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An Approach to Generation of Decision Rules Based on a Fuzzy Semi-equivalence Relation[1]

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Abstract: Classical classification and clustering based on equivalence relations are very important tools in decision-making. An equivalence relation is usually determined by properties of objects in a given domain. When making decision, anything that can be spoken about in the subject position of a natural sentence is an object, properties of which are fundamental elements of the knowledge of the given domain. This gives the possibility of representing the concept related to a given domain. In general, the information about a set of the objects is uncertain or incomplete. Various approaches representing uncertainty of a concept were proposed. In particular, Zadeh′s fuzzy set theory and Pawlak′s rough set theory have been most influential on this research field. Zadeh characterizes uncertainty of a concept by introducing a membership function and a similarity (fuzzy equivalence) relation of a set of objects. Pawlak then characterizes uncertainty of a concept by union of some equivalence classes of an equivalence relation. As one of particular important and widely used binary relations, equivalence relation plays a fundamental role in classification, clustering, pattern recognition, polling, automata, learning, control inference and natural language understanding, etc. An equivalence relation is a binary relation with reflexivity, symmetry and transitivity. However, in many real situations, it is not sufficient to consider equivalence relations only. In fact, a lot of relations determined by the attributes of objects do not satisfy transitivity. In particular, information obtained from a domain of objects is not transitive, when we make decision based on properties of objects. Moreover, the information about symmetry of a relation is mostly uncertain. So, it is needed to approximately make decision and reasoning by indistinct concepts. This provokes us to explore a new class of relations, so-called class of fuzzy semi-equivalence relations. In this paper we introduce the notion of fuzzy semi-equivalence relations and study its properties. In particular, a constructive method of fuzzy semi-equivalence classes is presented. Applying it we present approaches to the fuzzyfication of indistinct concepts approximated by fuzzy relative and semi-equivalence classes, respectively. And an application of the fuzzy semi-equivalence relation theory to generate decision rules is outlined.

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1 Introduction

The concepts of equivalence and similarity play basic roles in many fields of pure and applied sciences, in particular, classical classification and clustering based on equivalence relations are very important tools in decision-making, since we need searching for similar behavior based on classification and/or clustering, when making decision. This includes application of many technologies, e.g. pattern recognition, diagnosis, learning, control inference and natural language understanding etc. An equivalence relation is usually determined by properties of objects in a given domain. When making decision, anything that can be spoken about in the subject position of a natural sentence is an object, properties of which are fundamental elements of the knowledge of the given domain; then concepts are more complex elements of knowledge [Orlowska and Pawlak, 1984]. This gives the possibility of representing concepts related to a given domain.

In general, the information about a set of the objects is uncertain or incomplete. We observe that, in most practical cases, the a priori data as well as the criteria, by which the performance of a making-decision system are judged, are far from being precisely specified or having accurately-known probability distributions. To cope with the analysis of man-machine systems of various types (e.g. economic systems, biological systems, social systems and political systems etc.) and to deal effectively with such systems, it is needed a radically different kind of mathematics. For representing indistinguishability of a concept, it is important that how to classify/cluster objects and to select corresponding criteria, for example, feature of objects. The binary classification into conferring and non-conferring of product/parts are often used for dealing with sources of uncertainty or imprecise condition, while feature selection is meant here to refer to the problem of dimensionality reduction of data, which initially contain a great number of features (or characters): one hopes to choice “optimal” subsets of the original features which still contain the information essential. Thus, for dealing with problems with uncertainty or incompleteness involving equivalence, similarity, clustering, preference pattern, etc. various approaches representing uncertainty of a concept were proposed, many of which are based on some extensions of classical set theory. In particular, Zadeh’s fuzzy set theory [Zadeh, 1965] and Pawlak′s rough set theory. In particular, Zadeh′s fuzzy set theory [Zadeh 1965] and Pawlak′s rough set theory [Pawlak, 1994] have been most influential on this research field. Zadeh characterized uncertainty of a concept by introducing a membership function from a set of the objects to the real interval [0,1] (possibly any algebraic construct with a partial order, e.g. a lattice, a Boolean algebra etc.) . He also studied similarity relation among objects of a set for fuzzy classification and cluster [Zadeh, 1971]. A similarity relation is essentially a generalization of the concept of an equivalence relation. Furthermore, Zadeh thought that the remarkable human capability to perform a wide variety of physical and mental tasks without any measurements and any computation is computing with words. Therefore, he presented the notion of computing with words (so-called soft computing), which plays a key role in human recognition, decision and execution processes [Zadeh, 1996; 1999]. On the other hand, Pawlak then characterizes uncertainty of a concept by union of some equivalence classes of an equivalence relation [6]. He and his followers Orlowska et al. [Orlowska and Pawlak, 1984; Orlowska, 1998] used the concept of rough set as a formal tool for modeling and processing incomplete information in information systems. As in soft computing, this also makes making-decision more plastic. It seems that there is a connection between the both theories since they all address the problem of information granulation: the theory of fuzzy sets is centered upon fuzzy information granulation, whereas rough theory is focused on crisp information granulation. In particular, the fact that the focus of rough set theory moved from the notion of indistinguishability to one of similarity shows that these two theories have become much closer to each other. Dubois and Prade [1992] are one of the first who investigated the problem of fuzzification of a rough set. Then a more general approach to this issue was proposed recently by Radzikowska and Kerre [2002] .

It is worth to point out that an equivalence relation R on a nonempty set should satisfy reflexivity (R(x,x) holds), symmetry (R(x,y)=R(y,x)) and transitivity ($R \circ R \subseteq R$). However, in many real situations, it is not sufficient to consider equivalence relations only. In fact, a lot of relations determined by the attributes of objects do not satisfy transitivity. Moreover, the information about symmetry of a relation is mostly uncertain. For example, we want to determine appropriate strategies by a friendship or neighbor relation. Clearly they are both reflexive and symmetric. Unfortunately, neither friendship nor neighbor relations among a group of persons are transitive. Similarly, in general, the strength of a fuzzy relation is not transitive yet. Thus a question arises: can we represent (characterize) indistinct concepts based on such relations. Wu Xuemou [1981] and Zhang Mingyi [1984; 1989] introduced the theory of semi-equivalence relations (i.e. a relation satisfying reflexivity and symmetry) such that solutions to the above question become possible. According to Wu Xuemou, let R be a semi-equivalence relation. Then a subset Q of G is a semi-equivalence class (w.r.t. R) if Q is a maximal subset of G such that $G^2 \subseteq R$. Zhang Mingyi gave approximations of uncertain concepts by unions of some semi-equivalence classes. This provokes us to explore a new class of relations, so-called class of fuzzy semi-equivalence relations, and study its properties. Considering fuzziness of a relation, study on fuzzification of approximations based on a semi-equivalence relation is naturally interesting from either theoretic or practical point of view.

In this paper we present the fuzzification of approximations of indistinct concepts characterized by fuzzy semi-equivalence classes. At first, we briefly recall basic notions of semi-equivalence relations. Then the concept of fuzzy semi-equivalence relation is introduced and its properties are showed. In particular, a constructive method of fuzzy semi-equivalence classes is presented. Approximations of indistinct concepts under a fuzzy semi-equivalence relation are discussed using two ways based on fuzzy relative and fuzzy semi-equivalence classes, respectively. And approaches to a set of decision rules generated by a fuzzy equivalence relation are outlined. Finally, concluding remarks, comparing of relative work and some options of further work complete the paper.

2 Semi-equivalence relation and Es-approximations

In this section, we recall notions and results on semi-equivalence relations, proofs of which are given in the appendix. At first, we give notions of a semi-equivalence relation, its relative classes and semi-equivalence classes.

