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Constructive Urysohn Universal Metric Space

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Abstract: We construct the Urysohn metric space in constructive setting without choice principles. The Urysohn space is a complete separable metric space which contains an isometric copy of every separable metric space, and any isometric embedding into it from a finite subspace of a separable metric space extends to the whole domain.
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Category: G.0

1 Introduction

In 1927, Pavel Samuilovich Urysohn proved that up to isometric isomorphism there exists a unique complete separable metric space $\mathbb{U}$ with the following *extension property*: for any metric space $\mathbf{X}$, any finite subspace $\mathbf{F} \subseteq \mathbf{X}$ and any isometric embedding $f : \mathbf{F} \to \mathbb{U}$ there exists an isometric embedding $\mathbf{X} \to \mathbb{U}$ which extends f. This property in particular implies *universality*: any separable metric space isometrically embeds into $\mathbb{U}$. A metric space satisfying these properties is called the *Urysohn space*, and has later found many applications of which there is a nice overview in [HN08].

Urysohn's proof [Ury27], as well as other authors' subsequent ones [Kat88, Hol92], relied on classical principles. The exception is the computable version of the Urysohn space by Hiroyasu Kamo [Kam05], but our version is in more general constructive setting[1].

In the usual constructive treatment of metric spaces [BB85], the Axiom of Countable Choice (abbr. AC_0) is assumed (in particular, it is required for completions of metric spaces by Cauchy sequences to work). However, we use it only in Lemma 20, and even that result is later generalized in choiceless environment. Thus we do not assume any choice principles in this paper at all.

The remainder of the Introduction fixes and explains the setting, definitions and notation; it also contains a few technical lemmas useful later on. Sections 2 and 3 examine the countable version of the Urysohn space, and the actual Urysohn space is then presented as its completion in Section 4.

[1] In fact, constructive results such as the one in this paper yield corresponding theorems about computability when interpreted in suitable realizability topoi, as was demonstrated by, e.g., [Bau00] and [Lie04].

1.1 Natural Numbers, Finiteness and Countability

The existence of the set of natural numbers $\mathbb{N} = \{0, 1, 2 \ldots\}$, subject to Peano axioms, is assumed. One may verify that relations $=, <, \leq$ on $\mathbb{N}$ are decidable.

For any $n \in \mathbb{N}$ we introduce the notation

$$\mathbb{N}_{<n} = \{0, 1, \ldots, n-1\} = \{k \in \mathbb{N} \mid k < n\}\,,$$

$$\mathbb{N}_{\geq n} = \{n, n+1, n+2, \ldots\} = \{k \in \mathbb{N} \mid k \geq n\}\,.$$

We call a set A *finite* when there exists a surjective map $\mathbb{N}_{<n} \to A$ for some $n \in \mathbb{N}$ — i.e. we can enumerate the elements of A with the first few natural numbers. Note that the empty set $\emptyset$ is finite by this definition since we can take $n = 0$. In fact, any finite set is either empty or inhabited; consider any surjection $\mathbb{N}_{<n} \to A$ and decide whether n equals or is greater than 0.

Whenever we have a map $a\colon N \to A$ where N is a subset of natural numbers, we write simply a_k instead of $a(k)$ for the value of a at $k \in N$. We can thus write a finite set as $A = \{a_0, a_1, \ldots, a_{n-1}\}$ if we consider the surjection $a\colon \mathbb{N}_{<n} \to A$ fixed. However, in this list some elements can potentially repeat since we only require a to be a surjection, not a bijection. Therefore, contrary to the classical intuition, for a finite set A there need not exist $n \in \mathbb{N}$ such that A would have exactly n elements (in the sense that there is a bijection between A and $\mathbb{N}_{<n}$). In fact, this happens precisely when A has decidable equality (since in that case we can remove the repetitions of elements in the list).[2]

Next, we call a set A *infinite* when there exists an injective map $\mathbb{N} \to A$. Clearly, a set which contains an infinite subset, is itself ifinite. The dual notion, the existence of a surjection $\mathbb{N} \to A$, means we have an enumeration of elements of A by natural numbers. Thus, we might be tempted to call it countability, but it turns out to be more convenient to include into definition not necessarily inhabited sets (in particular, we want the empty set $\emptyset$ to be countable). The general definition is: a set A is *countable* when there exists a surjective map $\mathbb{N} \to 1 + A$ (here, $1 = \{*\}$ denotes a singleton set while $+$ stands for binary disjoint union, i.e. binary coproduct). However, inhabitedness is the only issue here; there is a surjection $\mathbb{N} \to A$ if any only if A is both inhabited and countable.

Any finite set is countable since we have a surjection $\mathbb{N} \to 1 + \mathbb{N}_{<n}$ for every $n \in \mathbb{N}$. Any decidable subset D of a countable set C is countable since we can adjust the surjection $f\colon \mathbb{N} \to 1 + C$ to obtain the surjection $g\colon \mathbb{N} \to 1 + D$, defined by

$$g_n = \begin{cases} f_n & \text{if } f_n \in D, \\ * & \text{if } f_n \notin D. \end{cases}$$

[2] Some authors reserve the word ‘finite’ only for sets in bijection with $\mathbb{N}_{<n}$ while what we call finite they term *finitely enumerated*.

Classically, if we have both an injection and a surjection from one set to another, they must be in bijective correspondence. Constructively this does not hold in general, but one special case in which it does is rather useful.

Lemma 1. *If an infinite countable set has decidable equality, it is in bijection with* $\mathbb{N}$*.*

Proof. Easy (basically, remove the repetitions in the enumeration). □

From natural numbers one constructs the integers $\mathbb{Z}$ and rational numbers $\mathbb{Q}$ in the standard way. Both sets have decidable $=, <, \leq$; in fact they are in bijection with $\mathbb{N}$.

1.2 Real Numbers, Metric Spaces and Completion

Once we have the rationals, we may construct the set of real numbers $\mathbb{R}$ (along with its basic arithmetic operations) as Dedekind reals [TVD88, Ric08]. For $a \in \mathbb{R}$ we denote its lower cut by L_a and its upper cut by U_a, i.e. $a = (L_a, U_a)$. We define $a < b$ to mean that the intersection $U_a \cap L_b$ is inhabited (intuitively, a is less than b when there is a rational number greater than a and less than b). We define $a \leq b$ as any of the following equivalent statements: $L_a \subseteq L_b$, $U_a \supseteq U_b$, $L_a \cap U_b = \emptyset$, $\neg(b < a)$.

We denote the open and the closed intervals by

$$]a, b[= \{x \in \mathbb{R} \mid a < x < b\}, \qquad [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$

The relation $\leq$ is a partial order which makes $\mathbb{R}$ a lattice, with binary minima and maxima given as

$$\min\{a, b\} = (L_a \cap L_b, U_a \cup U_b) \qquad \text{and} \qquad \max\{a, b\} = (L_a \cup L_b, U_a \cap U_b)$$

(of course, we may then compute the minimum and maximum of any inhabited finite subset of reals).

Contrary to the case of previous number sets, none of the relations $=, <, \leq$ is decidable on $\mathbb{R}$ in general. However, we have the following lemma.

