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Spatial Reasoning with Integrated Qualitative-Metric Fuzzy Constraint Networks

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Abstract: Qualitative Spatial Reasoning can be greatly improved if metric information can be represented and reasoning can be performed on it; moreover, modelling vagueness and uncertainty in both qualitative and metric relations allows reasoning in a more flexible way about data coming from real world.

In this paper Rectangle Algebra is integrated with a bi-dimensional Point Algebra by defining a set of 25 Point-Region relations, in this way a Spatial Qualitative Algebra (SQA) among point and regions is obtained. Besides, SQA is extended to deal with uncertain data by means of the Fuzzy Sets Theory. Fuzzy metric information is represented using pyramidal possibility distributions, and transformation functions that allow passing from qualitative to metric information and vice versa are provided.

Key Words: qualitative spatial reasoning, fuzzy relations, approximate reasoning
Category: I.2.4

1 Introduction

Spatial Reasoning plays a central role in many Artificial Intelligence applications such as robot navigation, visual object recognition, intelligent image information systems, query processing in geographic databases; new challenges have arisen also in Cell Biology [Pool, 2003]. As in the case of other qualitative reasoning formalisms, there are basically two approaches to build a model suitable for spatial reasoning: model physical space and the objects within it or model the relationships between the objects [Frank, 1992]; a set of qualitative relations may be incomplete and even inconsistent, and the consistent integration of such information relies on the algebraic properties of the qualitative relations. Spatial Reasoning can be formulated using the framework of Constraint Satisfaction Problems (CSPs), for example TCSPs have been defined for reasoning about time [Dechter et al., 1991].

Different aspects of space can be considered; here, the more common classification of spatial relations into topological relations and directional relations is taken into account. Also distances are considered, but in the usual context of metrics, not as qualitative relations. Topological spatial relations are those that are invariant under continuous transformations, such as rotation or scaling. Directional relations are defined between a reference object and a primary object with respect to a fixed frame of reference, usually determined by a predefined entity such as the North Pole. Topological information is commonly represented using extended regions as basic entities, while orientation is based on points. In this paper an algebra dealing with relations between regions, namely the Rectangle Algebra [Balbiani et al., 1998] is combined with the algebra of Cardinal Directions [Frank, 1992]. This allows to obtain a more expressive algebra, which will be called Spatial Qualitative Algebra (SQA); besides, also metric information about distances has been added.

In Temporal Reasoning the most classical model of integration between qualitative and quantitative constraints was proposed in [Meiri, 1996] who defined an extended Temporal CSP able to deal with both types of information using an unique constraint network. In Spatial Reasoning Condotta [Condotta, 2000] proposed to manage these two types of information using distinct CSPs. This paper applies the idea of Meiri to spatial constraints, that is an unique constraint network for both qualitative and metric information.

Realistic applications usually contain information pervaded by vagueness and uncertainty. This kind of notions can be dealt in the framework of Fuzzy Constraint Satisfaction Problem (FCSP) [Dubois et al., 1996] where constraints are satisfied to a degree, rather than satisfied or not satisfied, and the acceptability of a potential solution is a gradual notion. The spatial constraints taken into account are extended in a fuzzy way by associating a preference degree to each basic relation of the qualitative relations and a pyramidal preference distribution to each metric constraint.

In [Section 2] Rectangle Algebra and Cardinal Directions Algebra are extended to the fuzzy case, in [Section 3] the metric spatial constraints are defined. In [Section 4] the Point-Region relations are introduced allowing to build the Spatial Qualitative Algebra, then integration of qualitative and metric temporal constraints in a fuzzy framework is presented. Finally, [Section 5] discusses a first implementation of a constraint solver for the new framework and describes a simple scenario to show its expressiveness.

2 Qualitative constraints

2.1 The Fuzzy Rectangle Algebra fRA

Balbiani et al. [Balbiani et al., 1998] define the Rectangle Algebra as an extension of the well-known Allen's Interval Algebra (IA) [Allen, 1983] to the bidimensional space [see 1]. Interval Algebra models the relative position between any two intervals using a set of thirteen basic (or atomic) relations $\mathcal{I}$, namely: before, meets, overlaps, starts, during, finishes (b, m, o, s, d, f) together with their inverses (bi, mi, oi, si, di, fi) and the basic relation equal (eq). Analogously, the domain considered in the Rectangle Algebra is the set of rectangles

[1] extensions to multiple dimensions are possible [Mitra and Ligozat, 2002]

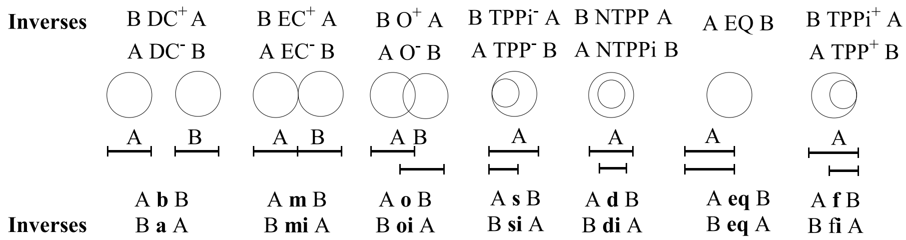

Figure 1: similarities between IA and extended RCC.

with sides parallel to the axes of some orthogonal basis in $\mathbb{R}^2$; this domain is called REC. A basic relation between two rectangles (atomic RA-relation) is a pair (r_x, r_y) of basic IA-relations: the x-axis relation and the y-axis relation; their set is called $\mathcal{A}_{rec}$. In this way, there are $13^2 = 169$ possible basic relations between any two given rectangles. If a and b are two rectangles in REC then $a\ (r_x, r_y)\ b$ is a basic RA-constraint which is satisfied if and only if both the IA-constraints r_x and r_y are satisfied.