Definition 2.1 [Wu Xuemou, 1981] A relation R on a nonempty set G is called a semi-equivalence relation if it is reflexive and symmetric. Let $E_S[G]=\{R \mid R$ is a semi-equivalence relation on G$\}$ and $E[G]=\{R \mid R$ is an equivalence relation on G $\}$.

Definition 2.2 [Wu Xuemou, 1981] For any $R\in E_S[G]$ and any $a\in G$, $[a]_R=\{b\in G \mid aRb\}$ is called a relative class of G w.r.t. R and the family $G_R=\{[a]_R \mid a\in G\}$ of sets is called the relative quotient of G by R. A subset Q of G with $Q^2\subseteq R$ and maximal w.r.t. inclusion relation (i.e. $Q=\max\{A\subseteq G \mid A^2\subseteq R\}$) is called a semi-equivalence class of G w.r.t. R and $G/R =\{Q \mid Q$ is a semi-equivalence class of G w.r.t. R$\}$ is called the quotient of G by R.

Important properties of semi-equivalence relations are as follows:

Theorem 2.1 [Zhang Mingyi, 1984] $(E_S[G], \cup, \cap\}$ is a complete lattice, where G^2 (complete relation) is the greatest element and I (equality relation) is the least element.

Theorem 2.2 [Zhang Mingyi, 1984] For any given $R\in E_S[G]$ and $a\in G$,

(1) if $a\in Q\in G/R$ then $Q\subseteq[a]_R$.

(2) $|G/R| \leq |G_R|$, where $|B|$ is the cardinal of B for any set B. A

(3) For any given $R\in E[G]$ ($E_S[G]$) and $Q\in G/R$, the restriction $R{\restriction}Q$ of R to Q is an (semi-) equivalence relation on Q.

(4) $\cup_{a\in G} [a]_R =G$, $\cup G/R=G$.

(5) For any $R\in E_S[G]$, $\cup_{Q\in G/R} Q^2= \cup_{a\in G} [a]_R{}^2$.

From the above properties, we can get an algorithm for computing the quotient set G/R. Note that equivalence classes are nonempty and pairwise disjoint, whereas semi-equivalence classes overlap each other possibly.

Then the (approximately) definability of a set under a semi-equivalence relation and its algebraic construction are given.

Definition 2.3 [Zhang Mingyi, 1984] For any $R\in E_S[G]$, a subset A of G is R-definable if there are $Q_i\in G/R$ ($i\in I$, I is an index set, possibly empty) such that $A=\cup\{Q_i \mid i\in I\}$. Let $Def[G]=\{A \mid A\subseteq G$ and A is R-definable$\}$. A set $Q\in G/R$ is R-selective if Q is a singleton set.

Theorem 2.3 [Zhang Mingyi, 1984] (Def [G], $\cup$) is a complete upper semi-lattice. In general, Def [G] is closed under neither the intersection nor the complement c.

Definition 2.4 [Zhang Mingyi, 1989] For any $R\in E_S[G]$ and $A\subseteq G$, we say that

(1) the set $\overline{A} =\cap\{B \mid A\subseteq B$ and $B\in Def[G]\}$ is an exterior Es-approximation of A, and $\underline{A} = \cup\{B \mid B\subseteq A$ and $B\in Def[G]\}$ is an interior Es-approximation of A;

(2) a set A is approximately R-definable if $\overline{A}\neq G$ and $\underline{A}\neq\varnothing$;

(3) a set A is internally indefinable if $\underline{A}=\varnothing$ and A is externally indefinable if $\overline{A}=G$. A is totally indefinable if $\overline{A}=G$ and $\underline{A}=\varnothing$.

Roughly speaking, R-definability gives us a possibility to answer such membership question as whether x belongs to A precisely. When R is specialized as an equivalence relation, the above definitions and results can be naturally translated into corresponding ones in rough set theory.

3 Fuzzy Semi-equivalence Relation

In this section, we introduce the notions of a fuzzy semi-equivalence relation as a generalization of a semi-equivalence relation. In addition, fuzzy Es-approximations and their some basic properties are derived. In [Zadeh, 1965], a fuzzy (binary) relation R was defined as a fuzzy collection of ordered pairs. In the following, the symbols $\vee$ and $\wedge$ stand for max and min respectively. For convenience, we identify a fuzzy relation R and its membership function μR. The concept of a similarity relation defined in [Zadeh, 1971] is essentially a generalization of the concept of an equivalence relation. More specifically,

Definition 3.1 [Zadeh, 1971] Given a nonempty universe G, call a fuzzy relation R on G (i.e. a fuzzy subset of G^2: $G^2 \rightarrow [0,1]$), a fuzzy equivalence (similarity) relation if R satisfies:

(1) reflexivity: $R(a,a)=1$ for any $a\in G$;

(2) symmetry: $R(a,b)=R(b,a)$ for any $a, b\in G$;

(3) transitivity: $R\circ R\subseteq R$ or, more explicitly, $R(x,z)\geq\vee_{y\in G}(R(x,y)\wedge R(y,z))$.

Definition 3.2 [Zadeh, 1971] The height of a fuzzy relation R , denoted by h(R) is defined by $h(R)=\vee_{x\in G\vee y\in G} R(x,y)$. A fuzzy relation is subnormal if $h(R)<1$ and normal if $h(R)=1$. For any α in [0,1], an α-level-set of a fuzzy relation R from X to Y , denoted by R_α , is a non-fuzzy set in $X\times Y$ such that $R_\alpha=\{(x,y) \mid R(x,y)\geq\alpha\}$. A subnormal non-fuzzy set, denoted by αR_α, is defined as $\alpha R_\alpha(x,y) = \alpha\bullet R_\alpha(x,y)$, where the notation $\bullet$ stands for the algebra product.

From the above definition an immediate and yet important consequence is that any fuzzy relation admits of the resolution $R=\cup_\alpha \alpha R_\alpha$, $0<\alpha\leq 1$[Zadeh, 1971]. Based on this consequence we get the following basic property of fuzzy equivalence relations:

Theorem 3.1 [Zadeh, 1971] Let $R=\cup_\alpha \alpha R_\alpha$, $0<\alpha\leq 1$, be the resolution of a similarity relation. Then each R_α is an equivalence relation on G. Conversely, if

(1) the R_α , $0<\alpha\leq 1$, is a nested sequence of distinct equivalence relations on G, with $\alpha_1>\alpha_2 \Leftrightarrow R_{\alpha_1} \subset R_{\alpha_2}$,

(2) R_1 is non-empty and $Dom(R_\alpha)=Dom(R_1)$,

then for any choice of α's in (0,1] which includes $\alpha=1$, R is a similarity relation on G.

In most real-world situations, the strength of a fuzzy relation is not transitive. So, we introduce the concept of a fuzzy semi-equivalence relation.

Definition 3.3 A fuzzy semi-equivalence relation R on G is a fuzzy relation such that R satisfies reflexivity and symmetry. A fuzzy relative class $[a]_R$ with $a\in G$ as a representative is a fuzzy subset of G defined by $[a]_R(x)=R(a,x)$ for all $x\in G$ and the fuzzy relative quotient of G by R is the family of fuzzy sets $G_R=\{[a]_R \mid a\in G\}$. A fuzzy semi-equivalence class Q is a fuzzy subset of G such that $Q=\max\{T \mid T$ is a normal fuzzy subset of G and $T^2 \subseteq R\}$, where T^2 is the algebra product:$T^2(x,y)=T(x)\bullet T(y)$. The quotient of G by R is the family of fuzzy sets $G/R=\{Q \mid Q$ is a fuzzy semi-equivalence class of G w.r.t. R$\}$.

From the above definition, it is easy to get the following result, which shows relationship between fuzzy semi-equivalence and fuzzy relative classes.

Corollary 3.2 For any $Q\in G/R$ if $Q(a)=1$ for some $a\in G$ then $Q\subseteq[a]_R$.

