{2}-$dense, for each $i \in \{1,\dots,n\}$ there exists $u'_i \in \mathcal{F}^p(A)$ such that $d(u_i,u'_i) < \frac{r}{2}$. Then, for each $i \in \{1,\dots,n\}$, Lemma 28 and $u_i \sim_{\mathrm{iso}} \alpha_{\leq p}$ imply $u'_i \sim_{\mathrm{iso}}^{<r} \alpha_{\leq p}$. Suppose $v \in \mathcal{F}^p(A)$ is such that $v \sim_{\mathrm{iso}}^{\leq r} \alpha_{\leq p}$. Then $d(v,u_i) < \varepsilon$ for some $i \in \{1,\dots,n\}$ which implies $d(v,u'_i) < \varepsilon + \frac{r}{2} < \varepsilon + \varepsilon = 2\varepsilon$. Therefore $u'_1,\dots,u'_n$ is a $(p,r,2\varepsilon)-$basis for A in (X,d,α). □

It is easy to prove the following lemma.

Lemma 30. *Let (X,d) be a metric space.*

(i) Let $r > 0$, let A be an $r-$dense set in (X,d) and let $f : X \to X$ be an isometry. Then $f(A)$ is also $r-$dense.

(ii) If $x,y \in \mathcal{F}^p(X)$ and $\varepsilon > 0$ are such that y is $\varepsilon-$dense and $d(x,y) < \varepsilon$, then x is $2\varepsilon-$dense.

Theorem 31. *Let (X,d,α) be an effectively compact computable metric space such that there exist only finitely many isometries of the metric space (X,d). Let β be an effective separating sequence in (X,d). Then $\beta \sim \alpha$.*

The rest of this section is the proof of Theorem 31.

Let $f_1, \dots, f_n$ be all isometries $X \to X$, $f_i \neq f_j$, $i \neq j$. As in the proof of Proposition 26 we conclude that there exist a positive rational number ε_0 and $p_0 \in \mathbf{N}$ such that

$$d((f_i \circ \alpha)_{\leq p_0}, (f_j \circ \alpha)_{\leq p_0}) > 9\varepsilon_0, \tag{18}$$

for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Choose $a_1, \dots, a_n \in \mathcal{F}^{p_0}(\{\alpha_k \mid k \in \mathbf{N}\})$ and $b_1, \dots, b_n \in \mathcal{F}^{p_0}(\{\beta_k \mid k \in \mathbf{N}\})$ so that for each $i \in \{1, \dots, n\}$

$$d(a_i, (f_i \circ \alpha)_{\leq p_0}) < \varepsilon_0,\ d(b_i, (f_i \circ \alpha)_{\leq p_0}) < \varepsilon_0.$$

Clearly $d(a_i, b_i) < 2\varepsilon_0$, $\forall i \in \{1, \dots, n\}$. It follows from Lemma 16 that

$$d(a_i, a_j) > 7\varepsilon_0,\ d(b_i, b_j) > 7\varepsilon_0, \tag{19}$$

for all $i, j \in \{1, \dots, n\}$, $i \neq j$.

Lemma 32. *Let $x, y_1, \dots, y_n \in \mathcal{F}^{p_0}(X)$ and $m \in \{1, \dots, n\}$ be such that $d(x, y_m) < \varepsilon_0$, such that for each $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $d(y_i, (f_j \circ \alpha)_{\leq p_0}) < \varepsilon_0$ and such that $d(y_i, y_j) > 4\varepsilon_0$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$. Then*

- **(i)** *there exists $l \in \{1, \dots, n\}$ such that $d(x, b_l) < 3\varepsilon_0$ and $d(y_m, a_l) < 2\varepsilon_0$;*
- **(ii)** *if $i, l' \in \{1, \dots, n\}$ are such that $d(x, b_{l'}) < 3\varepsilon_0$ and $d(y_i, a_{l'}) < 2\varepsilon_0$, then $i = m$.*

Proof. (i) There exists $l \in \{1, \dots, n\}$ such that $d(y_m, (f_l \circ \alpha)_{\leq p_0}) < \varepsilon_0$. This and $d((f_l \circ \alpha)_{\leq p_0}, a_l) < \varepsilon_0$ give $d(y_m, a_l) < 2\varepsilon_0$. In the same way $d(y_m, b_l) < 2\varepsilon_0$ which, together with $d(x, y_m) < \varepsilon_0$, gives $d(x, b_l) < 3\varepsilon_0$.