An RA-constraint $R = \bigcup_i \{(r_{x,i}, r_{y,i})\}$ is satisfiable if and only if there exist two rectangles a and b satisfying one of the basic RA-relations in R. In Rectangle Algebra the usual operations of inversion, intersection and composition are defined. All the operations are performed on pairs of unions of basic relations; recall that the projected basic relations are IA relations, so the usual operations can be easily defined [Balbiani et al., 1998], for example the inverse of relation $R = \bigcup_i \{(r_{x,i}, r_{y,i})\}$ is $R^{-1} = \bigcup_i \{(r_{x,i}^{-1}, r_{y,i}^{-1}) : (r_{x,i}, r_{y,i}) \in R\}$.

Composition between atomic IA relations has been defined in [Allen, 1983] by means of a transitivity table which has an entry for each possible combination of atomic relations pairs.

An RA-network is a graph $G = (V, E)$ given by a set of variables V which represents rectangles and a set M of RA-constraints between the variables in V. An RA-network $\mathcal{N}$ with variables $V = \{v_1, \ldots v_n\}$ is consistent if and only if there exists a solution given by n rectangles $(a_1, \ldots, a_n), a_i \in REC^n$ such that all RA-constraints are satisfied by the assignment $v_i = a_i, i = 1, \ldots, n$. The Rectangle Algebra has the same complexity of the IA, as far as the consistency problem of an RA-network is concerned.

IA relations are somewhat similar to mono-dimensional Region Connection Calculus (RCC) relations over regular regions [Randell et al., 1992], and to give to the spatial constraints a more intuitive meaning a sort of "orientation" in RCC relations has been introduced in relations DC, EC, O and TPP; in this way the analogy is more clear and 13 atomic relations $\mathcal{R}$ corresponding to the

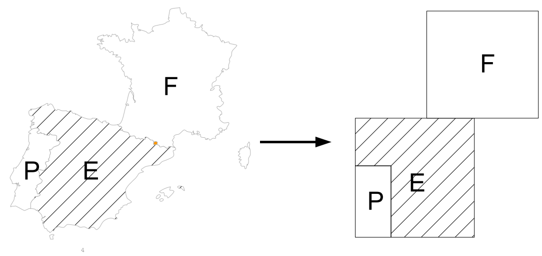

Figure 2: "Portugal is almost West w.r.t. Spain, but also partially South and partially South-West of it".

13 Allen's atomic relations $\mathcal{I}$ can be devised, as shown in [Fig. 1]. In the following, $\mathcal{R}$ relations will be used.

Definition 1. the set $\mathcal{R}$ is the set of the atomic relations $\{DC^-$, DC^+, EC^-, EC^+, O^-, O^+, TPP^-, TPP^+, $TPPi^-$, $TPPi^+$, $NTPP$, $NTPPi$, $EQ\}$.

Flexibility and uncertainty can be introduced in Rectangle Algebra in a way similar to that proposed in [Guesgen et al., 1994]. The definition of RA-constraints is relaxed, by assigning to every atomic relation $r_i \in \mathcal{R}$ a degree $\alpha_i \in [0,1]$, which tells the preference degree of the corresponding assignment among the others; in this way a fuzzy Rectangle Algebra fRA can be defined.

Definition 2. Let a and b be two rectangles in REC, then a fRA constraint is defined as

$$R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$$

where $r_{z,i}, z \in \{x, y\}, i = \{1, \ldots, 13\}$ are $\mathcal{R}$ relations and $\alpha_i \in [0,1]$ are the preference degrees of $r_{z,i}$. Each disjunct $(r_{x,i}, r_{y,i})[\alpha_i]$ is an atomic fuzzy RA relation.

As usual, when the preference degree is zero the corresponding $\mathcal{R}$ relations are not specified, when 1 it is omitted. Using preference degrees the expressiveness of usual RA-relations can be increased, for example it is possible to denote the fact that *Portugal is West w.r.t Spain but partially also South and South-West.*

Example 1. the position of Portugal w.r.t. Spain can be expressed using the fRA constraint (see [Fig. 2])

$$P\{(EC^-, EQ), (EQ, EC^-)[0.2], (EC^-, EC^-)[0.5]\}E$$

The preference degree of the first pair (EC^-, EQ) is 1 and it has been omitted, the other two pairs have preference degrees less than 1 but greater than zero, so their preference degrees have been written. The remaining combinations have not been specified at all, since their preference degrees are zero. Notice that, according to the Fuzzy Set Theory, the preference degrees have not to sum up to 1 as in the case of the Probability Theory.

With respect to classical RA now preference degrees have to be taken into account, therefore intersection and union have to combine them, and they will be called conjunctive and disjunctive combination respectively. Among the different T-Norms and T-Conorms that can be used [Klement et al., 2000], in the following *min* and *max* will be considered. The operations between fRA constraints are defined as follows:

Definition 3. given a fRA relation $R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$, the inverse relation R^{-1} is defined as

$$R^{-1} = \bigcup_i \{(r^{-1}_{x,i}, r^{-1}_{y,i})[\alpha_i] : (r'_{x,i} r_{y,i})[\alpha_i] \in R\}$$

where $r^{-1}_{x,i}$ and $r^{-1}_{y,i}$ are the inverses of the basic relations as reported in [Fig. 1].

Example 2. if $R = ((DC^-, EQ)[0.3])$ then is $R^{-1} = ((DC^+, EQ)[0.3])$.

Definition 4. given two fRA relations $R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$ and $S = \bigcup_j \{(s_{x,j}, s_{y,j})[\beta_j])\}$ the disjunctive combination between R and S is defined as

$$\begin{aligned} R \oplus S = \bigcup\nolimits_i \{ \; & (r_{x,i}, r_{y,i})[\gamma_i] : (r_{x,i}, r_{y,i})[\alpha_i] \in R \wedge \\ & (s_{x,j}, s_{y,j})[\beta_j] \in S \wedge r_{x,i} = s_{x,j} \wedge \\ & r_{y,i} = s_{y,j}, \gamma_i = \max(\alpha_i, \beta_j)\} \end{aligned}$$

Example 3. the disjunctive combination of the fRA relations
$R = \{(DC^-, EQ)[0.3], (NTPPi, EQ)[0.7]\}$ and
$S = \{(DC^-, EQ)[0.5], (NTPPi, DC^+)[0.7]\}$ is

$$\begin{aligned} T = \{ \; & (DC^-, EQ)[0.5],)(NTPPi, EQ)[0.7], \\ & (NTPPi, DC^+)[0.7]\} \end{aligned}$$

.