Proof

From $Q\in G/R$ and $Q(a)=1$, we have $Q(x)=Q^2(a,x)\leq R(a,x)=[a]_R(x)$, which means that $Q\subseteq[a]_R$.

Fuzzy relative classes and fuzzy semi-equivalence classes both are just similar classes when R is specified to be a fuzzy equivalence relation on G. Usually we wonder whether there is $a\in G$ such that $Q(a)=1$ for each $Q\in G/R$. In general, this is not true if G is infinite. Consider the following example:

Example 1 Suppose $G=\{a_i \mid i\geq 1\}$, $R(a_i,a_j)=1$ if i=j and $R(a_i,a_j)=(i/i+1)\bullet(j/j+1)$ if $i\neq j$ for i, $j\geq 1$. Clearly, R is a fuzzy semi-equivalence relation on G. We construct an element Q in G/R as follows : $Q(a_i)=i/i+1$ for any $i\geq 1$. It is easy to see that Q is normal and $Q^2\subseteq R$. In face, $\vee_i Q(a_i)=1$, $Q^2(a_i,a_i) = (i/i+1)\bullet(i/i+1)<R(a_i,a_i)$ for any $i\geq 1$ and $Q^2(a_i,a_j)=R(a_i,a_j)$ for $i\neq j$. Now we show the maximality of Q. If there is a fuzzy subset P of G such that $P^2\subseteq R$ and $Q\subset P$, then there is $k\geq 1$ such that $Q(a_k)<P(a_k)$. So, $R(a_k,a_{k+1})=Q^2(a_k,a_{k+1})=Q(a_k)\bullet Q(a_{k+1})<P(a_k)\bullet P(a_{k+1})=P^2(a_k,a_{k+1})$, which contradicts the assumption $P^2\subseteq R$. Hence $Q\in G/R$ and there is no $a\in G$ such that $Q(a)=1$.

However, a positive result holds: given $Q\in G/R$, there is $a\in G$ such that $Q(a)=1$ by the normality of Q. Furthermore, we can construct a fuzzy subset $Q\in G/R$ such that $Q(a)=1$ for any $a\in G$, that is, we have the following result:

Theorem 3.3 Suppose that R is a fuzzy semi-equivalence relation on G. For any $a\in G$ there is $Q\in G/R$ such that $Q(a)=1$. More precisely, For any $a\in G$, a fuzzy subset Q of G can be constructed as follows such that $Q\in G/R$ and $Q(a)=1$:

$Q(a_1)=1$; for $i\geq 1$, $Q(a_{i+1})=\wedge_{j\leq i}\{R(a_j,a_{i+1})/Q(a_j) \mid 0<Q(a_j)\}$,

where $\{a_i \mid i\geq 1\}$ is any enumeration of all elements in G with $a=a_1$.

Proof

It is clear that $Q(a_i)\leq 1$ for any $i\geq 1$and $Q(a_i)\leq R(a_j,a_i)/Q(a_j)$ for any j<i with $Q(a_j)>0$. Hence, $Q^2(a_i,a_i)\leq R(a_i,a_i)$ and $Q^2(a_i,a_j)=Q(a_i)\bullet Q(a_j)\leq R(a_j,a_i)=R(a_i,a_j)$. Clearly, $Q^2(a_i,a_j)\leq R(a_i,a_j)$ for any j such that $Q(a_j)=0$. Furthermore, $Q^2(a_i,a_j)\leq R(a_i,a_j)$ for any j<i. By symmetry we have $Q^2(a_i,a_j)\leq R(a_i,a_j)$ for any j:j>i. Therefore, $Q^2\subseteq R$. To prove the maximality of Q, we suppose there is a fuzzy subset P of G such that $P^2\subseteq R$ and $Q\subset P$. Then there is i>1 such that $P(a_j)=Q(a_j)$ for any $j:1\leq j<i$ and $Q(a_i)<P(a_i)$. Since $Q(a_i)=\wedge_{j<i}\{R(a_j,a_i)/Q(a_j) \mid 0<Q(a_j)\}$, we have $Q(a_i)=R(a_k,a_i)/Q(a_k)$ for some k<i. Hence $R(a_k,a_i)=Q(a_k)\bullet Q(a_i)<P(a_k)\bullet P(a_i)=P^2(a_k,a_i)$, which contradicts the assumption $P^2\subseteq R$. Thus, $Q\in G/R$ and $Q(a)=1$.

Note that, for different enumeration of elements in G, Q constructed as above is different possibly.

Example 1 (continue) Under the enumeration $\{a_i \mid i\geq 1\}$ of elements in G, we have $Q\in G/R$, where $Q(a_1)=1$ and Q(ai)=R(a1,ai) for any i>1. In general, for any given $i\geq 1$, if P is a fuzzy subset of G such that $P(a_i)=1$and $Q(a_j)=R(a_i,a_j)$ for $j\neq i$, then $P\in G/R$.

Example 2 Assume that G= {a, b, c} and the fuzzy semi-equivalence relation R on G is as follows:

	a	b	c
a	1	0.3	0.2
b	0.3	1	0.7
c	0.2	0.7	1

Clearly, R is not transitive since $R\circ R(a,c)=0.3 > 0.2=R(a,c)$. Then

GR={{(a,1), (b,0.3), (c,0.2)}, {(a,0.3), (b,1), (c,0.7)}, {(a,0.2), (b,0.7), (c,1)}

G/R= {{(a,1), (b,0.3), (c,0.2)}, {(a,0.3), (b,1), (c, 2/3)}, {(a,2/7), (b,1), (c,0.7)}, {(a,o.2), (b,0.7), (c,1)}}.

Corollary 3.4 $\cup_{a\in G}[a]_R=G$ and $\cup$ G/R=G.

Proof

By Definition 3.3 we have $(\cup_{a\in G}[a]_R)(x)=\vee_{a\in G} [a]_R(x)=[x]_R(x)=1$ for any $x\in G$. Hence, $\cup_{a\in G}[a]_R=G$. By Theorem 3.3 we have $(\cup(G/R)(x)=\vee_{Q\in G/R}Q(x)=1$for any $x\in G$. Therefore, $\cup$ G/R=G.

For a fuzzy semi-equivalence class we have the following properties, which are similar to those for a fuzzy equivalence relation.

Theorem 3.5 If R is a fuzzy semi-equivalence relation on G then Q^2 is a symmetric fuzzy relation and $Q^2\subseteq Q^2\circ Q^2$ for any $Q\in G/R$.

Proof

It is clear that $Q^2(x,y)= Q(x)\bullet Q(y)= Q(y)\bullet Q(x) =Q^2(y,x)$. And

$$
\begin{aligned}
Q^2\circ Q^2(x,y)&=\vee_{z\in G}\{Q^2(x,z)\wedge Q^2(z,y)\}\\
&=\vee_{z\in G} \{Q(x)\bullet Q(z)\wedge Q(z)\bullet Q(y)\}\\
&=\vee_{z\in G} Q(z)\bullet(Q(x)\wedge Q(y))\\
&=(Q(x)\wedge Q(y))\bullet\vee_{z\in G}Q(z) \quad \text{(since Q is normal)}\\
&= Q(x)\wedge Q(y)\\
&\geq Q(x)\bullet Q(y)\\
&= Q^2(x,y)
\end{aligned}
$$

As in [Zadeh, 1971], A_α denotes an α-level set of a fuzzy A of G by $A_\alpha =\{x \mid A(x)\geq\alpha\}$and αA_α denotes a subnormal non-fuzzy set by $\alpha A_\alpha(x)=\alpha\bullet A_\alpha(x)$,where $0<\alpha\leq 1$. By these nations, an immediate and yet interesting consequence is the following result:

Theorem 3.6 Any fuzzy relative class $[x]_R$ of x ($x\in G$) w.r.t. a fuzzy semi-equivalence relation R admits of the resolution $[x]_R =\cup_{0<\alpha\leq 1}$ α, where $[x]_R$ is the relative class of R_α (the α–level set of R), the representative of which is x. Any fuzzy semi-equivalence class Q of G w.r.t. a fuzzy semi-equivalence relation R admits of the resolution $Q=\cup_{0<\alpha\leq 1}\alpha Q_\alpha$.