Lemma 2. *Consider the following statements for a subset* $A \subseteq \mathbb{R}$*.*

1. $<$ *is decidable on* A*.*
2. $\leq$ *is decidable on* A*.*
3. $=$ *is decidable on* A*.*

Then[3] *1.* $\Longrightarrow$ *2.* $\Longleftrightarrow$ *3.*

[3] But not always 2. $\Rightarrow$ 1. since that would imply the so-called *Markov Principle* (MP): for any sequence of digits $0, 1$, if not all terms are 0, then there is a term which is 1.

Proof. Because $\neg(b < a) \iff a \leq b \iff a = \min\{a, b\}$ and $a = b \iff a \leq b \wedge b \leq a$. □

The *absolute value* on reals is defined as $|a| = \max\{a, -a\}$. The following holds for all $a, b \in \mathbb{R}$: $|a| \geq 0$, $|-a| = |a|$, $|a| + |b| \geq |a + b|$, $|a| = 0 \iff a = 0$.

Once we have the real numbers, we can define metric spaces. We will start with a more general notion, and make a slight alteration to the standard definition because it will be later important for us to know over what values the distances may range.

Definition 3. Given a subset $A \subseteq \mathbb{R}$, an *A-valued pseudometric space* (or just an *A-pseudometric space*) $\mathbf{X} = (X, d_X)$ is a set X together with the map (*A-valued pseudometric*) $d_X \colon X \times X \to A$, subject to the following conditions for all $x, y, z \in X$.

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$ (symmetry)
- $d(x, y) + d(y, z) \geq d(x, z)$ (triangle inequality)

The value $d(x, y)$ is called the *distance* between x and y. From the above conditions it follows that distances are always nonnegative. If in addition the following condition

- $d(x, y) = 0 \implies x = y$ (nondegeneracy)

is satisfied for all $x, y \in X$, we call (X, d_X) an *A-valued metric space* (*A-metric space*) while d is an *A-valued metric.*

An $\mathbb{R}$-valued metric space is just the standard metric space. We call a $\mathbb{Q}$-valued metric space a *rational metric space.*

The *Euclidean metric* $d_E(a, b) = |a - b|$ makes the real numbers into a metric space. We will automatically consider any subset of $\mathbb{R}$ to be equipped with the (restriction of) Euclidean metric.

An *open ball* in the pseudometric space $\mathbf{X} = (X, d_X)$ with *center* $x \in X$ and *radius* $r > 0$ is the set $B(x, r) = \{y \in X \mid d_X(x, y) < r\}$. Note that every ball is inhabited since it contains its center.

A subspace $\mathbf{D}$ of a pseudometric space $\mathbf{X}$ is called *dense* in $\mathbf{X}$ when every open ball in $\mathbf{X}$ has an inhabited intersection with $\mathbf{D}$ (for example, $\mathbb{Q}$ is dense in $\mathbb{R}$). A pseudometric space is *separable* when it has a countable dense subspace.

Definition 4. A map $f \colon \mathbf{X} \to \mathbf{Y}$ between pseudometric spaces $\mathbf{X} = (X, d_X)$ and $\mathbf{Y} = (Y, d_Y)$ is an *isometry* when it preserves distances, i.e. $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$.

If the isometry is injective, we call it *isometric embedding*. Note that any isometry from a metric space is automatically an embedding because of nondegeneracy: $f(x) = f(y) \implies d_Y(f(x), f(y)) = 0 \implies d_X(x, y) = 0 \implies x = y$. If we have a fixed isometric embedding $\mathbf{X} \to \mathbf{Y}$, we can consider $\mathbf{X}$ as a (pseudo)metric subspace of $\mathbf{Y}$.

A bijective isometry is called an *isometric isomorphism*. Spaces which are isometrically isomorphic share all (pseudo)metric properties. If we have a fixed isomorphism $\mathbf{X} \to \mathbf{Y}$, we can for all purposes consider $\mathbf{X}$ and $\mathbf{Y}$ to be the same space.

A *dense isometry* is the isometry which has a dense image in the codomain.

If an isometry is defined only on some subspace of the domain, we call it a *partial isometry*. If this subspace is finite, we call the isometry also *finite*. If however the isometry is defined on the whole domain, we call it *total*. Thus we obtain the following concise definition of the Urysohn space.

Definition 5. A *Urysohn metric space* is any complete separable metric space, such that any finite partial isometry into it extends to a total isometry.

We still need to agree on the notion of completeness, however. Classically a space is complete when its every Cauchy sequence converges. Constructively there are two difficulties with this.

First of all, we need a *modulus of convergence* which describes how much of the sequence must we take to obtain the desired proximity. A Cauchy sequence with a given modulus of convergence is called a *regular Cauchy sequence*. Common choices for the modulus of convergence of the sequence a in the space $\mathbf{X} = (X, d_X)$ are $d_X(a_m, a_n) \leq \frac{1}{m} + \frac{1}{n}$ (a bit unfortunate choice if we start counting natural numbers with 0) and $d_X(a_m, a_n) \leq 2^{-\min\{m,n\}}$; the latter can quickly be generalized by allowing a constant factor in front of the power of 2 (we use this version in the proof of Lemma 20; it basically just means we start counting from some later term of the sequence), or even allowing an arbitrary geometric sequence. All these choices yield equivalent definitions of completion.

Different sequences can converge to the same point, so we pronounce two Cauchy sequences a, b *equivalent* when their terms near according to the modulus of convergence, for example $|a_m - b_n| \leq 2^{-\min\{m,n\}}$ for all $m, n \in \mathbb{N}$. For a space $\mathbf{X} = (X, d_X)$ we define its *Cauchy completion* $\widehat{\mathbf{X}} = (\widehat{X}, d_{\widehat{X}})$ as follows: $\widehat{X}$ is the set of all equivalence classes of regular Cauchy sequences, and the distance between classes $[a], [b]$ is the limit of distances $d_X(a_n, b_n)$ (this turns out to be a well-defined metric). Even though completion is usually defined for metric spaces, it works equally well for pseudometric ones. Notice however, that the completion of any pseudometric space is always a metric space since the equivalence relation identifies the Cauchy sequences which would be at zero distance.

There is a natural dense isometry from a space to its completion, namely the one which ascribes to x the equivalence class of the constant Cauchy sequence with all terms x. We call a space *Cauchy complete* when this isometry is surjective. Since isometries between metric spaces are injective, we can consider any metric space as a subspace of its completion, and a metric space is complete if and only if the isometry into its completion is in fact an isometric isomorphism.

We now come to the second difficulty. As it is well known, completions of spaces by Cauchy sequences are not appropriate in a setting without countable choice because we might not be able to conclude that a completion of a space is complete[4]. The theory of completion appropriate for the choiceless environment is Richman's completion [Ric08] where we allow an inhabited set of approximations at each stage, not merely a single term (for details, see Subsection 4.1). In the presence of countable choice, both versions of completion yield isomorphic spaces[5].

Whichever theory of completion we use (whether Cauchy completion with AC_0, or Richman's completion), we have the following standard lemma.