Definition 5. given two fRA relations $R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$ and $S = \bigcup_j \{(s_{x,j}, s_{y,j})[\alpha_j])\}$ the conjunctive combination between R and S is defined as

$$\begin{aligned} R \otimes S = \bigcup\nolimits_i \{ \; & (r_{x,i}, r_{y,i})[\gamma_i] : (r_{x,i}, r_{y,i})[\alpha_i] \in R \wedge \\ & (s_{x,j}, s_{y,j})[\beta_j] \in S \wedge r_{x,i} = s_{x,j} \wedge \\ & r_{y,i} = s_{y,i}, \gamma_i = \min(\alpha_i, \beta_j)\} \end{aligned}$$

Example 4. the conjunctive combination of the fRA relations $R = \{(DC^-, EQ)[0.3], (NTPPi, EQ)[0.7]\}$ and $S = \{(DC^-, EQ)[0.5], (NTPPi, DC^+)[0.7]\}$ is

$$T = \{(DC^-, EQ)[0.3]\}$$

Definition 6. given two fRA relations $R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$ and $S = \bigcup_j \{s_{x,j}, s_{y,j})[\beta_j])\}$ the composition between R and S is defined as

$$\begin{aligned} R \circ S \; = & \bigcup \{(t_{x,i}, t_{y,i})[\min(\alpha_h, \beta_k)] \\ & h, k : (r_{x,h}, r_{y,h}) \circ (s_{x,k}, s_{y,k}) = (t_{x,i}, t_{y,i})\} \end{aligned}$$

composition between atomic relations is performed as in the case of classical Rectangle Algebra taking into account the correspondences between $\mathcal{I}$ and $\mathcal{R}$.

Example 5. the composition of $R = \{(DC^-, EQ)[0.3]\}$ and $S = \{(NTPPi, DC^+)[0.7]\}$

$$T = \{(DC^-, DC^+)[0.3]$$

The fRA is an algebra, that is a set of relations closed under certain operations. It is easy to see that inversion is closed, since every atomic relation in fRA has an inverse. Also combinations give relations belonging to fRA, in fact the resulting relations are formed by atoms in $\mathcal{R}$ coming from both or either operands. In composition the disjunction of relations coming from the classical composition of atomic relations is used, while preference degrees are computed by means of max and min functions. Since classical RA is closed under composition also fRA is closed.

2.2 The Fuzzy Cardinal Directions Algebra

When qualitative spatial positions between two points have to be described, a natural way is to model them using cardinal directions. Frank suggested methods for describing the cardinal direction of a point with respect to a reference point in a geographic space, i.e., directions are in the form of "North", "East", "South", and "West" depending on the granularity [Frank, 1992]. He distinguishes between two different methods for determining the different sectors corresponding to the single directions: the cone-based method and the projection-based method. The projection-based system consists of nine acceptance areas, one for each of the directions plus a neutral zone EQ: $\mathcal{C} = \{E, NE, N, NW, W, SW, S, SE, EQ\}$. The projection-based approach describes these relations in terms of the Point Algebra (PA) [Vilain et al., 1989] by specifying a point algebraic relation for each of the two axes separately. This provides the projection-based approach with a formal semantics and allows to define the Cardinal Directions Algebra, or CDA.

Pair	Name	Pair	Name
		inverses	
$(>,=)$	E	$(<,=)$	W
$(>,>)$	NE	$(<,<)$	SW
$(=,>)$	N	$(=,<)$	S
$(<,>)$	NW	$(>,<)$	SE
$(=,=)$	EQ	$(=,=)$	EQ

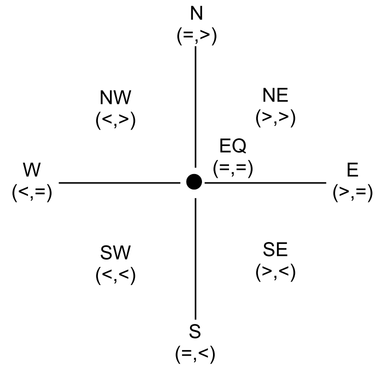

Figure 3: Point-point spatial relations and their graphical representation.

Flexibility and uncertainty can be introduced also in CDA by assigning to every atomic relation $r_i \in \mathcal{C}$ a degree $\alpha_i \in [0,1]$, which tells the preference degree of the corresponding assignment among the others, obtaining in this way a fuzzy Cardinal Directions Algebra, or $fCDA$.

Definition 7. Let a and b be two points, then a $fCDA$ constraint is defined as

$$R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$$

where $r_{z,i}, z \in \{x,y\}, i = \{1,2,3\}$ are basic Point Algebra relations $\{<,>,=\}$ and $\alpha_i \in [0,1]$ are the preference degrees of $r_{z,i}$. Each disjunct $(r_{x,i}, r_{y,i})[\alpha_i]$ is an atomic $fCDA$ relation.

Due to the limited number of basic relations involved, each pair of relations can be interpreted in a more natural way, as shown in [Fig. 3].

The operations on $fCDA$ constraints are defined in a way analogous to what is done for fuzzy RA, with the only difference that now the atomic relations belong to PA^2 and no more to $\mathcal{R}^2$.

3 Fuzzy Spatial Metric constraints

In [Condotta, 2000] metric spatial knowledge is represented by means of two constraint networks $\langle V, C \rangle$, one for each coordinate, which limit the possible distances between the variables in V. In this paper the constraints still limit the distances between the variables, but there is an unique constraint network for both coordinates. The variables therefore take values on $\mathbb{R}^2$. Besides, a fuzzy relation $R_P(C_{ij}) : \mathbb{R} \times \mathbb{R} \to [0,1]$ is associated to each constraint C_{ij} between variables v_i and v_j in V. $[R_P(C_{ij})](d_x, d_y)$ indicates to what extent an assignment $(v_j|_x - v_i|_x, v_j|_y - v_i|_y) = (d_x, d_y)$ satisfies the constraint C_{ij}.