Proof

We identify $[x]_{R_\alpha}$ and $\alpha[x]_{R_\alpha}$ with their membership function, i.e.

$[x]_{R_\alpha}$ (y) =1, if $y\in [x]_{R_\alpha}$ and $[x]_{R_\alpha}$ (y) =0, otherwise;

$\alpha[x]_{R_\alpha}$ (y)=α, if y∈ $[x]_{R_\alpha}$ and $\alpha[x]_{R_\alpha}$ (y)=0, otherwise.

Therefore $\cup_\alpha \alpha[x]_{R_\alpha}$ (y)=$\vee_\alpha \alpha$ $[x]_{R_\alpha}$ (y)= $\vee_{\alpha\le[x]_{R_\alpha}} \alpha = [x]_{R_\alpha}$ (y), which in turn implies $[x]_{R_\alpha} = \cup_\alpha \alpha[x]_{R_\alpha}$.

Similarly, we have $\cup_{0<\alpha\le 1}$ αQ_α(x)= $\vee_{0\prec\alpha\le 1}$ αQ_α(x)= $\vee_{\alpha\le Q_\alpha(x)}$ α = Q(x) for any x∈ G and hence Q= $\cup_{0\prec\alpha\le 1}$ αQ_α.

By the above theorem we get the following basic property of a fuzzy semi-equivalence relation, which is similar to Theorem 3.1.

Theorem 3.7 Let R= $\cup_{0\prec\alpha\le 1}$ αR_α be the resolution of a fuzzy semi-equivalence relation R on G. Then each R_α is a semi-equivalence relation on G. Conversely, if the R_α, 0<α≤1, is a nested sequence of distinct semi-equivalence relations on G, with $\alpha_1>\alpha_2 \Leftrightarrow R_{\alpha_1} \subset R_{\alpha_2}$, R_1 non-empty and Dom(R_α)=Dom(R_1), then for any choice of α's in (0,1] which includes α=1, R is a semi-equivalence relation on G.

Proof

We identify R_α and αR_α with their membership function respectively, i.e.

R_α(x,y) = 1 if (x,y)∈ R_α and R_α (x,y) = 0, otherwise;

αR_α(x,y)=α if (x,y)∈ R_α and α R_α(x,y)=0, otherwise.

"⇒" First, from R(x,x)=1 for any x∈ G, we have R_α(x,x)=1 for any x∈ G and hence R_α is reflexive for all α in (0,1]. Next, for each α∈ (0,1], let R_α(x,y)=1, which implies R(x,y)≥α and hence, by symmetry of R, that R(y,x)≥α. Consequently, R_α(y,x)=1 and R_α is symmetry.

"⇐" Since R_1 is non-empty, R_1(x,x)=1 for any x∈ Dom(R_1)=G. Noting that R(x,y) = $\vee_{0\prec\alpha\le 1}$ α R_α(x,y) for any x,y∈ G, it is clear that the symmetry of R_α for each α∈ (0,1] implies the symmetry of R. Hence, R is fuzzy semi-equivalent.

Zadeh [Zadeh, 1971] pointed out that the similarity classes of a similarity relation are not disjoint, in general. He gave a more general property (Proposition 5 of [Zadeh, 1971]) to characterize the counterpart of disjointness. We state it as follows:

Proposition [Zadeh, 1971] Let R be a similar relation in G={a_1,...a_n} characterized by a membership function R(a_i,a_j). With each a_i∈ G, we associate a similar class $[a_i]_R$ or simply $[a_i]$ characterized by $[a_i](a_j)$=R(a_i,a_j). Suppose $[a_i]$ and $[a_j]$ are arbitrary similar classes of R, the height of the intersection of $[a_i]$ and $[a_j]$ is bounded from above by R(a_i,a_j), that is h($[a_i]\cap[a_j]$)≤R(a_i,a_j).

Unfortunately, a result similar to the above proposition does not hold for the fuzzy relative classes of a fuzzy semi-equivalence relation, in general. Here we give a counterexample.

Example 2 (continue) It is clear h($[a]_R\cap [c]_R$)= R∘R(a,c)=0.3>0.2=R(a,c).

If we replace the algebra product T^2 by "min-product" T∗T (i.e. T∗T(x,y)= min{T(x),T(y)} (or T(x)∧T(y)) in Definition 3.3, then Corollaries 3.2 and 3.4 still hold. Theorem3.3 also is valid，but its proof will be anew given.

Theorem 3.3* Suppose that R is a fuzzy semi-equivalence relation on G. For any $a\in G$ there is $Q\in G/R$ (under the min-product operator$*$) such that $Q(a)=1$.

Proof

Suppose that $\{a_i \mid i\geq 1\}$ is any enumeration of all elements in G. No loss generality, let $a=a_1$. We define a fuzzy subset Q of G as follows:

$Q(a_1)=1$; for any $i\geq 1$, $Q(a_{i+1})$ is a maximal solution of the inequalities

$Q(a_{i+1})\wedge Q(a_j)\leq R(a_{i+1},a_j)$, where $j\leq i$ (*)

By induction on i, it is easy to show that there is $k\leq i$ such that $Q(a_{i+1})\wedge Q(a_k)= R(a_{i+1},a_k)$, since $Q(a_{i+1})$ is a maximal solution of inequalities (*). By the definition of Q, we have $Q(a_i)\leq 1$ for any $i\geq 1$ and $(Q*Q)(a_i,a_j)= Q(a_i)\wedge Q(a_j)\leq R(a_i,a_j)$ for any $j<i$. Since R is symmetrical, we have $(Q*Q)(a_i,a_j)\leq R(a_i,a_j)$ for any $j>i$. Hence, $Q*Q\subseteq R$.

If there is P such that $(P*P)\subseteq R$ and $Q\subset P$, then there is some $i>1$ such that $P(a_j)=Q(a_j)$ for any $j<i$ and $Q(a_i)<P(a_i)$. By the assertion proved previously, we have $Q(a_i)\wedge Q(a_k)=R(a_i,a_k)$ for some $k<i$. Therefore, $R(a_i,a_k)=Q(a_i)\wedge Q(a_k)<P(a_i)\wedge P(a_k)= (P*P)(a_i,a_k)$, which contradicts the assumption $P*P\subseteq R$. So, $Q\in G/R$ and $Q(a)=1$.

Moreover, we can get results similar to Theorems 3.6 and 3.7.

Example 1 (continue) Under the min-product $*$ and the enumeration $\{a_i \mid i\geq 1\}$ of elements in G, we also have $Q\in G/R$, where $Q(a_1)=1$ and $Q(a_i)=R(a_1,a_i)$ for any $i>1$. In general, for any given $i\geq 1$, if P is a fuzzy subset of G such that $P(a_i)=1$ and $Q(a_j)=R(a_i,a_j)$ for $j\neq i$, then $P\in G/R$.

Example 2 (continue) Under the min-product $*$, we have the quotient of G by R:

G/R={{(a,1), (b,0.3), (c,0.2)}, {(a,0.3), (b,1), (c,0.2)}, {(a,0.2), (b,1), (c,0.7)}, {(a,0.2), (b,0.7), (c,1)}}.

Different from Theorem 3.5, we have the following theorem.