Lemma 6. *Every uniformly continuous map* $f\colon \mathbf{X} \to \mathbf{Y}$ *where* $\mathbf{X}$ *is a metric space and* $\mathbf{Y}$ *is a complete metric space, uniquely extends to a uniformly continuous map* $\bar{f}\colon \widehat{\mathbf{X}} \to \mathbf{Y}$*. If* f *is an isometry, then so is* $\bar{f}$*.*

Corollary 7. *Let* $\mathbf{X}$ *be a metric space,* $\mathbf{D}$ *a dense metric subspace of* $\mathbf{X}$*, and* $\mathbf{Y}$ *a complete metric space. Then any isometry* $\mathbf{D}$ *uniquely extends to an isometry* $\mathbf{X} \to \mathbf{Y}$*.*

Proof. Notice that $\mathbf{X}$ and $\mathbf{D}$ have the same completion, so we can first extend the isometry to $\widehat{\mathbf{D}} \to \mathbf{Y}$ and then restrict it to $\mathbf{X} \to \mathbf{Y}$. □

We finish this subsection with an observation that any pseudometric space can be made into a metric space by identifying points at zero distance. Explicitly, if $\mathbf{X} = (X, d_X)$ is a pseudometric space, define $x \sim y \iff d_X(x, y) = 0$ for all $x, y \in X$. One may quickly verify that $\sim$ is an equivalence relation, so define $\widetilde{X}$ to be the set of all equivalence classes. For classes $[x], [y] \in \widetilde{X}$, let $d_{\widetilde{X}}([x], [y]) = d_X(x, y)$. It turns out that this is a well-defined metric on $\widetilde{X}$. Call the metric space $\widetilde{\mathbf{X}} = (\widetilde{X}, d_{\widetilde{X}})$ the *Kolmogorov quotient*[6] of $\mathbf{X}$.

We have the following facts about the Kolmogorov quotient.

- The natural quotient map is an isometry.

[4] This happens already in the simple (but crucial) case. The Cauchy completion of the rationals is the set of so-called *Cauchy reals* $\mathbb{R}_c$. In general, $\mathbb{R}_c$ is only a subspace of the Dedekind reals $\mathbb{R}$, but in the presence of AC_0 these two sets are isomorphic.

[5] In particular, Richman's completion of rationals yields Dedekind reals.

[6] In general topology, the Kolmogorov quotient of a topological space is constructed by identifying points which have the same neighbourhoods, thus obtaining a T_0 space. In pseudometric spaces, points with same neighbourhoods are precisely those at zero distance.

- The completion is in fact a Kolmogorov quotient of the pseudometric space of the regular Cauchy sequences in the case of Cauchy completion, and regular families in the case of Richman's completion.
- A pseudometric space and its Kolmogorov quotient have the same completion.

1.3 Choice of Distance Values

Notation 8 Throughout the paper, let $\mathbb{D}$ denote a countable dense additive subgroup of $\mathbb{R}$ with decidable equality. Here:

- *additive subgroup* of $\mathbb{R}$ means $0 \in \mathbb{D}$, $-a \in \mathbb{D}$ and $a + b \in \mathbb{D}$ for all $a, b \in \mathbb{D}$;
- *dense* means $\mathbb{D}$ intersects every inhabited open interval (i.e. $\mathbb{D}$ is dense in $(\mathbb{R}, d_E)$ in the metric sense).

Furthermore, let

- $\mathbb{D}_{\geq 0} = \{a \in \mathbb{D} \mid a \geq 0\}$,
- $\mathbb{D}^+ = \{a \in \mathbb{D}_{\geq 0} \mid a \neq 0\} = \{a \in \mathbb{D} \mid a \not\leq 0\} = \{a \in \mathbb{D} \mid \neg\neg(a > 0)\}$,
- $\mathbb{D}_{>0} = \{a \in \mathbb{D} \mid a > 0\}$.

Also, fix an element $\delta_{\mathbb{D}} \in \mathbb{D} \cap \,]0, \infty[$ (it exists because of density of $\mathbb{D}$).

The reason for this notation is because we will mostly work with metric spaces taking values in $\mathbb{D}$. Typical practical choices for $\mathbb{D}$ are rational numbers, dyadic rational numbers and (real) algebraic numbers.

Some remarks regarding $\mathbb{D}, \mathbb{D}_{\geq 0}, \mathbb{D}^+, \mathbb{D}_{>0}$.

- Clearly, $\delta_{\mathbb{D}} \in \mathbb{D}_{>0} \subseteq \mathbb{D}^+ \subseteq \mathbb{D}_{\geq 0} \subseteq \mathbb{D}$; in particular, the sets $\mathbb{D}_{>0}, \mathbb{D}^+, \mathbb{D}_{\geq 0}, \mathbb{D}$ are all inhabited. In fact, they are infinite since we have the map $\mathbb{N} \to \mathbb{D}_{>0}$, $n \mapsto (n+1) \cdot \delta_{\mathbb{D}}$.
- Decidability of $=$ implies decidability of $\leq$ by Lemma 2, and since decidable subsets of a countable set are countable, we see that by Lemma 1 the sets $\mathbb{D}^+, \mathbb{D}_{\geq 0}, \mathbb{D}$ are in bijection with $\mathbb{N}$.
- Also because of decidable $\leq$, the sets $\mathbb{D}_{>0}, \mathbb{D}^+, \mathbb{D}_{\geq 0}, \mathbb{D}$ are sublattices of $\mathbb{R}$ (meaning they are closed for minima and maxima). Consequently, $\mathbb{D}$ is also closed for taking absolute value (actually, so are $\mathbb{D}_{\geq 0}, \mathbb{D}^+, \mathbb{D}_{>0}$, though for a trivial reason).
- $\mathbb{D}$ has decidable $<$ if and only if $\mathbb{D}^+ = \mathbb{D}_{>0}$. We do not assume this condition since we do not need it, though it holds in practical cases.

– Restriction of the Euclidean metric makes $\mathbb{D}$ (and therefore also $\mathbb{D}_{\geq 0}$, $\mathbb{D}^+$, $\mathbb{D}_{>0}$) into a $\mathbb{D}$-valued metric space.

– Because of nondegeneracy of the metric, any $\mathbb{D}$-valued metric space must have decidable equality.

Contrary to finite spaces, general countable spaces need not be either empty or inhabited. We will find it rather useful to be able to, without loss of generality, assume inhabitedness of a space, so we prepare ourselves the following lemma.

Lemma 9.

– *Every countable (resp. separable) metric space isometrically embeds into an inhabited countable (resp. separable) metric space.*

– *If the countable metric space is $\mathbb{D}$-valued, it isometrically embeds into an inhabited $\mathbb{D}$-valued countable metric space.*

Proof. Let $\mathbf{C} = (C, d_C)$ be a countable space and $c \colon \mathbb{N} \to 1 + C$ a surjection. Define the map $d \colon (1+C) \times (1+C) \to \mathbb{R}$ as follows. Let $d(*,*) = 0$, and for any $a, b \in C$, let $d(a,b) = d_C(a,b)$. Finally, if a is any element of C, let $n \in \mathbb{N}$ be any index such that $a = c_n$, and let e be the first element in the list $c_0, c_1, \ldots, c_n$ which is in C. Define $d(*,a) = d(a,*) = \delta_{\mathbb{D}} + d_C(e,a)$.

The map d is a well-defined metric on the inhabited countable set $1 + C$. We verify only the relevant cases of the triangle inequality. Let $a, b, c \in C$. Then

$$d(a,b) + d(b,c) = d_C(a,b) + d_C(b,c) \geq d_C(a,c) = d(a,c),$$

$$d(*,a) + d(a,b) = \delta_{\mathbb{D}} + d_C(e,a) + d_C(a,b) \geq \delta_{\mathbb{D}} + d_C(e,b) = d(*,b),$$

$$d(a,*) + d(*,b) =$$

$$= \delta_{\mathbb{D}} + d_C(e,a) + \delta_{\mathbb{D}} + d_C(e,b) \geq d_C(a,e) + d_C(e,b) \geq d_C(a,b) = d(a,b).$$

The natural inclusion $C \to 1 + C$ is an isometric embedding. Clearly, if d_C is $\mathbb{D}$-valued, then so is d.