Trapezoidal possibility distributions usually adopted in Fuzzy Temporal Networks [Marín et al., 1997] are extended here to two orthogonal dimensions. In the following they will be called pyramidal distributions.

Definition 8. a pyramidal distribution is a pair of trapezoidal distributions plus an associated preference degree: $\langle T_x, T_y\rangle[\alpha], \alpha \in [0, 1]$

In particular the normalized [see 2] version of possibility distributions proposed in [Badaloni et al., 2004] is adopted; according to these authors, each T_z is described by a 4-tuple of values, each describing four characteristic points of the two orthogonal trapezoids in x and y.

Definition 9. a well-formed trapezoid T is a 4-ple $\lhd a, b, c, d \rhd$ where $a, b \in \mathbb{R} \cup \{-\infty\}$, $c, d \in \mathbb{R} \cup \{+\infty\}$, $\lhd$ is either **(** or **[** and $\rhd$ is either **)** or **]** . A trapezoidal distribution T is allowed if and only if it satisfies the following conditions:

- $a \leq b \leq c \leq d$
- if $a = -\infty$ then $b = -\infty \ \wedge \ \lhd$ is **(**
- if $a < b$ then $\lhd$ is **(**
- if $a = d$ then $\lhd$ is **[** $\wedge \ \rhd$ is **]**
- if $d = +\infty$ then $c = +\infty \ \wedge \ \rhd$ is **)**
- if $c < d$ then $\rhd$ is **)**

Definition 10. the set of well-formed pyramidal distributions is denoted by $\mathcal{P}$.

A metric constraint C_{ij}, is a disjunction of pyramidal distributions:

Definition 11. A metric constraint C_{ij} is a set of pyramidal distributions

$$C_{ij} = \{P_1 \cdots P_n\}$$

where $P_k = \langle T_{x,k}, T_{y,k}\rangle[\alpha_k]$.

The semantics of a constraint C_{ij} is identified by the possibility distribution

$$[R_P(C_{ij})](x, y) = \max_{k=1\cdots n} [R_P(P_k)](x, y)$$

corresponding to the disjunction of pyramidal distributions $R_P(P_k) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and

$$[R_{P_k}(C_{ij})](x, y) = min\{[R_{T_{x,k}}(C_{ij})](x), [R_{T_{y,k}}(C_{ij})](y)\}$$

[2] in the Fuzzy Set Theory a normalized possibility distribution is a distribution which contains at least a preference degree equal to 1, that is fully plausible.

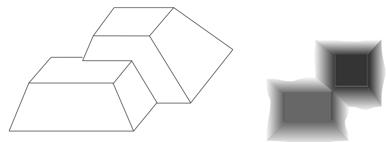

Figure 4: Example of spatial metric constraint.

where

$$[R_{T_{z,k}}(C_{ij})](z) = \begin{cases} 0 & \text{if } \; z < a_{z,k} \;\vee \\ & \quad (z = a_{z,k} \;\wedge\; \lhd \text{ is } (\,) \;\vee \\ & \quad (z = d_{z,k} \;\wedge\; \rhd \text{ is })\,) \;\vee \\ & \quad z > d_{z,k} \\ \frac{z-a_{z,k}}{b_{z,k}-a_{z,k}} & \text{if } \; a_{z,k} < z < b_{z,k} \\ \frac{z-d_{z,k}}{c_{z,k}-d_{z,k}} & \text{if } \; c_{z,k} < z < d_{z,k} \\ 1 & \text{otherwise} \end{cases}$$

Example 6. as an example of metric constraint, on the right of [Fig. 4] a region with an undefined boundary is represented and, on the left, a possible corresponding fuzzy constraint. The fuzzy constraint could be expressed, in relative coordinates, as

$$\{\langle(0,5,10,15),(0,3,8,11)\rangle[0.7],\langle(7,10,13,16),(5,8,11,14)\rangle[1.0]\}$$

The height of the pyramidal distribution is not necessarily normalized to 1, and this allows reasoning about preferences, truth of imprecise events, priorities and so on [Dubois et al., 1996]. For instance, the user can set the possibility degrees according to his own preferences using non-normalized distributions to indicate partial inconsistency of constraints coming from unreliable information sources.

3.1 Operations between metric constraints

For metric constraints the usual operations are provided:

Definition 12. given a metric constraint $C_{ij} = \{P_1, \cdots, P_m\}$ between variables v_i and v_j, each disjunct of the inverse constraint C_{ij}^{-1} is defined as

$$\begin{aligned} P_k^{-1} = \langle & \lhd_x - d_{x,k}, -c_{x,k}, -b_{x,k}, -a_{x,k} \rhd_x, \\ & \lhd_y - d_{y,k}, -c_{y,k}, -b_{y,k}, -a_{y,k} \rhd_y \rangle [\alpha_k] \end{aligned}$$

Definition 13. given two metric constraints $C_{ij} = \{P_1, \cdots, P_m\}$ between variables v_i and v_j and $C'_{jw} = \{P'_1, \cdots, P'_n\}$ between variables v_j and v_w, the constraint $C_{ij} \circ C'_{jw} = \bigcup_h P''_h$ is such that for any two disjuncts $P_k = \langle T_{x,k}, T_{y,k}\rangle[\alpha_k] \in C_{ij}$ and $P'_l = \langle T'_{x,l}, T'_{y,l}\rangle[\alpha_l] \in C'_{jw}$

$$P''_h = \langle T_{x,k} \circ T'_{x,l}, T_{y,k} \circ T'_{y,l}\rangle[min\{\alpha_k, \alpha_l\}]$$

where composition between trapezoidal distributions is defined as in [Badaloni et al., 2004].