Theorem 3.8 If R is a fuzzy semi-equivalence relation on G then $Q*Q$ is a symmetric and transitive fuzzy relation and $Q*Q= (Q*Q)\circ(Q*Q)$ for any $Q\in G/R$.

Proof

The symmetry of $Q*Q$ is clear. For any $x,y\in G$,

$$\begin{aligned}(Q*Q)\circ(Q*Q)(x,y) &= \vee_{z\in G}\{(Q*Q)(x,z)\wedge(Q*Q)(z,y)\}\\ &= \vee_{z\in G}\{Q(z)\wedge(Q(x)\wedge Q(y))\}\\ &= (Q(x)\wedge Q(y))\wedge(\vee_{z\in G} Q(z))\\ &= (Q*Q)(x,y) \quad (\vee_{z\in G} Q(z)=1 \text{ by the normality of } Q).\end{aligned}$$

Furthermore, the following result is obvious.

Theorem 3.9 Assume that Q is a fuzzy semi-equivalence class of G w.r.t. a fuzzy semi-equivalence relation R. For the 1-level-set Q_1 of Q, Q_1^2 (or Q_1*Q_1) is an (classical) equivalence relation on Q_1. Further, $Q_1^2=Q_1*Q_1=R\upharpoonright Q_1$.

Proof

By Definition 3.3 and the definition of 1-level-set of a fuzzy set, it is obvious that Q_1^2 (or Q_1*Q_1) is an equivalence relation on Q_1. Since $Q(x)=1$ for any $x\in Q_1$, it is clear that $Q_1^2(x,y)=Q_1*Q_1(x,y)=R\upharpoonright Q_1(x,y)$ for any $x,y\in Q_1$ and hence that $Q_1^2= Q_1*Q_1= R\upharpoonright Q_1$.

4 Approximation under fuzzy semi-equivalence relations

In this section we explore how to approximate an indistinct concept under a fuzzy semi-equivalence relation. Indistinct objects are those, by means of properties of which we are not able to distinguish them since information about them is incomplete or uncertain. In most real situations, relations representing properties of indistinct objects are fuzzy semi-equivalence relations. To deal with such situation we introduce the notion of fuzzy approximate definability of sets, which is an extension of rough sets [Orlowska and Pawlak, 1984; Dubois and Prade, 1992; Pawlak, 1994; Orlowska, 1998; Padzikowska and Kerre, 2002] and fuzzy rough sets [Wu Xuemou, 1981; Zhang Mingyi, 1984 and 1989].

Corollaries 3.2, 3.5 and theorem 3.3 (both under algebra product and min-product operator) told us that G/R is a refinement of G_R, when they are considered as fuzzy partitions. Therefore, we will give two approaches to approximate indistinct concepts, which are based on fuzzy relative classes and fuzzy semi-equivalence classes respectively.

4.1 Fuzzy relative rough approximation

For a nonempty universe G and a fuzzy semi-equivalence relation R on G, let G= $\{a_i \mid i\in I\}$ and F (G) be the class of all fuzzy subsets of G.

Definition 4.1 A fuzzy set $A\in F(G)$ is fuzzy relative rough (shortly, FRR) definable if there is $I_0\subseteq I$ such that $A=\cup\{[a_i]_R \mid i\in I_0\}$. Denote $Def_{FRR}[G]=\{A\mid A\in F(G)$ and A is FRR definable$\}$.

Clearly, the empty set and the universe G both are in $Def_{FRR}[G]$. And $[a_i]_R$ is FRR definable for each $i\in I$.

Definition 4.2 A fuzzy relative rough (shortly, FRR) approximation is a mapping Apr_{FRR}: $F(G)\to F(G)\times F(G)$ such that $Apr_{FRR}(A) =(\underline{A}_{FRR}, \overline{A}_{FRR})$ for every $A\in F(G)$, where $\underline{A}_{FRR} =\cup\{B\mid B\subseteq A$ and $B\in Def_{FRR}[G]\}$, $\overline{A}_{FRR} =\cap\{B\mid A\subseteq B$ and $B\in Def_{FRR}[G]\}$. The fuzzy set $\underline{A}_{FRR}$ (resp. $\overline{A}_{FRR}$) is called an FRR-interior (resp. FRR-exterior) approximation of A. A fuzzy set $A\in F(G)$ is interior FRR indefinable if $\underline{A}_{FRR} =\varnothing$ and A is exterior FRR indefinable if $\overline{A}_{FRR} =G$.

Definition 4.3 A pair $(D,E)\in F(G)\times F(G)$ is called an FRR set iff (D, E)= $Apr_{FRR}(A)$ for some $A\in F(G)$.

We always omit the subscripts "FRR" when it does not confuse. The following result shows an important algebraic property of FRR definable classes. .

Theorem 4.1 ($Def_{FRR}[G]$, $\cup$) is a complete upper semi-lattice, where G and $\varnothing$ are greatest and zero elements respectively, i.e. $A\cup\varnothing=A$ and $A\cup G=G$ for any $A\in Def_{FRR}[G]$.

Proof

It is easy to verify that $Def_{FRR}[G]$ is closed under union of a class of FRR definable sets and G, $\varnothing$ is its greatest and zero element, respectively.

From Theorem 4.1 and Definition 4.2 we immediately get

Corollary 4.2 $\underline{A} \in \mathrm{Def}_{FRR}[G]$.

Now we give some properties of FRR interior and exterior approximations.

Theorem 4.3 For a fuzzy semi-equivalence relation R on G and A, B$\in$F(G),

(1) A is FRR definable iff A= $\underline{A} = \overline{A}$.

(2) $\underline{A} \subseteq$ A$\subseteq$ $\overline{A}$.

(3) A$\subseteq$B $\Rightarrow \underline{A} \subseteq \underline{B}$, $\overline{A} \subseteq \overline{B}$.

(4) $\underline{\underline{A}} = \underline{A} = \overline{(\underline{A})}$, $\overline{\overline{A}} = \overline{A} = \underline{(\overline{A})}$.

(5) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $\underline{A} \bigcup \underline{B} \subseteq \underline{A \bigcup B}$.

Proof

The item 2 is clear by Definition 4.2.

(1) A is FRR-definable $\Leftrightarrow$ there is $I_0 \subseteq I$ s.t. A=$\cup\{[a_i]_R \mid i\in I_0\}$ $\Leftrightarrow$ $\underline{A}$ =A= $\overline{A}$ by Definitions 4.1 and 4.2.

(3) For any Q$\in \mathrm{Def}_{FRR}[G]$, if Q$\subseteq$A then Q$\subseteq$B. Therefore,

$\{Q \mid Q\subseteq A \text{ and } Q\in \mathrm{Def}_{FRR}[G]\}\subseteq\{Q \mid Q\subseteq B \text{ and } Q\in \mathrm{Def}_{FRR}[G]\}$.

This implies that $\underline{A} \subseteq \underline{B}$. On the other hand, for any Q$\in \mathrm{Def}_{FRR}[G]$, if B$\subseteq$Q then A$\subseteq$Q. Thus $\{Q \mid B\subseteq Q \text{ and } Q\in \mathrm{Def}_{FRR}[G]\}\subseteq\{Q \mid A\subseteq Q \text{ and } Q\in \mathrm{Def}_{FRR}[G]\}$. This shows that $\overline{A} \subseteq \overline{B}$.

(4) It is only needed to note that $\underline{A} \in$ Def[G] by Theorem 4.1. From item 1 we get $\underline{\underline{A}} = \underline{A} = \overline{(\underline{A})}$. Similarly, we have $\overline{\overline{A}} = \overline{A} = \underline{(\overline{A})}$.