(ii) Suppose $i, l' \in \{1, \dots, n\}$ are such that $d(x, b_{l'}) < 3\varepsilon_0$ and $d(y_i, a_{l'}) < 2\varepsilon_0$. Let l be as in (i). Inequalities $d(x, b_l) < 3\varepsilon_0$ and $d(x, b_{l'}) < 3\varepsilon_0$ imply $d(b_l, b_{l'}) < 6\varepsilon_0$ and we conclude from (19) that $l = l'$. Now from $d(y_m, a_l) < 2\varepsilon_0$ and $d(y_i, a_l) < 2\varepsilon_0$ we get $d(y_m, y_i) < 4\varepsilon_0$. Therefore $i = m$. □

Lemma 33. *Let $y_1, \dots, y_n$ be a $(p, r, \varepsilon)-$ basis for (X, d, α), where $p \geq p_0$ and $\varepsilon \leq \varepsilon_0$. Then*

- **(i)** *for each $i \in \{1, \dots, n\}$ there exists $j \in \{1, \dots, n\}$ such that $d(y_i, (f_j \circ \alpha)_{\leq p}) < \varepsilon$;*
- **(ii)** *$d((y_i)_{\leq p_0}, (y_j)_{\leq p_0}) > 7\varepsilon_0$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$;*
- **(iii)** *if $\alpha_{\leq p}$ is $\varepsilon-$dense, then the finite sequences $y_1, \dots, y_n$ are $2\varepsilon-$dense.*

Proof. (i) Let $k \in \{1,\dots,n\}$. Since $(f_k \circ \alpha)_{\leq p} \sim_{\text{iso}} \alpha_{\leq p}$, there exists $i_k \in \{1,\dots,n\}$ such that $d((f_k \circ \alpha)_{\leq p}, y_{i_k}) < \varepsilon$. If $k, k' \in \{1,\dots,n\}$ and $i_k = i_{k'}$, then

$$d((f_k \circ \alpha)_{\leq p}, (f_{k'} \circ \alpha)_{\leq p}) < 2\varepsilon \leq 2\varepsilon_0$$

which, together with (18), implies $k = k'$. Hence $\{1,\dots,n\} \to \{1,\dots,n\}$, $k \mapsto i_k$, is injective and therefore bijective.

(ii) Let $i, j \in \{1,\dots,n\}$, $i \neq j$. By (i) there exist $i', j' \in \{1,\dots,n\}$ such that $i' \neq j'$ and $d((f_{i'} \circ \alpha)_{\leq p}, y_i) < \varepsilon$, $d((f_{j'} \circ \alpha)_{\leq p}, y_j) < \varepsilon$. Then clearly

$$d((f_{i'} \circ \alpha)_{\leq p_0}, (y_i)_{\leq p_0}) < \varepsilon, \; d((f_{j'} \circ \alpha)_{\leq p_0}, (y_j)_{\leq p_0}) < \varepsilon.$$

We have $\varepsilon \leq \varepsilon_0$, so $d((y_i)_{\leq p_0}, (y_j)_{\leq p_0}) > 7\varepsilon_0$ by (18) and Lemma 16.