If $\mathbf{S}$ is a separable metric space with the countable subspace $\mathbf{C} = (C, d_C)$, then embed $\mathbf{C}$ into $(1 + C, d)$ as above, and continue into the completion of $(1 + C, d)$. By Corollary 7, this map extends to an isometry of the whole space $\mathbf{S}$. □

2 Countable Urysohn Space

The original idea for the construction of Urysohn metric space was to construct a countable rational Urysohn space, and then complete it. By a *countable rational Urysohn space* we mean a countable rational metric space, such that any finite partial isometry into it from another countable rational metric space extends to a total isometry. In particular, the countable rational Urysohn space is universal in the sense that any countable rational space isometrically embeds into it.

The constructive version of the countable rational Urysohn space is not new. In [HN08], the authors present what they call a *finite presentation* of it. Although they actually work in classical setting, minor adjustments[7] to their definitions and proofs make their arguments constructive.

We shall not follow their approach for several reasons, however. First of all, there is a far more straightforward way to build the rational Urysohn space which we present in this section. Secondly, their construction suffers from the same deficiency as ours will (see the beginning and the end of Section 3), but ours will be easier to rectify. Also, when constructing the countable Urysohn space, it is not truly important that the distances are rational. The crucial properties for distance values are those required for $\mathbb{D}$ given in Notation 8, and we will build a general countable $\mathbb{D}$-valued Urysohn metric space.

2.1 Construction

Extending a finite partial isometry to the whole countable space can be done inductively; in other words, it is sufficient to be able to extend the isometry one point at a time. We therefore begin with the following basic construction: adding points to a metric space, such that any finite isometry into this space extends by one point into the extended space. More precisely, for any given $\mathbb{D}$-valued metric space $\mathbf{X} = (X, d_X)$ and a number $n \in \mathbb{N}$, we would like a $\mathbb{D}$-valued metric space $E_n(\mathbf{X})$ and an isometric embedding $e_n^{\mathbf{X}} \colon \mathbf{X} \hookrightarrow E_n(\mathbf{X})$, such that for any given $\mathbb{D}$-valued metric space $\bar{\mathbf{F}} = (\{p_0, p_1, \ldots, p_{n-1}, p_n\}, d_{\bar{F}})$, any isometry $f \colon \mathbf{F} \to \mathbf{X}$ from the metric subspace $\mathbf{F} = (\{p_0, p_1, \ldots, p_{n-1}\}, d_F)$ of $\bar{\mathbf{F}}$ extends to an isometry $\bar{f} \colon \bar{\mathbf{F}} \to E_n(\mathbf{X})$ as in the diagram below.

$$\begin{tikzcd}
\mathbf{F} \arrow[r, "f"] \arrow[d, hook, "\subseteq"] & \mathbf{X} \arrow[d, hook, "e_n^{\mathbf{X}}"] \\
\bar{\mathbf{F}} \arrow[r, "\bar{f}"'] & E_n(\mathbf{X})
\end{tikzcd}$$

The idea is simply to add points with all possible prescribed distance from n (not necessarily distinct) points of X. More specifically, our new points will be tuples of the form $(x_0, d_0, x_1, d_1, \ldots, x_{n-1}, d_{n-1}) = (x_i, d_i)_{i \in \mathbb{N}_{<n}}$ which represents a

[7] As elements of the rational Urysohn space they take a certain kind of finite partially ordered rational metric spaces — what they call *(complete) triplets*. The crucial adjustment to make their presentation work constructively is to require the partial order of the triplets to be decidable.

point that is at distance d_i from the point x_i for all $i \in \mathbb{N}_{<n}$. Triangle inequalities give constraints when a point at such given distances can actually exist. It turns out that all relevant triangle inequalities can be encoded by $|d_i - d_j| \leq d(x_i, x_j) \leq d_i + d_j$ for all $i, j \in \mathbb{N}_{<n}$; here d denotes a distance which is already defined on points x_i. When this condition holds, we call the tuple $(x_i, d_i)_{i \in \mathbb{N}_{<n}}$ *permissible.*

Construction 10 *We are given a* $\mathbb{D}$*-valued metric space* $\mathbf{X} = (X, d_X)$ *and a natural number* $n \geq 1$*. We construct a new* $\mathbb{D}$*-valued metric space* $E_n(\mathbf{X})$ *and an isometric embedding* $e_n^{\mathbf{X}} \colon \mathbf{X} \hookrightarrow E_n(\mathbf{X})$*.*

Define

$$A_n(\mathbf{X}) = \Big\{ (x_i, d_i)_{i \in \mathbb{N}_{<n}} \in (X \times \mathbb{D}^+)^n \;\Big|\; \text{tuple } (x_i, d_i)_{i \in \mathbb{N}_{<n}} \text{ is permissible} \Big\}$$

and

$$E_n(\mathbf{X}) = \Big(X + A_n(\mathbf{X}), d \Big)$$

where for $x, y \in X$ and $a = (x_i, a_i)_{i \in \mathbb{N}_{<n}}, b = (y_j, b_j)_{j \in \mathbb{N}_{<n}} \in A_n(\mathbf{X})$:

- $d(x, y) \;=\; d_X(x, y),$
- $d(x, a) \;=\; d(a, x) = \min \{a_i + d_X(x_i, x) \mid i \in \mathbb{N}_{<n}\},$
- $d(a, b) \;=\; \begin{cases} \min \{a_i + d_X(x_i, y_j) + b_j \mid i, j \in \mathbb{N}_{<n}\} & \text{if } a \neq b, \\ 0 & \text{if } a = b. \end{cases}$

Observe the distance is well defined since both $\mathbb{D}^+$ and X (as a $\mathbb{D}$-valued metric space) have decidable equality.

$E_n(\mathbf{X})$ is a $\mathbb{D}$-metric space. Verifying the conditions is straightforward; here are some relevant cases of triangle inequality. Let $x, y \in X$ and $a = (x_i, a_i)_{i \in \mathbb{N}_{<n}}$, $b = (y_j, b_j)_{j \in \mathbb{N}_{<n}}$, $c = (z_k, c_k)_{k \in \mathbb{N}_{<n}} \in P_n(\mathbf{X})$. Assume a, b, c are pairwise distinct since if any two points appearing in the triangle inequality are equal, then the inequality surely holds.