(5) By item 3, we have $\overline{A}$, $\overline{B} \subseteq \overline{A \cup B}$. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \bigcup B}$. Conversely A$\subseteq$ $\overline{A}$ and B$\subseteq$ $\overline{B}$ from item 2. So, A$\cup$B$\subseteq$ $\overline{A} \cup \overline{B}$. Since $\overline{A} \cup \overline{B} \in$ Def[G] by Theorem 4.1, we get $\overline{A \bigcup B} \subseteq \overline{A} \cup \overline{B}$. Hence, $\overline{A \bigcup B} = \overline{A} \cup \overline{B}$. For the second part of the item, we have $\underline{A} \bigcup \underline{B} \subseteq \underline{A \bigcup B}$ since $\underline{A} \bigcup \underline{B} \subseteq$ A$\cup$B (by item 1) and $\underline{A} \bigcup \underline{B} \in$ Def[G].

Note that $\underline{A \bigcup B} \subseteq \underline{A} \bigcup \underline{B}$ does not hold, in general. For example, let A={(a,0.5), (b,0.5), (c,0.5)} and B={(a,0.1), (b,1), (c,1)} in Example 2. Then $\underline{A} = \underline{B}$ =$\varnothing$ but $\underline{A \bigcup B}$ = {(a,0.3), (b,1), (c,1)}.

Theorem 4.4 For any fuzzy semi-equivalence relation R on G and A$\in$F(G), B= $\underline{A}$ iff B is a maximal fuzzy subset of G such that B$\subseteq$A and B$\in$Def[G].

Proof

"IF" If B is a maximal fuzzy subset of G such that B$\subseteq$A and B$\in$Def[G], then $\underline{B} \subseteq \underline{A}$ is obvious. On the other hand, we have $\underline{A} \in$Def[G] by Corollary 4.2. So, B= $\underline{A}$ by the maximality of B.

"Only IF" It is clear from Corollary 4.2 and Definition 4.2.

4.2 Fuzzy semi-equivalence rough approximation

As we pointed out, the fuzzy semi-equivalence classes are more refined classes than fuzzy relative classes. Using fuzzy semi-equivalence classes, we can give definitions and results similar to those in the above subsection. All following results are true for quotient set G/R induced by either algebraic product or min-product operator. And their proofs can be given in a way similar to those in the preceding subsection. Here we only state corresponding notions and results.

For a nonempty universe G and a fuzzy semi-equivalence relation R, let $G/R=\{Q_i | i\in I\}$ and F(G) be the class of all fuzzy subsets of G.

Definition 4.4 A fuzzy set $A\in F(G)$ is fuzzy semi-equivalence rough (FSER) definable if there is $I_0\subseteq I$ such that $A=\cup\{Q_i | i\in I_0\}$. $Def_{FSER}[G]=\{A | A\in F(G)$ and A is FSER definable$\}$.

Definition 4.5 A FSER approximation is a mapping Apr_{FSER}: $F(G)\to F(G)\times F(G)$ such that $Apr_{FSER}(A)=(\underline{A}_{FSER}, \overline{A}_{FSER})$ for every $A\in F(G)$, where $\underline{A}_{FSER}=\cup\{B| B\subseteq A$ and $B\in Def_{FSER}[G]\}$ and $\overline{A}_{FSER}=\cap\{B | A\subseteq B$ and $B\in Def_{FSER}[G]\}$. The fuzzy set $\underline{A}_{FSER}$ (resp. $\overline{A}_{FSER}$) is called an FSER-interior (resp. FSER-exterior) approximation of A. A fuzzy set A is interior (exterior) FSER indefinable if $\underline{A}_{FSER}=\varnothing$ ($\overline{A}_{FSER}=G$).

Definition 4.6 A pair $(D,E)\in F(G)\times F(G)$ is called a FSER set if there is $A\in F(G)$ such that $(D,E)=Apr_{FSER}(A)$.

Theorem 4.5 $(Def_{FSER}[G], \cup)$ is a complete upper semi-lattice, where G and $\varnothing$ are greatest and zero elements respectively.

Corollary 4.6 $\underline{A}\in Def_{FSER}[G]$.

In the following theorem we omit the subscripts "FSER" in interior (exterior) FSER approximations and definable sets when they are confused from context.

Theorem 4.7 For a fuzzy semi-equivalence relation R on G and $A,B\in F(G)$,

(1) A is FSER definable iff $A=\underline{A}=\overline{A}$;

(2) $\underline{A}\subseteq A\subseteq\overline{A}$.

(3) $A\subseteq B\Rightarrow\underline{A}\subseteq\underline{B}$, $\quad\overline{A}\subseteq\overline{B}$.

(4) $\underline{\underline{A}}=\underline{A}=\overline{(\underline{A})}$, $\quad\overline{\overline{A}}=\overline{A}=\underline{(\overline{A})}$.

(5) $\overline{A\cup B}=\overline{A}\cup\overline{B}$, $\quad\underline{A}\bigcup\underline{B}\subseteq\underline{A\bigcup B}$.

Theorem 4.8 For any fuzzy semi-equivalence relation R on G and $A\in F(G)$, $B=\underline{A}$ iff B is a maximal fuzzy subset of G such that $B\subseteq A$ and $B\in Def[G]$.

Example 2 (continue)

Def_{FRR} [G]= GR$\cup\{\varnothing$, G, {(a,1), (b,1), (c,0.7)}, {(a,1), (b,0.7), (c,1)}, {(a,0.3), (b,1), (c,1)}};

Def_{FSER}[G]= G/R $\cup$ {$\varnothing$, G, {(a,1), (b,1), (c,2/3)}, {(a,1), (b,1), (c,0.7)}, {(a,1), (b,0.7), (c,1)}, {(a,2/7), (b,1), (c,0.7)}, {(a,0.3), (b,1), (c,1)}, {(a,2/7), (b,1), (c,1)}, (under algebraic product)

$Def_{FSER}[G]$= G/R $\cup$ {$\varnothing$, G, {(a,1), (b,1), (c,0.2)}, {(a,1), (b,1), (c,0.7)}, {(a,1), (b,0.7), (c,1)}, {(a,0.3), (b,1), (c,0.7)}, {(a,0.3), (b,1), (c,1)} (under min-product).

It is easy to verify that, in anyone of the above three cases, Def[G] is closed under neither intersection nor complement. In addition, for any non-normal fuzzy set A (i.e. A(x)<1 for all x$\in$G), we have $\underline{A}$=$\varnothing$. Let A={(a, 1), (b,0.7), (c, 0.7)}. Then $\underline{A}$= {(a,1,), (b,0.3), (c,0.2)}, $\underline{A}$ =A and $\underline{(\overline{A})}$={(a,1,), (b,0.3), (c,0.2)} (under either algebraic product or min-product).

For A={(a,0.2), (b,0.5), (c,0.5)} and B={(a,0.1), (b,1), (c,1)}, $\underline{A}$ = $\underline{B}$ =$\varnothing$ but $\underline{A \bigcup B}$= {(a,2/7), (b,1), (c,1)} (under algebraic product) or $\underline{A \bigcup B}$={(a,0.3), (b,1), (c,1)} (under min-product). This shows, in general, that = does hold in the second part of item 5 of Theorem 4.7.

5 Application

When constructing a decision system, the central problem is generating a set of decision rules and implementing approximate reasoning. The rough theory is based on approximate reasoning. It allows reducing original data and generating in automatic way the sets of decision rules. One enhancement of standard rough set classification does not well perform their task in an environment characterized by uncertain on incomplete data. In many real-word applications, we need deal with partially uncertain condition and decision attribute values in a decision system. Furthermore, relationship among these values with the same type, e.g. neighbor relation, often is not transitive. For example, in medicine, symptoms and diseases of a patient may be partially uncertain. In particular, diseases (decisions) made by doctor’s experiences are uncertain usually. This needs a flexible way to handle uncertain condition and decision attribute values in a decision system. Though rough sets can approximately represent rough concepts, it is based on an equivalence relation. So, application of fuzzy semi-equivalence relations for making decision appears to be natural and possible.