(iii) Suppose $\alpha_{\leq p}$ is $\varepsilon-$dense. Let $i \in \{1,\dots,n\}$. By (i) there exists $j \in \{1,\dots,n\}$ such that

$$d(y_i, (f_j \circ \alpha)_{\leq p}) < \varepsilon.$$

The fact that y_i is $2\varepsilon-$dense follows now from Lemma 30. □

Let $i \in \mathbf{N}$. By $\alpha[i]$ we denote the finite sequence

$$\alpha_{(i)_0}, \alpha_{(i)_1} \dots, \alpha_{(i)_{\overline{i}}}.$$

Proposition 34. *(i) Let $\mathcal{D}$ be the set of all $(i,j,m) \in \mathbf{N}^3$ such that $\overline{i} = \overline{j}$ and*

$$d(\alpha[i], \alpha[j]) < 2^{-m}.$$

Then D is r.e.

(ii) Let $\mathcal{A}$ be the set of all $(i,p,N) \in \mathbf{N}^3$ such that

$$\alpha[i] \sim_{\text{iso}}^{<2^{-N}} \alpha_{\leq p}.$$

Then $\mathcal{A}$ is r.e.

(iii) Let $\mathcal{V}$ be the set of all $(m,p,N,k,v_1,\dots,v_n) \in \mathbf{N}^{n+4}$ such that $(v_i)_j \leq m$ for each $i \in \{1,\dots,n\}$ and each $j \in \{0,\dots \overline{v_i}\}$ and such that

$$\alpha[v_1], \dots, \alpha[v_n] \textit{ is a } (p, 2^{-N}, 2^{-k}) - \textit{basis for } \{\alpha_0, \dots, \alpha_m\} \textit{ in } (X, d, \alpha).$$

Then $\mathcal{V}$ is r.e.

Proof. (i) Let $D = \{(i,j,m,l) \in \mathbf{N}^4 \mid d(\alpha_{(i)_l}, \alpha_{(j)_l}) < 2^{-m}\}$. Proposition 2(iv) implies that D is r.e. Let

$$D' = \{(i,j,m) \mid (i,j,m,l) \in D, \; \forall l \in \{0,\dots,\overline{i}\}\}.$$

It follows easily from Lemma 5 that D' is r.e. Now $\mathcal{D} = \{(i,j,m) \in \mathbf{N}^3 \mid \overline{i} = \overline{j}\} \cap D'$, hence $\mathcal{D}$ is r.e.

(ii) Let $A = \{(l, N, i, j) \in \mathbf{N}^4 \mid |d(\alpha_{(l)_i}, \alpha_{(l)_j}) - d(\alpha_i, \alpha_j)| < 2^{-N}\}$. By Proposition 2(iv) A is r.e. Let $A' = \{(l, p, N) \in \mathbf{N}^3 \mid (l, N, i, j) \in A, \forall i, j \in \{0, \dots, p\}\}$. Again, we conclude from Lemma 5 that A' is r.e. and the claim now follows from $\mathcal{A} = \{(l, p, N) \in \mathbf{N}^3 \mid \bar{l} = p\} \cap A'$.

(iii) Let $\zeta : \mathbf{N}^2 \to \mathbf{N}$ be the function of Lemma 7. We have that each element of $\mathcal{F}^p(\{0, \dots, m\})$ is of the form $(i)_0, \dots, (i)_{\bar{i}}$ for some $i \leq \zeta(m, p)$.

Let

$$\overline{A} = \{(l, p, N) \in \mathbf{N}^3 \mid \alpha[l] \text{ is not } 2^{-N} - \text{isometrically equivalent to } \alpha_{\leq p}\}.$$

Then for all $l, p, N \in \mathbf{N}$ we have $(l, p, N) \in \overline{A}$ if and only if

$$\bar{l} \neq p \text{ or } \left(\exists i, j \in \{0, \dots, p\} \text{ such that } |d(\alpha_{(l)_i}, \alpha_{(l)_j}) - d(\alpha_i, \alpha_j)| > 2^{-N}\right).$$

The set of all $(l, p, N) \in \mathbf{N}^3$ for which there exist $i, j \in \mathbf{N}$ such that

$$|d(\alpha_{(l)_i}, \alpha_{(l)_j}) - d(\alpha_i, \alpha_j)| > 2^{-N} \text{ and } i, j \in \{0, \dots, p\}$$

is r.e. by Proposition 2(iv) and Proposition 1(i). Therefore $\overline{A}$ is r.e.