The disjunctive and the conjunctive combinations correspond to the usual set-theoretic operations and can be obtained by reasoning about the orthogonal projections, which are both trapezoids.

All operations between pyramidal possibility distributions involve pairs of trapezoids and are applied independently on the orthogonal projections.

Definition 14. given two metric constraints $C_{ij} = \{P_1, \cdots, P_m\}$ and $C'_{ij} = \{P'_1, \cdots, P'_n\}$ between v_i and v_j, the constraint $C_{ij} \oplus C'_{ij} = \bigcup_h P''_h$ is such that for any two disjuncts $P_k = \langle T_{x,k}, T_{y,k}\rangle[\alpha_k] \in C_{ij}$ and $P'_l = \langle T'_{x,l}, T'_{y,l}\rangle[\alpha_l] \in C'_{ij}$

$$P''_h = \langle T_{x,k} \oplus T'_{x,l}, T_{y,k} \oplus T'_{y,l}\rangle[\max\{\alpha_k, \alpha_l\}]$$

Definition 15. given two metric constraints $C_{ij} = \{P_1, \cdots, P_m\}$ and $C'_{ij} = \{P'_1, \cdots, P'_n\}$ between v_i and v_j, the constraint $C_{ij} \otimes C'_{ij} = \bigcup_h P''_h$ is such that for any two disjuncts $P_k = \langle T_{x,k}, T_{y,k}\rangle[\alpha_k] \in C_{ij}$ and $P'_l = \langle T'_{x,l}, T'_{y,l}\rangle[\alpha_l] \in C'_{ij}$

$$P''_h = \langle T_{x,k} \otimes T'_{x,l}, T_{y,k} \otimes T'_{y,l}\rangle[\min\{\alpha_k, \alpha_l\}]$$

4 Qualitative and metric constraints

In [Condotta, 2000] Condotta proposed to build two constraint networks, one for qualitative constraints and the other for metric constraints. In this paper the idea used by Meiri to integrate temporal constraints [Meiri, 1996] is adopted: a single network for both types of constraints.

4.1 Relations between Points and Regions

The first step to integrate metric and qualitative information is to define an algebra that includes all the combinations that can occur between a point and a (rectangular) region. There is therefore the need to find relations that link points with regions. An intuitive way to do this is to extend in two dimensions the Point-Interval relations coming from the temporal context. Being 5 the atomic mono-dimensional relations between a point and an interval ([Fig. 5]), in the spatial case there will be $5^2 = 25$ atomic relations. In this paper the mono-dimensional relations corresponding to the projections of a spatial relation on an orthogonal axis will be named in a different way w.r.t. Meiri's Point-Interval relations:

Definition 16. the set of atomic Point Region relations is defined on the set $\mathcal{PR} = \{E^-, T^-, I, T^+, E^+\}$.

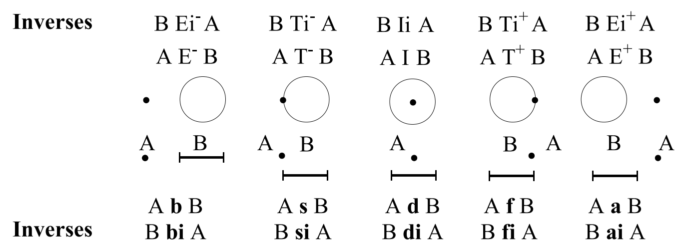

Figure 5: Point-Region spatial relations.

Flexibility and uncertainty can be introduced in PR relations by assigning to every atomic relation $r_i \in \mathcal{PR}$ a degree $\alpha_i \in [0,1]$, which tells the preference degree of the corresponding assignment among the others, and obtaining in this way a set of fuzzy Point-Region relations, or fPR.

Definition 17. Let a be a point and $b \in REC$, then a fPR constraint is defined as

$$R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i])\}$$

where $r_{z,i}, z \in \{x,y\}, i = \{1,2,3\}$ are $\mathcal{PR}$ relations and $\alpha_i \in [0,1]$ are the preference degrees of $r_{z,i}$. Each disjunct $(r_{x,i}, r_{y,i})[\alpha_i]$ is an atomic fPR relation.

Also fPR relations can be interpreted in a more natural way; besides the atomic relations in $\mathcal{C}$, with their 9 standard names, 16 additional relations have been added; they have been named as in [Tab. 1].

4.2 The Fuzzy Spatial Qualitative Algebra ($fSQA$)

Once the fuzzy Point-Region relations have been defined, an algebra that encloses all the fuzzy relations between Points and Regions can be defined; it will be called Fuzzy Spatial Qualitative Algebra or $fSQA$.

Definition 18. the Fuzzy Spatial Qualitative Algebra $fSQA$ is given by

$$fRA \cup fCDA \cup fPR$$

wherefRA is the fuzzy Rectangle Algebra, $fCDA$ the fuzzy Cardinal Directions Algebra and fPR is the fuzzy Point Region set.

The $fSQA$ algebra is closed under the inversion, intersection and composition operations; inversion and intersection for $\mathcal{A}_{rec}$ and $fCDA$ relations concern operands coming from the same algebra, and these have already been defined before. As far as fPR relations are involved, inverse fPR relations are denoted by adding a suffix "i" to the corresponding relations in $\mathcal{PR}$ (for example, given a region a and a point b if $a\ {}_tNE\ b$ then $b\ {}_tNE_i\ a$), while disjunctive and conjunctive combination operations are defined as follows.