$$\begin{gathered} d(a, x) + d(x, y) = \min \{a_i + d_X(x_i, x) \mid i \in \mathbb{N}_{<n}\} + d_X(x, y) = \\ = \min \{a_i + d_X(x_i, x) + d_X(x, y) \mid i \in \mathbb{N}_{<n}\} \geq \\ \geq \min \{a_i + d_X(x_i, y) \mid i \in \mathbb{N}_{<n}\} = d(a, y) \end{gathered}$$

$$\begin{gathered} d(x, a) + d(a, y) = \\ = \min \{a_i + d_X(x_i, x) \mid i \in \mathbb{N}_{<n}\} + \min \{a_j + d_X(x_j, y) \mid j \in \mathbb{N}_{<n}\} = \\ = \min \{a_i + d_X(x_i, x) + a_j + d_X(x_j, y) \mid i, j \in \mathbb{N}_{<n}\} \geq \\ \geq \min \{d_X(x_i, x) + d_X(x_i, x_j) + d_X(x_j, y) \mid i, j \in \mathbb{N}_{<n}\} \geq d(x, y) \end{gathered}$$

$$\begin{aligned}
&d(a,b)+d(b,x)=\\
&=\min\{a_i+d_X(x_i,y_j)+b_j \mid i,j\in\mathbb{N}_{<n}\}+\min\{b_k+d_X(y_k,x)\mid k\in\mathbb{N}_{<n}\}=\\
&=\min\{a_i+d_X(x_i,y_j)+b_j+b_k+d_X(y_k,x)\mid i,j,k\in\mathbb{N}_{<n}\}\geq\\
&\geq\min\{a_i+d_X(x_i,y_j)+d_X(y_j,y_k)+d_X(y_k,x)\mid i,j,k\in\mathbb{N}_{<n}\}\geq\\
&\geq\min\{a_i+d_X(x_i,x)\mid i\in\mathbb{N}_{<n}\}=d(a,x)
\end{aligned}$$

$$\begin{aligned}
&d(a,x)+d(x,b)=\\
&=\min\{a_i+d_X(x_i,x)\mid i\in\mathbb{N}_{<n}\}+\min\{b_j+d_X(y_j,x)\mid j\in\mathbb{N}_{<n}\}=\\
&=\min\{a_i+d_X(x_i,x)+b_j+d_X(y_j,x)\mid i,j\in\mathbb{N}_{<n}\}\geq\\
&\geq\min\{a_i+b_j+d_X(y_j,x_i)\mid i,j\in\mathbb{N}_{<n}\}=d(a,b)
\end{aligned}$$

$$\begin{aligned}
&d(a,b)+d(b,c)=\min\{a_i+d_X(x_i,y_j)+b_j\mid i,j\in\mathbb{N}_{<n}\}+\\
&+\min\{b_k+d_X(y_k,z_l)+c_l\mid k,l\in\mathbb{N}_{<n}\}=\\
&=\min\{a_i+d_X(x_i,y_j)+b_j+b_k+d_X(y_k,z_l)+c_l\mid i,j,k,l\in\mathbb{N}_{<n}\}\geq\\
&\geq\min\{a_i+d_X(x_i,y_j)+d_X(y_j,y_k)+d_X(y_k,z_l)+c_l\mid i,j,k,l\in\mathbb{N}_{<n}\}\geq\\
&\geq\min\{a_i+d_X(x_i,z_l)+c_l\mid i,l\in\mathbb{N}_{<n}\}=d(a,c)
\end{aligned}$$

Finally, for the isometric embedding $e_n^{\mathbf{X}}\colon \mathbf{X}\hookrightarrow E_n(\mathbf{X})$, take the natural inclusion $X\hookrightarrow X+A_n(\mathbf{X})$. □

It should be obvious that the above defined $E_n(\mathbf{X})$ solves the given problem. Indeed, if $p_n=p_i$ for some $i\in\mathbb{N}_{<n}$, define $\bar{f}(p_n)=f(p_i)$, and otherwise define $\bar{f}(p_n)=\big(f(p_i),d_{\bar{F}}(p_n,p_i)\big)_{i\in\mathbb{N}_{<n}}$. This works because it is surely a permissible tuple, and we furthermore have $d((x_i,d_i)_{i\in\mathbb{N}_{<n}},x_j)=d_j$ for all $j\in\mathbb{N}_{<n}$ and permissible tuples $(x_i,d_i)_{i\in\mathbb{N}_{<n}}\in A_n(\mathbf{X})$ since $a_j+d_X(x_j,x_j)=a_j$ and $a_j-a_i\leq d_X(x_i,x_j)$.

The above choice of the metric is not the only possible one, and indeed later (in Subsection 3.2) we consider alternative constructions. We have the following constraints when defining the metric d: it has to extend d_X, and we require $d((x_i,d_i)_{i\in\mathbb{N}_{<n}},x_j)=d_j$. For other distances we actually have much leeway, though triangle inequalities provide upper and lower bounds. The above construction presents the simplest choice by taking the maximal allowable distances.

Lemma 11. *Given a $\mathbb{D}$-valued metric space $\mathbf{X}=(X,d_X)$ and a surjection $f\colon\mathbb{N}\to X$, we may for any $n\in\mathbb{N}_{\geq1}$ construct a bijection between $\mathbb{N}$ and the underlying set of $E_n(\mathbf{X})$ in a canonical way.*

Proof. Fix a bijection $g\colon \mathbb{N} \to \mathbb{D}^+$. Define the surjection $h\colon \mathbb{N}^{2n} \to A_n(\mathbf{X})$ by

$$h\Big((a_i, b_i)_{i \in \mathbb{N}_{<n}}\Big) = \begin{cases} \Big(f(a_i), g(b_i)\Big)_{i \in \mathbb{N}_{<n}} & \text{if } \Big(f(a_i), g(b_i)\Big)_{i \in \mathbb{N}_{<n}} \text{ permissible,} \\ (f(0), \delta_{\mathbb{D}})_{i \in \mathbb{N}_{<n}} & \text{otherwise} \end{cases}$$

(note that the permissibility condition is decidable because $\mathbb{D}$ has decidable $\leq$). There is a canonical choice of a bijection $\mathbb{N} \xrightarrow{\cong} \mathbb{N} + \mathbb{N}^{2n}$, so the composition

$$\mathbb{N} \xrightarrow{\cong} \mathbb{N} + \mathbb{N}^{2n} \xrightarrow{f+h} X + A_n(\mathbf{X})$$

is a surjection. There is also an injection $\mathbb{N} \to X + A_n(\mathbf{X})$ which maps $m \in \mathbb{N}$ to $(f(0), g(m))_{i \in \mathbb{N}_{<n}}$. Since $E_n(\mathbf{X})$ has decidable equality as a $\mathbb{D}$-valued metric space, Lemma 1 now supplies us with the desired bijection. $\square$

We define inductively the following sequence of $\mathbb{D}$-valued metric spaces. Let $\mathbb{U}_0^{\mathbb{D}} = 1$ with its only possible metric. Let $\mathbb{U}_n^{\mathbb{D}} = E_n(\mathbb{U}_{n-1}^{\mathbb{D}})$ for $n \in \mathbb{N}_{\geq 1}$.

Lemma 12.

1. *All $\mathbb{U}_n^{\mathbb{D}}$ are inhabited countable spaces. In fact, there is a sequence of surjective maps $(e_n\colon \mathbb{N} \to \mathbb{U}_n^{\mathbb{D}})_{n \in \mathbb{N}}$, such that for all $n \geq 1$ the maps e_n are bijections.*

2. *There exists an increasing sequence of natural numbers $(a_n)_{n \in \mathbb{N}}$ and a sequence of injective maps $(j_n\colon \mathbb{U}_n^{\mathbb{D}} \to \mathbb{D}^{a_n})_{n \in \mathbb{N}}$, such that for all $n \in \mathbb{N}$, the map j_{n+1} is an extension of j_n in the sense that the following diagram commutes.*

$$\begin{tikzcd} \mathbb{U}_n^{\mathbb{D}} \arrow[r, "j_n"] \arrow[d, hook] & \mathbb{D}^{a_n} \arrow[d, hook] \\ \mathbb{U}_{n+1}^{\mathbb{D}} \arrow[r, "j_{n+1}"'] & \mathbb{D}^{a_{n+1}} \end{tikzcd}$$

   Here we consider $\mathbb{D}^{a_n}$ included into $\mathbb{D}^{a_{n+1}}$ in the standard way, i.e. $\mathbb{D}^{a_n} \cong \mathbb{D}^{a_n} \times \{0\}^{a_{n+1}-a_n} \subseteq \mathbb{D}^{a_{n+1}}$.