Let V be the set of all $(i, k, v_1, \dots, v_n) \in \mathbf{N}^{n+2}$ such that

$$(i, v_1, k) \in \mathcal{D} \text{ or } (i, v_2, k) \in \mathcal{D} \text{ or } \dots \text{ or } (i, v_n, k) \in \mathcal{D}.$$

Then V is r.e. as the union of r.e. sets.

Let F be the set of all $(i, m, p) \in \mathbf{N}^3$ such that $\bar{i} = p$ and $(i)_j \leq m$ for each $j \in \{0, \dots, \bar{i}\}$. Clearly, F is recursive. We also have that the set $G = \{(m, v_1, \dots, v_n) \in \mathbf{N}^{n+1} \mid (v_i)_j \leq m, \forall i \in \{1, \dots, n\}, \forall j \in \{0, \dots \overline{v_i}\}\}$ is recursive.

Finally, let us prove that $\mathcal{V}$ is r.e. We have $(m, p, N, k, v_1, \dots, v_n) \in \mathcal{V}$ if and only if $(m, v_1, \dots, v_n) \in G$, $(v_1, p, N) \in \mathcal{A}$, $\dots$, $(v_n, p, N) \in \mathcal{A}$ and

$$\forall x \in \mathcal{F}^p(\{\alpha_0, \dots, \alpha_m\}) : \text{ if } x \sim_{\text{iso}}^{\leq 2^{-N}} \alpha_{\leq p}, \text{ then } d(\alpha[v_j], x) < 2^{-k} \text{ for some } j \tag{20}$$

However, (20) is equivalent to the following: for each $i \in \{0, \dots, \zeta(m, p)\}$

$$(i, m, p) \notin F \text{ or } (i, N) \in \overline{A} \text{ or } (i, k, v_1, \dots, v_n) \in V. \tag{21}$$

Let V' be the set of all $(m, p, N, k, v_1, \dots, v_n)$ such that (21) holds for each $i \in \{0, \dots, \zeta(m, p)\}$. The fact that F is recursive and $\overline{A}$ and V r.e. implies, together with Lemma 5, that V' is r.e. We have $(m, p, N, k, v_1, \dots, v_n) \in \mathcal{V}$ if and only if $(m, v_1, \dots, v_n) \in G$, $(v_1, p, N) \in \mathcal{A}$, $\dots$, $(v_n, p, N) \in \mathcal{A}$ and $(m, p, N, k, v_1, \dots, v_n) \in V'$. Therefore $\mathcal{V}$ is r.e. $\square$

For $i \in \mathbf{N}$ let us denote by $\beta[i]$ the finite sequence $\beta_{(i)_0}, \beta_{(i)_1} \dots, \beta_{(i)_{\bar{i}}}$.

Lemma 35. *Suppose $\varphi, \psi : \mathbf{N} \to \mathbf{N}$ are recursive functions such that for each $k \in \mathbf{N}$ the finite sequence $\alpha[\varphi(k)]$ is $2^{-k}-$dense in (X, d) and such that $\overline{\varphi(k)} = \overline{\psi(k)}$,*

$$d\left(\beta[\psi(k)], \alpha[\varphi(k)]\right) < 2^{-k},$$

$\forall k \in \mathbf{N}$. Then $\alpha \sim \beta$.