Definition 19. given two fPR relations $R = \bigcup_i\{(r_{x,i}, r_{y,i})[\alpha_i])\}$ and $S = \bigcup_j\{(s_{x,j}, s_{y,j})[\beta_j])\}$ the disjunctive combination between R and S is defined as

$$\begin{aligned} R \oplus S = \bigcup_i\{&(r_{x,i}, r_{y,i})[\gamma_i] : (r_{x,i}, r_{y,i})[\alpha_i] \in R \wedge \\ &(s_{x,j}, s_{y,j})[\beta_j] \in S \wedge r_{x,i} = s_{x,j} \wedge \\ &r_{y,i} = s_{y,j}, \gamma = \max(\alpha, \beta)\} \end{aligned}$$

Definition 20. given two fPR relations $R = \bigcup_i\{(r_{x,i}, r_{y,i})[\alpha_i])\}$ and $S = \bigcup_j\{(s_{x,j}, s_{y,j})[\beta_j])\}$ the conjunctive combination between R and S is defined as

$$\begin{aligned} R \otimes S = \bigcup_i\{&(r_{x,i}, r_{y,i})[\gamma_i] : (r_{x,i}, r_{y,i})[\alpha_i] \in R \wedge \\ &(s_{x,j}, s_{y,j})[\beta_j] \in S \wedge r_{x,i} = s_{x,j} \wedge \\ &r_{y,i} = s_{y,j}, \gamma = \min(\alpha, \beta)\} \end{aligned}$$

The composition operation may involve operands belonging to different algebras, and therefore it is defined in terms of a combined composition table which takes into account all possible combinations between a point and a region; preference degrees are again obtained by means of a "max-min" weighting. [Tab. 2] shows all these combinations; the symbol "$\emptyset$" denotes illegal combinations.

Definition 21. given two $fSQA$ relations $R = \bigcup_i\{(r_{x,i}, r_{y,i})[\alpha_i])\}$ and $S = \bigcup_j\{s_{x,j}, s_{y,j})[\beta_j])\}$ the composition between R and S is defined as

$$\begin{aligned} R \circ S\ = &\bigoplus (t_{x,i}, t_{y,i})[\min(\alpha_h, \beta_k)] \\ &h, k : (r_{x,h}, r_{y,h}) \circ (s_{x,k}, s_{y,k}) = (t_{x,i}, t_{y,i}) \end{aligned}$$

where the composition between atomic relations $(r_{x,j}, r_{y,j})$ and $(s_{x,k}, s_{y,k})$ is given by [Tab. 2].

Table T_{CDA} is the transitivity table of CDA Algebra [Frank, 1992]. T_{RA} the transitivity table of the Rectangle Algebra [Balbiani et al., 1998], which can be replaced by a double look-up in the IA transitivity table [Allen, 1983], one for each of the two orthogonal components of an RA relation.

The remaining tables are analogous to those proposed in [Meiri, 1996] considering the correspondences between temporal and spatial relations discussed before ([Fig. 1] and [Fig. 5]) but have not been reported here for space limits.

Table 1: relations for fPR interpretation.

Pair	**Name**	**Pair**	**Name**	**Pair**	**Name**
(E^+, I)	E	(E^+, T^-)	ESE	(T^+, I)	${}_tE$
(E^+, E^+)	NE	(E^+, T^+)	ENE	(T^+, T^+)	${}_tNE$
(I, E^+)	N	(T^+, E^+)	NNE	(I, T^+)	${}_tN$
(E^-, E^+)	NW	(T^-, E^+)	NNW	(T^-, T^+)	${}_tNW$
(E^-, I)	W	(E^-, T^+)	WNW	(T^-, I)	${}_tW$
(E^-, E^-)	SW	(E^-, T^-)	WSW	(T^-, T^-)	${}_tSW$
(I, E^-)	S	(T^-, E^-)	SSW	(I, T^-)	${}_tS$
(E^+, E^-)	SE	(T^+, E^-)	SSE	(T^+, T^-)	${}_tSE$
(I, I)	In				

4.3 Transformation functions

Having defined the possibility distributions of the metric constraints as a combination of two trapezoidal distributions along two orthogonal axes and having said that (fuzzy) CDA relations are formed by PA relations along orthogonal axes, the transformation functions introduced originally in [Meiri, 1996] and extended then in [Badaloni et al., 2004] in order to be applied to trapezoidal distributions can be easily defined.

More specifically a (qualitative) $fCDA$ constraint can be transformed in a metric constraint by applying the QUAN^{fuz} to both its components (which are PA^{fuz} relations)

Definition 22. given a $fCDA$ constraint $R = \bigcup_i \{(r_{x,i}, r_{y,i})[\alpha_i]\}$ the function $fQUAN2(R) : fCDA \to \mathcal{P}$ is defined as

$$\bigcup_i \{\langle \mathrm{QUAN}^{fuz}(r_{x,i}), \mathrm{QUAN}^{fuz}(r_{y,i})\rangle [\alpha_i]\}$$

Example 7. The $fCDA$ constraint $R = \{(<,<)[0.5], (<,=)[0.3]\}$ becomes a pyramidal distribution $fQUAN2(R) = \{\langle(0, 0, +\infty, +\infty), (0, 0, +\infty, +\infty)\rangle[0.5], \langle(0, 0, +\infty, +\infty), [0, 0, 0, 0]\rangle[0.3]\}$.

Table 2: transitivity table of SQA

	CDA	**PR**	**RP**	**RA**
CDA	T_{CDA}	$T_{CDA \circ PR}$	$\emptyset$	$\emptyset$
PR	$\emptyset$	$\emptyset$	$T_{PR \circ RP}$	$(T$
RP	$(T_{CDA \circ PR})^T$	$T_{RP \circ PR}$	$\emptyset$	$\emptyset$
RA	$\emptyset$	$\emptyset$	$(T_{PR \circ RA})^T$	T_{RA}

On the other hand, if $C = \bigcup_i \langle T_{i,x}, T_{i,y}\rangle[\alpha_i]$ is a metric constraint then it can be transformed in a qualitative constraint by applying the QUAL^{fuz} to both components $T_{i,x}$ and $T_{i,y}$ of each disjunct (which are trapezoids).

Definition 23. the function $fQUAL2(R) : \mathcal{P} \rightarrow fSQA$ is defined as

$$fQUAL2(R) = \langle \bigcup_i \{\mathrm{QUAL}^{fuz}(T_{i,x}[\alpha_i])\}, \bigcup_i \{\mathrm{QUAL}^{fuz}(T_{i,y}[\alpha_i])\rangle$$

Example 8. The metric constraint $C = \{\langle(-5, 5, 10, 15), (0, 3, 6, 9)\rangle[0.5]\}$ becomes the $fCDA$ constraint $fQUAL2(C) = \langle(<[0.5], =[0.25], >[0.25]), (<[0.5])\rangle$.