In this section, we illustrate application of fuzzy semi-equivalence relation theory in medicine by outline an approach to making decision from given case histories. This is an improvement of our original method in [Zhang Mingyi and Li Danning, 1986].

Let S=(U, C, D, V) be a decision system, where U, C, D and V are pairwise disjoint sets; U={o_1,...,0_n} is a finite set of patients or cases (objects); V={v_1,...,v_q} is a class of attribute values with different types; and C={c_1,...c_m} is a finite set of symptoms (condition) such that, for c$\in$C, c:U$\rightarrow$V(c) (V(c) is the value of symptom c) ; D ={d_1,...,d_k} is a set of diseases (decisions) such that, for d$\in$D, d:U$\rightarrow$V(d) (V(d) is the value of disease d). Values of a symptom are given by case history (an oral account) of a patient or an instrument, while values of a disease are given by a doctor. The two sets C and D characterize each object o in U, that is, for each o in U, there are two corresponding fuzzy sets c and d, which characterize the patient o.

The fuzzy-rough (-relative) sets adopt the concepts of a fuzzy semi-equivalence relation to approximately discern training instances according to some criteria. The objects x and y are indiscernible for each attribute on a subset B of attributes, if their values for each attribute in B are closed according to a certain distance. One uses the lower and upper approximations for B on X to derive certain and possible decision rules.

Our approach is outlined as follows:

Step 1. To get a fuzzy subset of U by the set of attribute values, we use canonical methods to translate values with the same type of all symptoms into a number in [0,1], i.e. we establish fuzzy subsets of $U\times C$ ($U\times D$). For each $o\in U$, c(o) and d(o) are memberships of symptom c and disease d for patient o, respectively. c(o) = 0 (=1) represents patient o has no (certainly has) symptom c. Similarly, we can understand what does d(o) mean. Then, for each $o\in U$, C(o) and D(o) are fuzzy subsets of $\{o\}\times C$ and $\{o\}\times D$, which characterize symptoms and diseases of patient o, respectively.

Step 2. A fuzzy semi-equivalence relation on U based on symptom attribute values is defined as following: given any $o_i,o_j\in U$ and $d\in D$, for each $c\in C$, $R(o_i,o_j)(c)\in[0,1]$ is determined by a distance of some type between $c_i(o)$ and $c_j(o)$, e.g.

$$R(o_i,o_j)(c)=1-\mathrm{Abs}(c_i(o)-c_j(o)) / (c_i(o)+c_j(o)).$$

Based on the set C of symptom-attribute values, we establish the similar (semi-equivalence) relation RC on U such that, for each $c\in C$, $RC(o_i,o_j)(c)= R(o_i,o_j)(c)/Q$, where o_i, $o_j \in U$, $\alpha_{(c,d)}$ is the weight of symptom c for disease d such that

$$\Sigma_{c\in C,\ d\in D}\ \alpha_{(c,d)} = 1 \text{ and } Q=\Sigma_{c\in C,\ d\in D}\ \alpha_{(c,d)}\bullet R(o_i,o_j)(c).$$

Furthermore, we construct fuzzy relative classes U_{RC} and fuzzy semi-equivalence classes U/ RC.

Step 3. Define a fuzzy semi-equivalence relation RD on D such that, for any $o_i,o_j\in U$,

$$RD(o_i,o_j)(d) =1-\mathrm{Abs}(d(o_i)-d(o_j)) / (d(o_i)+d(o_j)).$$

Using it we construct D_{RD} and D/ RD.

Step 4. Generate decision rules of form “If Then”. For each RC fuzzy relative (semi-equivalence) class, use the conjunction of symptom-attribute values of the class representative as the condition of a rule and the disjunction of lower (or upper) approximations of the class representatives under the relation RD as the decision of corresponding rule respectively. Usually, we can reduce a set of decisions by some level λ, i.e. for $D'\subseteq D$, let $D'(o,\lambda)=\{d\in D \mid d(o)\geq\lambda\}$.

Step 5. (It is unnecessary when only one of fuzzy relative and fuzzy semi-equivalence relations is used). Select an appreciate relation between fuzzy relative and fuzzy semi-equivalence relations by a lot of real applications.

Remark There are many methods for dealing with multi-criteria decision, e.g., Yager’s Ordered Weighted Averaging (OWA) operators and their variants, reduced multivariate polynomial pattern classifier et al. [Kar-Annt et al., 2004; Muller et al., 2007; Pelatz et al., 2007; Skowron et al., 2007]. They are useful for determining the weight of symptom c for disease d in Step 2. In our forthcoming paper, we will discuss it. In addition, other problems are considered. Firstly, we give a method to alter these rules by justifying or testing these generated rules using new cases (patients). Then when adding new conditions (symptoms) we use gradual approximations of indistinct concepts under fuzzy semi-equivalence relations to refine the generated rules.

As a simplified example for illustrating the above stated application, we consider the following

Example 2 (continue) G={a,b,c} is the set of patients in a given case history and each element in G is determined by its set of attribute (symptoms) values. Sometimes, for convenient, we make a patient (or a class of patients) equivalent to a disease suffered by the patient or to treatments made for the patient). R is an alike relation among patients, which is fuzzy semi-equivalent and a pair of alike degrees between their corresponding attribute values determines the alike degree between two patients. We consider the set of fuzzy semi-equivalence classes under the min-product operator, since Q*Q is transitive for each $Q\in G/R$. For any $x\in G$, Q(x) represents degrees of belongingness of x to Q. For example, Q(a)=1 means that x is certainly in Q (or a is a core of Q, i.e. the distance between a and the borders of G equals to zero), and Q(b)=0.3 represents that b is close to the borders of G with distance 1−0.3. So, treatments made for each definable set $A\in Def_{FSER}[G]$ are known by the case history. These treatments are dependent on the order of all value Q(x), that is, they are base principally on these cores and attributes with bigger alike degrees. Decision rules then can be made as follows: given a patient o, we establish a fuzzy set $A\in F(G)$ such that, for $x\in G$, A(x) is the alike degrees between patients a and x determined by their corresponding attributes.

(1) if $\underline{A}_{FSER}=\varnothing$ then the treatments for A can be only made by $\overline{A}_{FSER}$ when $\overline{A}_{FSER}\neq G$ (this means that the treatments for A contain at the most treatments for $\overline{A}_{FSER}$); otherwise, A is a new disease. For instance, if A(x)=0.8 for any $x\in G$, then $\overline{A}_{FSER}=G$ and A is a new disease.

(2) if $\underline{A}_{FSER}\neq\varnothing$, then the treatments for A contain at the least treatments for $\underline{A}_{FSER}$.

(3) if $\underline{A}_{FSER}\neq\varnothing$ and $\overline{A}_{FSER}\neq G$, then the treatments for A should range form the treatments for $\underline{A}_{FSER}$ to the treatments for $\overline{A}_{FSER}$.

6 Discussion and Further Work

In the paper, we used Vojtas' Truth-function fuzzy logic in a narrow sense [Vojtas, 2001]. Assume L is a multi-sorted predicate language with or without function symbols and restrict our declarative semantics only on Herbrand models U^A_L, a Herbrand universe of sort A--- all ground terms as crisp. Let B_L be the Herbrand base of the language L. All fuzzy predicates will be interpreted by a mapping from B_L to the unit interval [0,1]. We call f: $B_L\to[0,1]$ a fuzzy interpretation of our language. For ground atoms $p\in B_L$, f(p) is its truth value. For arbitrary formula φ and an evaluation of all sorts of variables e^A: $Var^A\to U^A_L$, the truth value f(φ)[e] is calculated along the complexity of formulas using truth functions of connectives and quantifiers:

$f(\neg\varphi)=1-f(\varphi)$.