Proof. Let $i, k \in \mathbf{N}$. Then there exists $j \in \mathbf{N}$ such that $d(\alpha_i, \alpha_{(\varphi(k))_j}) < 2^{-k}$, $0 \leq j \leq \overline{\varphi(k)}$. It follows from Proposition 2(iv) and Proposition 1(i) that there exists a recursive function $h : \mathbf{N}^2 \to \mathbf{N}$ such that $d(\alpha_i, \alpha_{(\varphi(k))_{h(i,k)}}) < 2^{-k}$ and $0 \leq h(i, k) \leq \overline{\varphi(k)}$, $\forall i, k \in \mathbf{N}$. Therefore for all $i, k \in \mathbf{N}$ we have

$$d\left(\alpha_i, \beta_{(\psi(k))_{h(i,k)}}\right) < 2 \cdot 2^{-k}.$$

It follows that α is a recursive sequence in (X, d, β), hence $\alpha \sim \beta$. □

We are now ready to prove Theorem 31. Let $\varphi : \mathbf{N} \to \mathbf{N}$ be a recursive function such that $X = \bigcup_{i=0}^{\varphi(k)} B(\alpha_i, 2^{-k})$, $\forall k \in \mathbf{N}$. For $k \in \mathbf{N}$ let

$$A_k = \{\alpha_0, \ldots, \alpha_{\varphi(k)}\}.$$

Then A_k is $2^{-k}-$dense for each $k \in \mathbf{N}$. Let $k_0 \in \mathbf{N}$ be such that $2^{-k_0} < \varepsilon_0$.

Let $k \in \mathbf{N}$. By Proposition 26 there exist $p, N \in \mathbf{N}$, where $p \geq \max\{\varphi(k + k_0), p_0\}$, and a proper $(p, 2^{-N}, 2^{-(k+k_0+2)})-$basis $u_1, \ldots, u_n$ for (X, d, α). It is clear that then $u_1, \ldots, u_n$ is also a proper $(p, 2^{-N'}, 2^{-(k+k_0+2)})-$basis for (X, d, α) for each $N' \geq N$. Thus we may assume that $N \geq k + k_0 + 2$.

The set A_{N+2} is $\frac{2^{-N}}{2}-$dense in (X, d) and we have $\frac{2^{-N}}{2} < 2^{-N} \leq 2^{-(k+k_0+2)}$. By Lemma 29 there exists a $(p, 2^{-N}, 2^{-(k+k_0+1)})-$basis $u'_1, \ldots, u'_n$ for A_{N+2}. Since $u'_1, \ldots, u'_n \in \mathcal{F}^p(A_{N+2})$, there exist $v_1, \ldots, v_n \in \mathbf{N}$ such that $u'_1 = \alpha[v_1]$, $\ldots$, $u'_n = \alpha[v_n]$ and such that $(v_i)_j \leq \varphi(N + 2)$ for each $i \in \{1, \ldots, n\}$ and each $j \in \{0, \ldots \overline{v_i}\}$.

Hence we have the following conclusion: for each $k \in \mathbf{N}$ there exist p, N, v_1, $\ldots$, $v_n \in \mathbf{N}$ such that

$$p \geq \max\{\varphi(k + k_0), p_0\},\ N \geq k + k_0 + 2,\ (v_i)_j \leq \varphi(N + 2), \tag{22}$$

$\forall i \in \{1, \ldots, n\}$, $\forall j \in \{0, \ldots \overline{v_i}\}$, and such that

$$\alpha[v_1], \ldots, \alpha[v_n] \text{ is a } (p, 2^{-N}, 2^{-(k+k_0+1)}) - \text{basis for } A_{N+2} \text{ in } (X, d, \alpha). \tag{23}$$

Therefore, by Proposition 34(iii) and Proposition 1(ii), there exist recursive functions $\widetilde{p}, \widetilde{N}, \widetilde{v_1}, \ldots, \widetilde{v_n} : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$ (22) and (23) hold when

$$p = \widetilde{p}(k),\ N = \widetilde{N}(k),\ v_1 = \widetilde{v_1}(k),\ \ldots,\ v_n = \widetilde{v_n}(k). \tag{24}$$