It can be noticed that the explicit representation of different kinds of extremes, open, closed or unbounded, is essential to define such transformation functions, since they require generalized trapezoids.

By means of the concepts introduced above, spatial networks whose variables can represent both points and rectangular regions, and whose edges are accordingly labelled by qualitative and quantitative fuzzy spatial constraints can be modelled.

In particular, as in [Badaloni et al., 2004], Point-Point metric constraints are maintained in a numerical form as long as possible, while Region-Region and Point-Region constraints are necessarily qualitative, that is they are modelled as fRA and fPR relations, respectively.

On the basis of these considerations it is possible to define the operations involving all kinds of constraints introduced so far. Since metric constraints can be defined only between points, the definitions of the operations between qualitative and metric constraints can be limited, without loss of generality, to the following cases:

Definition 24. given a metric constraint C_{ij} and a qualitative constraint C'_{ij} between variables v_i and v_j their disjunctive combination is

$$C_{ij} \oplus C'_{ij} = C_{ij} \otimes fQUAN2(C'_{ij})$$

Definition 25. given a metric constraint C_{ij} and a qualitative constraint C'_{ij} between variables v_i and v_j their conjunctive combination is

$$C_{ij} \otimes C'_{ij} = C_{ij} \otimes fQUAN2(C'_{ij})$$

Definition 26. given a metric constraint C_{ij} between variables v_i and v_j and a qualitative constraint $C'_{jk} \notin fPR$ between variables v_j and v_k their metric composition is

$$C_{ij} \circ C'_{jk} = C_{ij} \circ fQUAN2(C'_{jk})$$

Definition 27. given a metric constraint C_{ij} between variables v_i and v_j and a qualitative constraint $C'_{jk} \in fPR$ between variables v_j and v_k their qualitative composition is

$$C_{ij} \circ C'_{jk} = fQUAL2(C_{ij}) \circ C'_{jk}$$

In this last case since only $fCDA$ relations can be transformed in metric constraints the operation must be performed in a qualitative way. By means of these operations, an integrated qualitative-metric Fuzzy Spatial Constraint Network $\mathcal{N} = (V, E)$ can be defined: V is a set of points and regions and E is a set of qualitative and metric fuzzy spatial constraints between them.

5 Reasoning about space

5.1 Algorithms and complexity

Given a qualitative-metric Fuzzy Spatial Constraint Network (FSCN), the most interesting reasoning tasks are finding an optimal solution, determining the degree of consistency and finding the minimal network. The network can be modelled as an instance of the Constraint Satisfaction Problem (CSP) and solved using “generate and test” and backtracking algorithms which, however, are very inefficient (these algorithms are exponential). To improve efficiency of the backtracking algorithm it is possible to use a forward checking step to eliminate inconsistent or redundant relations. Moreover, it is possible to prune sub-trees of the search space that cannot lead to a satisfaction degree better than the current optimal one, thus obtaining eventually a Branch & Bound algorithm.

An algorithm that can be applied in the forward checking phase is Path-Consistency (PC) [Dechter et al., 1991], which is polynomial. This method is complete if the set of relations is closed under the operations of inversion, intersection and composition, as in the case of $fSQA$, and that the composition is not *weak*; the last requirement is satisfied by singleton labelled networks or by strongly convex relations [Balbiani et al., 1999].

Given a constraints network $\mathcal{N} = (V, C)$, PC method consists in replacing, for every triple of variables v_i, v_j, v_k, the constraints C_{ij} with the relation obtained by applying the relaxation

$$C_{ij} \otimes (C_{ik} \circ C_{kj}) \tag{1}$$

until a fix point is reached; if the empty relation is found the network is not consistent.

A prototype of a constraint solver for FSCNs has been implemented in Prolog using Constraint Handling Rules (CHRs) [Frühwirth, 1998], an extension of the Constraint Logic Programming (CLP) which facilitate the definition of constraint theories and algorithms to solve them. CHRs have been developed for many of the major Prolog distributions.

Relations and composition tables have been implemented as Prolog facts, which are automatically indexed by the language, while for Path Consistency CHR rules have been exploited. The intersection part of Formula (1) corresponds to the following simplification rule in CHR:

$$\begin{aligned} ctr(I, J, Rel1), ctr(I, J, Rel2) \Leftrightarrow & \\ & \text{inters}(Rel1, Rel2, Rel3) \\ | \ & ctr(I, J, Rel3) \end{aligned}$$

which means that if two relations $Rel1$ and $Rel2$ between the variables I and J match the head of the rule on the left of the symbol "$\Leftrightarrow$" and the guard predicate between the symbols "$\Leftrightarrow$" and "$|$" is satisfied, then the new relation $R3$ replaces the matched relations between the same variables.

The composition part of Formula (1) has been implemented with the following propagation rule:

$$\begin{aligned} ctr(I, J, Rel1), ctr(J, K, Rel2) \Rightarrow & \\ & I < J, K \neq I, K \neq J, \text{compos}(Rel1, Rel2, Rel3) \\ | \ & ctr(I, J, Rel3) \end{aligned}$$

this means that if two relations $Rel1$ and $Rel2$ between the variables I, K and K, J respectively match the head of the rule on the left of the symbol "$\Rightarrow$" and the guard predicates between the symbols "$\Rightarrow$" and "$|$" are satisfied, then the new relation $R3$ is added between variables I and J.

The non-determinism implicit in Prolog language has been used to automatically backtrack from inconsistent assignments during computation.