$f(\varphi\vee\psi)=\max(f(\varphi), f(\psi))$.
$f(\varphi\wedge\psi)=\min(f(\varphi), f(\psi))$.
$f(\varphi\rightarrow\psi)=\max(1-f(\varphi), f(\psi))$.
$f(\forall x f(\varphi))=\inf\{f(\varphi)[e'] \mid e'=_x e\}$, where $e'=_x e$ means that e' can differ from e only at x.
$f(\varphi)=\inf\{f(\varphi)[e] \mid e \text{ arbitrary}\}$.

In a similar way, we can discuss corresponding classes of fuzzy semi-equivalence relations and approximations based on them, when we choose some of the following types of connectives by practice or experience:

The Lukasiewicz connectives---

$\vee L(x,y)=\min(1, x+y)$ $\wedge L(x, y)=\max(0, x+y-1)$
$\neg L(x)=1-x$ $\rightarrow L(x, y)=\min(1,1-x+y)$

The Godel intuitionist connectives---

$\vee G(x,y)=\max(x, y)$ $\wedge G(x, y)=\min(x, y)$
$\neg G(0)=1$, $\neg G(x)=0$ for $x>0$ $\rightarrow G(x, y)=y$ if $x>y$ else 1

The product logic---

$\vee P(x,y)=x+y-x\bullet y$ $\wedge P(x, y)= x\bullet y$
$\neg P(0)=1$, $\neg P(x)=0$ for $x>0$ $\rightarrow P(x, y)=\min(1, y/x)$ if $x>y$ else 1

Modeling imprecise and incomplete information is one of the main research topics in the area of knowledge representations. Fuzzy set theory and rough set theory have been most influential on this research field. Wu and Zhang considered a general class of relations, so-called semi-equivalence relation. In this paper we present a more general approach to this issue. By introducing the notions of fuzzy semi-equivalence relation, we propose a broad class of fuzzy rough sets, and define three classes of fuzzy rough sets taking into account three classes of quotients: relative quotient as well as semi-equivalence quotients under algebraic product and min-product respectively. The study of gradual approximations of indistinct concepts under fuzzy semi-equivalence relations and its application to generating decision rules as well as revision of generated rules is an option of our further work.

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Appendix

Proof of Theorem 2.1

Clearly, for any $R_i(i\geq 2)\in E_S[G]$ and any $x,y\in G$, $(x,x)\in\cup_i R_i$. If $(x,y)\in\cup_i R_i$, then there is some $R_k\subseteq\cup_i R_i$ such that $(x,y)\in R_k$. By the symmetry of R, we have $(y,x)\in R_k\subseteq\cup_i R_i$. So, $\cup_i R_i\in E_S[G]$. If $(x,y)\in\cap_i R_i$, then $(x,y)\in R_i$ for any i. Hence $(y,x)\in R_i$ for any i and $(y,x)\in\cap_i R_i$. Therefore, $\cap_i R_i\in E_S[G]$. It is easy to verify that G and I are the greatest and least elements respectively.

Proof of Theorem 2.2

(1) For any $x\in Q$, $(a,x)\in Q^2\subseteq R$ by Definition 2.2 . So, $x\in [a]_R$, which means $Q\subseteq[a]_R$.

(2) It is obvious when G is a countable infinite universe. For any finite G, we show $|G/R|\leq|G_R|$ by induction on cardinal n of G.

Base For n=1 it is clear.

Step Suppose that $|G/R|\leq|G_R|$ for n=k. We consider the case where n=k+1. Let $P=G-\{a\}$ for some $a\in G$. Then $|P|=k$ and $|P/R|\leq|P_R|=k$ by induction hypothesis. For any $Q\in G/R$, if $a\notin Q$ then $Q\subseteq P$ and $Q\in P/R$. If $a\in Q$ and $|Q|>1$, then $Q-\{a\}\in P/R$; if $Q=\{a\}$ then $Q\notin P/R$. Hence, we have $|G/R|\leq|P/R|$ or $|G/R|-1\leq|P/R|$. This implies that $|G/R|\leq|P/R|+1\leq|P_R|+1=k+1=|G_R|$.

(3) It is obvious.

(4) Since $\cup G/R\subseteq G$, it is sufficient to prove that, for any $x\in G$, $x\in\cup_{a\in G}[a]_R$. For simplicity, let $G=\{a_i | i\geq 1\}$. No loss generality, we assume $x=a_1$. Define $Q=\{b_i | i\geq 1\}\subseteq G$ as follows:

$b_1=a_1$, for $i\geq 1$,

$b_{i+1}=a_j$, where $a_j\in G-\{b_1,...b_i\}$ such that $\{b_1,..., b_i, a_j\}^2\subseteq R$.

It is easy to show that $Q\in G/R$. So, for any $x\in G$, there is $Q\in G/R$ such that $x\in Q$. Then we have $x\in\cup G/R$. Therefore $G\subseteq\cup G/R$, which implies $\cup G/R=G$. It is obvious that $\cup_{a\in G}[a]_R=G$ by Definition 2.2.

(5) From (1) and (4) we have $\cup_{a\in G}[a]_R{}^2\subseteq\cup_{Q\in G/R}Q^2$. for some $a\in Q$, if $Q\in G/R$, then we have $Q\subseteq[a]_R$ by (1). Therefore, $Q^2\subseteq\cup_{a\in G}[a]_R{}^2$. Thus, $\cup_{Q\in G/R}Q^2=\cup a\in G[a]R^2$.

Proof of Theorem 2.3

For any class of R-definable sets $\{A_i | A_i\subseteq G$ and A_i is R-definable, $i\geq 1\}$, we have that $\cup_i A_i=\cup_i\cup_j\{Q_{ij} | Q_{ij}\in R/G\}$, where $A_i=\cup_j\{Q_{ij} | Q_{ij}\in R/G, j\geq 1\}$. So, $\cup_i A_i = \cup_{i,j}\{Q_{ij} | Q_{ij}\in R/G\}$. By Definition 2.3, $\cup_i A_i\in Def[G]$.

For the assertion that Def [G] is closed under neither the intersection nor the complement c generally, we consider the following example.

Example 3. Let G be the set constituted by six people and R be a relation on G defined as follows:

R	o_1	o_2	o_3	o_4	o_5	o_6
o_1	1	0	1	1	0	0
o_2	0	1	0	1	1	1
o_3	1	0	1	0	0	0
o_4	1	1	0	1	0	1
o_5	0	1	0	0	1	0

o_6 0 1 0 1 0 1

Here, o_iRo_j shows that o_i and o_j are familiar with each other for any i, $j\in\{1, \ldots, 6\}$. And We have

$G/R= \{\{o_1, o_3\}, \{o_2, o_4, o_6\}, \{o_2, o_5\}\}$.

$GR=\{\{ o_1,o_3,o_4\},\{o_2,o_4,o_5,o_6\}, \{o_1,o_3\},\{o_1,o_2,o_4,o_6\}, \{o_2, o_5\},\{o_2, o_4, o_6\}\}$.

$\cup G/R=\cup_{a\in G}[a]_R=G$

$\cup_{Q\in G/R}$ Q2=$\cup_{a\in G}$ $[a]_R^2$.

$Def[G]=\{\{o_1,o_3\}, \{o_2,o_4,o_6\}, \{o_2,o_5\}, \{o_1,o_2,o_3,o_4,o_6\},\{o_1,o_2,o_3,o_5\}, \{o_2$,o4,o5,o6}}.

$\overline{A}$=G and A={o1,o3} for A={o1, o2, o3}.

{o2, o_4, $o_6\}\cap\{o_2, o_5\}=\{o_2\}\notin$ Def[G].

$\{o_2, o_4, o_6\}$c=$\{o_1, o_3, o_5\}\notin$ Def[G].