Let $\mathcal{B}$ be the set of all $(i,p,N) \in \mathbf{N}^3$ such that

$$\beta[i] \sim_{\text{iso}}^{<2^{-N}} \alpha_{\leq p}.$$

Then $\mathcal{B}$ is r.e. and we get this in the same way as we get that the set $\mathcal{A}$ in Proposition 34(ii) is r.e. Since the sequence β is dense in (X,d), we easily conclude that for each $k \in \mathbf{N}$ there exists $i \in \mathbf{N}$ such that

$$\beta[i] \sim_{\text{iso}}^{<2^{-\widetilde{N}(k)+1}} \alpha_{\leq \widetilde{p}(k)}. \tag{25}$$

Therefore Proposition 1(ii) implies that there exists a recursive function $\psi : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$ (25) holds when $i = \psi(k)$.

Now we come to the crucial part of the proof. Let $k \in \mathbf{N}$. Let $p, N, v_1, \dots, v_n$ be defined by (24) and let $i = \psi(k)$. Since (23) holds, Lemma 27 implies that

$$\alpha[v_1], \dots, \alpha[v_n] \text{ is a } (p, 2^{-(N+1)}, 2^{-(k+k_0)}) - \text{basis for } (X,d,\alpha). \tag{26}$$

Now (25) and (24) imply that

$$d(\beta[\psi(k)], \alpha[\widetilde{v_m}(k)]) < 2^{-(k+k_0)} \tag{27}$$

for some $m \in \{1, \dots, n\}$.

Since $p \geq \varphi(k+k_0)$, $\alpha_{\leq p}$ is $2^{-(k+k_0)}-$dense and by Lemma 33(iii) the finite sequences $\alpha[v_1]$, $\dots$, $\alpha[v_n]$ are $2 \cdot 2^{-(k+k_0)}-$dense. Now, if $n = 1$, i.e. if there are no isometries $X \to X$ apart from the identity, then $m = 1$ and (27) together with Lemma 35 gives $\alpha \sim \beta$. Of course, n can be greater than 1 and so we have to determine somehow for which $m \in \{1, \dots, n\}$ (27) holds.

Using Lemma 33, we conclude from Lemma 32 that there exists $l \in \{1, \dots, n\}$ such that

$$d(\beta[\psi(k)]_{\leq p_0}, b_l) < 3\varepsilon_0 \text{ and } d(\alpha[\widetilde{v_m}(k)]_{\leq p_0}, a_l) < 2\varepsilon_0.$$

For $j, j' \in \{1, \dots, n\}$ let

$$C_{j,j'} = \{x \in \mathbf{N} \mid d(\beta[\psi(x)]_{\leq p_0}, b_j) < 3\varepsilon_0 \text{ and } d(\alpha[\widetilde{v_{j'}}(x)]_{\leq p_0}, a_j) < 2\varepsilon_0\}.$$

Hence, we have that for each $x \in \mathbf{N}$ there exist $j, j' \in \{1, \dots, n\}$ such that $x \in C_{j,j'}$. Since the set $C_{j,j'}$ is r.e. for all $j, j' \in \{1, \dots, n\}$, what we see similarly as in the proof of Proposition 34, we easily get that there exist recursive functions $\lambda, \tau : \mathbf{N} \to \mathbf{N}$ such that $x \in C_{\lambda(x),\tau(x)}$, $\forall x \in \mathbf{N}$. For $x = k$ we have $k \in C_{\lambda(k),\tau(k)}$, hence

$$d(\beta[\psi(k)]_{\leq p_0}, b_{\lambda(k)}) < 3\varepsilon_0 \text{ and } d(\alpha[\widetilde{v_{\tau(k)}}(k)]_{\leq p_0}, a_{\lambda(k)}) < 2\varepsilon_0.$$

It follows from Lemma 32 that $\tau(k) = m$. So (27) implies

$$d(\beta[\psi(k)], \alpha[\widetilde{v_{\tau(k)}}(k)]) < 2^{-(k+k_0)}$$

and we conclude from Lemma 35 that $\alpha \sim \beta$. Hence Theorem 31 is proved.

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