Proof. 1. By induction. Let e_0 be the unique map $\mathbb{N} \to \mathbb{U}_0^{\mathbb{D}}$. Now suppose the conditions are satisfied for $n \in \mathbb{N}$. Since we have a surjection $e_n\colon \mathbb{N} \to \mathbb{U}_n^{\mathbb{D}}$, Lemma 11 supplies us with the bijection $e_{n+1}\colon \mathbb{N} \to \mathbb{U}_{n+1}^{\mathbb{D}}$.

2. Let $a_0 = 0$ and j_0 the unique map between singletons $\mathbb{U}_0^{\mathbb{D}}$, $\mathbb{D}^0$. Assume now that we already have a_{n-1} and j_{n-1}. Define $k_n\colon A_n(\mathbb{U}_{n-1}^{\mathbb{D}}) \to \mathbb{D}^{n(1+a_{n-1})}$ as the composition

$$A_n(\mathbb{U}_{n-1}^{\mathbb{D}}) \hookrightarrow (\mathbb{U}_{n-1}^{\mathbb{D}} \times \mathbb{D}^+)^n \xrightarrow{\cong} (\mathbb{U}_{n-1}^{\mathbb{D}})^n \times (\mathbb{D}^+)^n \hookrightarrow (\mathbb{D}^{a_{n-1}})^n \times \mathbb{D}^n = \mathbb{D}^{n(1+a_{n-1})}$$

(clearly, k_n is an injection). Let $a_n = n(1+a_{n-1})+1$ and define $j_n \colon \mathbb{U}^{\mathbb{D}}_{n-1} + A_n(\mathbb{U}^{\mathbb{D}}_{n-1}) \to \mathbb{D}^{a_n}$ by the diagram below.

$$\begin{tikzcd}
\mathbb{U}^{\mathbb{D}}_{n-1} \arrow[r, "j_{n-1}"] \arrow[d, hook] & \mathbb{D}^{a_{n-1}} \arrow[r, "\cong"] & \mathbb{D}^{a_{n-1}} \times \{0\}^{a_{n+1}-a_n} \arrow[d, hook] \\
\mathbb{U}^{\mathbb{D}}_{n-1} + A_n(\mathbb{U}^{\mathbb{D}}_{n-1}) \arrow[rr, dotted, "j_n"] & & \mathbb{D}^{a_n} \\
A_n(\mathbb{U}^{\mathbb{D}}_{n-1}) \arrow[u, hook] \arrow[r, "k_n"'] & \mathbb{D}^{n(1+a_{n-1})} \arrow[r, "\cong"'] & \mathbb{D}^{n(1+a_{n-1})} \times \{\delta_{\mathbb{D}}\} \arrow[u, hook]
\end{tikzcd}$$

Since $\delta_{\mathbb{D}} \neq 0$, j_n is an injection. □

We have the diagram of inclusions $\mathbb{U}^{\mathbb{D}}_0 \hookrightarrow \mathbb{U}^{\mathbb{D}}_1 \hookrightarrow \mathbb{U}^{\mathbb{D}}_2 \hookrightarrow \mathbb{U}^{\mathbb{D}}_3 \hookrightarrow \ldots$; we define $\mathbb{U}^{\mathbb{D}}$ to be its direct limit (= colimit).

Proposition 13.

1. *The colimit $\mathbb{U}^{\mathbb{D}}$ exists as a set, and has decidable equality.*
2. *There is a bijection between $\mathbb{N}$ and $\mathbb{U}^{\mathbb{D}}$.*
3. *The set $\mathbb{U}^{\mathbb{D}}$ is also the coproduct $1 + A_1(\mathbb{U}^{\mathbb{D}}_0) + A_2(\mathbb{U}^{\mathbb{D}}_1) + A_3(\mathbb{U}^{\mathbb{D}}_2) + \ldots$*

Proof. 1. Denote $\mathbb{D}^\infty = \left\{(a_n)_{n\in\mathbb{N}} \in \mathbb{D}^{\mathbb{N}} \;\middle|\; \exists\, m \in \mathbb{N} \,.\, \forall\, n \in \mathbb{N}_{\geq m} \,.\, (a_n = 0)\right\}$ and let $c_n \colon \mathbb{D}^n \to \mathbb{D}^\infty$ be the standard inclusions which append the zero sequence at the end of finite sequences. Recall notation from Lemma 12(2) and its proof. For all $n \in \mathbb{N}$ define $i_n \colon \mathbb{U}^{\mathbb{D}}_n \to \mathbb{D}^\infty$ as $i_n = c_{a_n} \circ j_n$. The union of images of maps i_n is the desired colimit. It has decidable equality because it is a subset of $\mathbb{D}^\infty$ which has decidable equality.

2. The set $\mathbb{U}^{\mathbb{D}}$ is infinite because by Lemma 12(1) already the set $\mathbb{U}^{\mathbb{D}}_1$ is infinite. It is countable because $\mathbb{N}\times\mathbb{N}$ is, and by the same lemma we have a surjection $(i_n \circ e_n)_{n\in\mathbb{N}} \colon \mathbb{N} \times \mathbb{N} \to \mathbb{U}^{\mathbb{D}}$. Now use Lemma 1.

3. Obvious.

□

For technical reasons we constructed the colimit $\mathbb{U}^{\mathbb{D}}$ as a subset of $\mathbb{D}^\infty$, but in practice we consider $\mathbb{U}^{\mathbb{D}}$ simply as the union of the increasing sequence of sets $\mathbb{U}^{\mathbb{D}}_n$. In particular, we have the following proposition.

Proposition 14.

1. *There is a metric* $d_{\mathbb{U}^{\mathbb{D}}} \colon \mathbb{U}^{\mathbb{D}} \times \mathbb{U}^{\mathbb{D}} \to \mathbb{D}$ *which is the extension of the metrics on spaces* $\mathbb{U}^{\mathbb{D}}_n$*, and makes* $\mathbb{U}^{\mathbb{D}}$ *into a* $\mathbb{D}$*-valued metric space which is the colimit of* $\mathbb{D}$*-metric spaces* $\mathbb{U}^{\mathbb{D}}_n$*.*
2. *There is a map* $age \colon \mathbb{U}^{\mathbb{D}} \to \mathbb{N}$ *which maps an element* $x \in \mathbb{U}^{\mathbb{D}}$ *to the index* $n \in \mathbb{N}$*, such that* x *first appears in* $\mathbb{U}^{\mathbb{D}}_n$*.*

Proof. 1. Obvious.

2. Use Proposition 13(3).

□

2.2 Urysohn Properties

We now proceed to show that $\mathbb{U}^{\mathbb{D}}$ is the countable $\mathbb{D}$-valued Urysohn metric space.