5.2 Application example

As application example, the problem of siting a nuclear power plant will be considered. In the siting of a nuclear power plant, the aim is to protect the plant against external threats as well as to minimize any environmental detriments and threats that might arise from it [AA.VV., 2000]. Other factors to be considered include: impact on land use, socio-economic impacts, traffic arrangements,

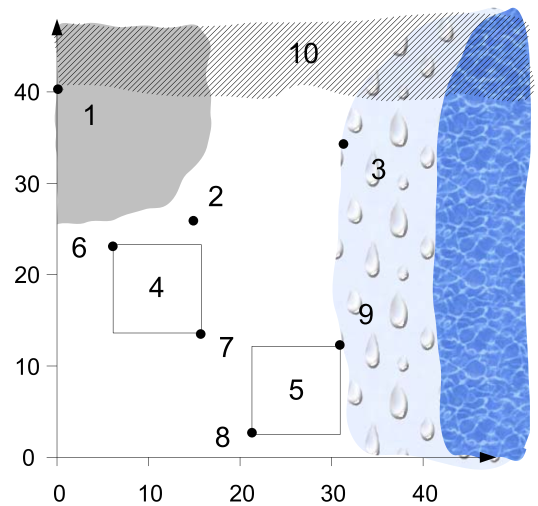

Figure 6: map for the application example.

reliable electric power transfer to the national grid and specific factors relating to the security of supply of electric power.

The problem will be simplified in order to show the expressiveness of the reasoning system without introducing too much complexity, therefore in the scenario just the following constraints will be considered:

- the plant site must be surrounded by a protective zone extending to about 5km;
- it must be at least 5km far from populated areas;
- it must not be in a seismic area;
- it must not be in an area prone to floods.

The map that will be used is shown in [Fig. 6], where Point 1 is the center of a populated area whose radius extends for about 15km and Region 10 is a seismic area. The aim is to find, if possible, two areas 4 and 5 for the plant and the nuclear waste storage, both located at least 5km away from the boundary of the populated area and from zones prone to floods; moreover, both areas must be in non-seismic locations. The total area must be enclosed by a square with sides each 30-35Km.

Starting from information given, 10 significant entities plus an origin can be identified and therefore the problem can be modelled in a graph with 11 vertices:

0. the origin for the relative coordinates;

1. the center of the populated area;
2. the maximal extension of the populated area;
3. the reference point for reasoning about zones prone to flood;
4. the first area, far from population and seismic zones;
5. the second area, far from zones prone to flood;

6-9. the boundaries of areas 4 and 5;
10. the seismic area.

There are 17 constraints that describe the scenario and that can be derived from the map; in the following the symbol "I" will indicate the disjunction of all the possible basic relations, that is indeed a constraint that does not limit anything. Some examples:

1. "*the plant site must be surrounded by a protective zone extending to about 5km*" (constraint between two points)*:*

$$6 \; \{\langle(8, 10, 10, 12), (-12, -10, -10, -8)\rangle\} \; 7$$

2. *"Area 5 must not be in a seismic zone"* (constraint between a point and a region):

$$9 \; \{(I, DC^-)\} \; 10$$

3. *"Area 5 must be at least 10km away from zones prone to floods"* (semi-unbounded constraint between two points):

$$9 \; \{\langle(8, 10, +\infty, +\infty), (8, 10, +\infty, +\infty)\rangle\} \; 3$$

4. *"The total area must be enclosed by a square with sides each 30-35Km"* (constraint between two points)*:*

$$6 \; \langle(28, 30, 35, 37), (18, 20, 25, 27)\rangle\} \; 9$$

5.3 Solving the problem

The solutions of the FSCN problem modelled above can be obtained applying a Branch & Bound algorithm, as said in Subsection 5.1. The inferred absolute coordinates are:

- $0 \; \{\langle(-\infty, -\infty, -5, 1), (-\infty, -\infty, 0, 8)\rangle\} \; 6$
- $0 \; \{\langle(-\infty, -\infty, 5, 13), (-\infty, -\infty, -10, 0)\rangle\} \; 7$
- $0 \; \{\langle(-\infty, -\infty, 15, 21), (-\infty, -\infty, 10, 24)\rangle\} \; 8$
- $0 \; \{\langle(-\infty, -\infty, 25, 29), (-\infty, -\infty, 20, 36)\rangle\} \; 9$

A possible solution is represented in [Fig. 7]; dotted rectangles represent the ranges of the areas extremes (core values).

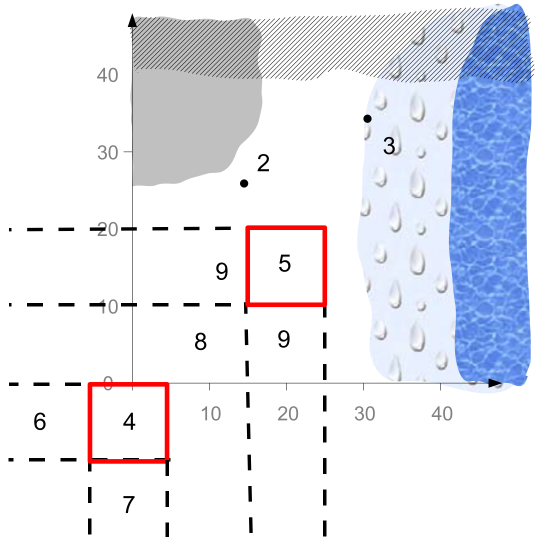

Figure 7: a solution of the considered scenario.

6 Conclusions

A general constraints satisfaction framework for Spatial Reasoning able to manage fuzzy spatial constraints involving qualitative points qualitative regions and metric points has been presented.

Rectangle Algebra and Cardinal Direction Algebra have been extended with the Fuzzy Sets Theory and a new set of 25 Point-Region relations has been defined in order to build an integrated Spatial Qualitative Algebra (SQA) which involves points and regions. Metric spatial constraints can be imposed between points and are modelled using fuzzy pyramidal possibility distributions. Metric and qualitative constraints are managed within a single constraint network and are transformed one into another when needed.

As the small application example reported at the end has shown, metric constraints allow describing more expressive scenarios while maintaining flexibility useful to take into account impreciseness and vagueness.

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