Lemma 15. *Let* $n \in \mathbb{N}$*. If we are given a finite* $\mathbb{D}$*-valued metric space* $\bar{\mathbf{F}} = (\{p_0, p_1, \ldots, p_{n-1}, p_n\}, d_{\bar{F}})$*, its finite subspace* $\mathbf{F} = (\{p_0, p_1, \ldots, p_{n-1}\}, d_F)$ *and an isometry* $f \colon \mathbf{F} \to \mathbb{U}^{\mathbb{D}}$*, then there exists a canonical choice of an isometry* $\bar{f} \colon \bar{\mathbf{F}} \to \mathbb{U}^{\mathbb{D}}$ *which extends* f*.*

Proof. If $n = 0$, let $\bar{f}(p_0)$ be the only element in $\mathbb{U}^{\mathbb{D}}_0 \subseteq \mathbb{U}^{\mathbb{D}}$. Now assume $n \geq 1$. If p_n is equal to any element of $\{p_0, p_1, \ldots, p_{n-1}\}$ (recall that $\bar{\mathbf{F}}$, as a $\mathbb{D}$-metric space, has decidable equality), we have $\bar{\mathbf{F}} = \mathbf{F}$ and $\bar{f} = f$. Finally, if p_n differs from all previous p_i-s, let

$$m = 1 + \max\Big\{age(f(p_0)), age(f(p_1)), \ldots, age(f(p_{n-1}))\Big\};$$

then $f(p_0), f(p_1), \ldots, f(p_{n-1}) \in \mathbb{U}^{\mathbb{D}}_{nm-1}$. Define the extension as

$$\bar{f}(p_n) = \Big(f(p_{i \bmod n}), d_{\bar{F}}(p_{i \bmod n}, p_n)\Big)_{i \in \mathbb{N}_{<nm}} \in \mathbb{U}^{\mathbb{D}}_{nm} \subseteq \mathbb{U}^{\mathbb{D}}.$$

□

Theorem 16 (Extension). *Let* $\mathbf{C} = (C, d_C)$ *be a countable* $\mathbb{D}$*-valued metric space,* $\mathbf{F} = (F, d_F)$ *a finite subspace of* $\mathbf{C}$*, and* $f \colon \mathbf{F} \to \mathbb{U}^{\mathbb{D}}$ *an isometry. Then there exists (a canonical choice of) an isometry* $\bar{f} \colon \mathbf{C} \to \mathbb{U}^{\mathbb{D}}$ *which extends* f*.*

Proof. Observe that by Lemma 9 we may without loss of generality assume C is inhabited. Permute the elements of C as necessary to obtain a surjection $e \colon \mathbb{N} \to C$, such that $F = \{e_0, \ldots, e_{n-1}\}$ for some $n \in \mathbb{N}$. Use induction and Lemma 15 to obtain the desired isometry. □

Corollary 17 (Universality). *Every countable* $\mathbb{D}$*-valued metric space isometrically embeds into* $\mathbb{U}^{\mathbb{D}}$*.*

Proof. In Theorem 16, take $F = \emptyset$ and for f the unique map $\emptyset \to \mathbb{U}^{\mathbb{D}}$. $\square$

Finally, we verify uniqueness of the countable Urysohn space using the standard back-and-forth method.

Proposition 18 (Uniqueness). *Up to isometric isomorphism there exists a unique countable $\mathbb{D}$-valued metric space with the property that any finite partial isometry from a countable $\mathbb{D}$-metric space into it extends to a total isometry in the canonical way.*

Proof. Any metric space with the above properties must have decidable equality (because it is $\mathbb{D}$-valued) and must be infinite (since there are infinite countable $\mathbb{D}$-metric spaces, for example $\mathbb{D}$ itself, which by Corollary 17 embed into it). By Lemma 1, such a space must be in bijection with $\mathbb{N}$.

So let $\mathbb{U}'$, $\mathbb{U}''$ be spaces with the given properties, and let $b' \colon \mathbb{N} \to \mathbb{U}'$, $b'' \colon \mathbb{N} \to \mathbb{U}''$ be bijections. We define maps $f \colon \mathbb{U}' \to \mathbb{U}''$, $g \colon \mathbb{U}'' \to \mathbb{U}'$ (which are to be mutually inverse isometries) inductively as follows (by a slight abuse of notation we will use the same letters for sets different at every stage, much like as if we wrote a computer program).

Start with sets $F = G = \emptyset$. At each step these will be decidable finite subsets of $\mathbb{U}'$ and $\mathbb{U}''$ respectively, and will measure where f and g have already been defined; in particular we will have $f(F) = G$, $g(G) = F$. One inductive step consists of the following.

- Let $i \in \mathbb{N}$ be the first number for which $b'(i) \notin F$. Extend the isometry $f \colon F \to \mathbb{U}''$ to the isometry $f \colon F \cup \{b'(i)\} \to \mathbb{U}''$. Let the new F be the previous $F \cup \{b'(i)\}$, and let $a = f(b'(i))$. Extend $g \colon G \to \mathbb{U}'$ to an isometry $g \colon G \cup \{a\} \to \mathbb{U}'$ (notice that $a \notin G$) by defining $g(a) = b'(i)$. Let the new G be the previous $G \cup \{a\}$.
- Do the same with the roles f and g reversed: pick the smallest number $j \in \mathbb{N}$ for which $b''(j) \notin G$, extend g to an isometry $G \cup \{b''(j)\} \to \mathbb{U}'$, extend f as an inverse, and enlarge F and G accordingly.

Since b', b'' are (in particular) surjections, the maps f and g are total. By construction, they are mutually inverse isometries, so they are isometric isomorphisms between $\mathbb{U}'$ and $\mathbb{U}''$. $\square$

3 Examination of Countable Urysohn Space

3.1 Towards Real Urysohn Space

Can we now construct the real Urysohn space as the completion of $\mathbb{U}^{\mathbb{D}}$? Following the example of the countable case, the crucial first step is the adaptation of Lemma 15 for real distances. The obvious idea how we could do it goes as follows: approximate the points in the completion by points in $\mathbb{U}^{\mathbb{D}}$, approximate the distances with values in $\mathbb{D}^+$, and use Lemma 15 for the approximations. Presumably, we obtain an approximation of the new point in the completion of $\mathbb{U}^{\mathbb{D}}$ which extends the given finite isometry.

Unfortunately this does not work. In fact, the extensions of the approximations almost never approximate a single point. The reason for this is because in Construction 10 we take the maximal distances possible, even if the differences between numbers are small. For example, the distance between $(*, \delta_{\mathbb{D}}), (*, \delta_{\mathbb{D}} + \epsilon) \in \mathbb{U}^{\mathbb{D}}_1$ tends to $2\delta_{\mathbb{D}}$ instead of 0 when $\epsilon \in \mathbb{D}^+$ tends to 0.

By its nature, the (countable) Urysohn space must possess infinitely many points at a given positive distance from any particular point. In order for the above idea to work, we need to choose extensions in such a way that small differences in numbers yield nearby extensions. We succeed in this in Construction 24.

Nevertheless, the above construction of $\mathbb{U}^{\mathbb{D}}$ is not useless. First of all, it will serve as basis for later constructions. Second, if we are satisfied with less than the full real Urysohn space, it still works. For example, it presents a very direct and simple construction of the countable Urysohn space, and as we are about to see, we can still salvage universality with a trick and the Axiom of Countable Choice.

Proposition 19. *Any metric space* $\mathbf{X}$ *which has a countable dense* $\mathbb{D}$*-valued subspace* $\mathbf{C}$ *isometrically embeds into the completion of* $\mathbb{U}^{\mathbb{D}}$*.*

Proof. Embed $\mathbf{C}$ into $\mathbb{U}^{\mathbb{D}}$ by Corollary 17, and extend the embedding to the isometry $\mathbf{X} \to \widehat{\mathbb{U}^{\mathbb{D}}}$ by Corollary 7. □

In practice this usually already suffices since we can take $\mathbb{D}$ to be the set of real algebraic numbers, and the metric spaces we most often deal with, say, in analysis, have countable dense subspaces with algebraic distances. However, if we assume countable choice, then the following lemma together with the previous proposition enables us to embed general separable metric spaces into $\widehat{\mathbb{U}^{\mathbb{D}}}$.

Lemma 20. *Assume* AC_0 *and suppose* $2^{-n} \in \mathbb{D}$ *for all* $n \in \mathbb{N}$*, as well as that the inequality* $<$ *on* $\mathbb{D}$ *is decidable*[8]*. Then any separable metric space isometrically embeds into a metric space with a* $\mathbb{D}$*-valued dense metric subspace in bijection with* $\mathbb{N}$*.*

Proof. Let $\mathbf{X} = (X, d_X)$ be a separable metric space with the countable dense subspace $\mathbf{C} = (C, d_C)$. By Lemma 9 we may without loss of generality assume

[8] These are not strong additional assumptions on $\mathbb{D}$, and they hold in practical cases anyway. Actually, the claim of the lemma holds in general, without any added conditions (this follows from Corollary 30), but the proof of it is a deal less direct.

that X and C are inhabited, and there is a surjection $c\colon \mathbb{N} \to C$. We will construct a metric d on the set $\mathbb{N} \times \mathbb{D}^+$ (which is in bijection with $\mathbb{N}$); intuitively, the elements $(n, a) \in \mathbb{N} \times \mathbb{D}^+$ will approximate c_n, and the smaller the "level" a, the better the approximation, so $\mathbf{C}$ will isometrically embed into the completion of $(\mathbb{N} \times \mathbb{D}^+, d)$ "at level 0".

Define the *lexicographic order* on $\mathbb{N} \times \mathbb{D}^+$ by

$$(n, a) \prec (m, b) \iff n < m \text{ or } (n = m \text{ and } a < b).$$

Observe that for all $(n, a), (m, b) \in \mathbb{N} \times \mathbb{D}^+$, exactly one of the following cases holds:

$$(n, a) = (m, b) \quad \text{or} \quad (n, a) \prec (m, b) \quad \text{or} \quad (m, b) \prec (n, a).$$

To simplify notation, define the sum between $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ as $x + A = \{x + a \mid a \in A\}$; in particular, for intervals this means $x + \,]a, b[\, = \,]x + a, x + b[$.

For any $(n, a), (m, b) \in \mathbb{N} \times \mathbb{D}^+$, define $d\big((n, a), (m, b)\big)$ to be

- 0 if $(n, a) = (m, b)$;
- some element of the set
$$|a - b| + \Big(d_C(c_n, c_m) + \,]\max\{a, b\}, a + b[\Big) \cap \mathbb{D}$$
if $(n, a) \prec (m, b)$ (it exists because $\max\{a, b\} < a + b$ and because of the density condition for $\mathbb{D}$);
- the already defined $d\big((m, b), (n, a)\big)$ if $(m, b) \prec (n, a)$.

By countable choice (since the set of pairs $(n, a) \prec (m, b)$ is in bijection with $\mathbb{N}$) this defines a map $d\colon (\mathbb{N} \times \mathbb{D}^+) \times (\mathbb{N} \times \mathbb{D}^+) \to \mathbb{D}$.

We verify the triangle inequality for d. Assume $(n, a), (m, b), (l, c) \in \mathbb{N} \times \mathbb{D}^+$ are pairwise distinct since otherwise the triangle inequality surely holds for them.

$$d\big((n, a), (m, b)\big) + d\big((m, b), (l, c)\big) \geq$$
$$\geq |a - b| + d_C(c_n, c_m) + \max\{a, b\} + |b - c| + d_C(c_m, c_l) + \max\{b, c\} \geq$$
$$\geq |a - c| + d_C(c_n, c_l) + a + c \geq d\big((n, a), (l, c)\big)$$

The other conditions for $(\mathbb{N} \times \mathbb{D}^+, d)$ to be a $\mathbb{D}$-valued metric are obvious. Let $\mathbf{N} = (N, d_N)$ be its Cauchy completion.

We now define the map $e\colon X \to N$. Take any $x \in X$. By countable choice there exists a map $u\colon \mathbb{D}^+ \to \mathbb{N}$, such that $d_X(x, c_{u(a)}) \leq a$ for all $a \in \mathbb{D}^+$. Define $e(x) = \lim_{n\to\infty}(u(2^{-n}), 2^{-n})$.

Take any $x, y \in X$ and maps $u, v\colon \mathbb{D}^+ \to \mathbb{N}$, such that $d_X(x, c_{u(a)}) \leq a$ and $d_X(y, c_{v(a)}) \leq a$ for all $a \in \mathbb{D}^+$. Then for any $i, j \in \mathbb{N}$,

$$d\Big((u(2^{-i}), 2^{-i}), (v(2^{-j}), 2^{-j})\Big) \leq |2^{-i} - 2^{-j}| + d_C(c_{u(2^{-i})}, c_{v(2^{-j})}) + 2^{-i} + 2^{-j} =$$

$$= 2 \cdot 2^{-\min\{i,j\}} + d_C(c_{u(2^{-i})}, c_{v(2^{-j})}) \leq$$
$$\leq 2 \cdot 2^{-\min\{i,j\}} + d_X(c_{u(2^{-i})}, x) + d_X(x, y) + d_X(y, c_{v(2^{-j})}) \leq$$
$$\leq 2 \cdot 2^{-\min\{i,j\}} + 2^{-i} + d_X(x, y) + 2^{-j} \leq 4 \cdot 2^{-\min\{i,j\}} + d_X(x, y)$$

and

$$d\Big((u(2^{-i}), 2^{-i}), (v(2^{-j}), 2^{-j})\Big) \geq d_C(c_{u(2^{-i})}, c_{v(2^{-j})}) \geq$$
$$\geq d_X(x, y) - d_X(x, c_{u(2^{-i})}) - d_X(c_{v(2^{-j})}, y) \geq$$
$$\geq d_X(x, y) - 2^{-i} - 2^{-j} \geq d_X(x, y) - 2 \cdot 2^{-\min\{i,j\}}.$$

Together:

$$d_X(x, y) - 2^{1-\min\{i,j\}} \leq d\Big((u(2^{-i}), 2^{-i}), (v(2^{-j}), 2^{-j})\Big) \leq d_X(x, y) + 2^{2-\min\{i,j\}}.$$

The special case $x = y$, $u = v$ of this inequality shows $(u(2^{-n}), 2^{-n})_{n \in \mathbb{N}}$ is a regular Cauchy sequence in $(\mathbb{N} \times \mathbb{D}^+, d)$, and so its limit $e(x)$ is a well-defined element of N. Taking just $x = y$ shows that $e(x)$ does not depend on the choice of the map u. In full generality, the above inequality implies e is an isometric embedding. □

3.2 Alternative Constructions

There are a lot of possible variations of how to construct a countable Urysohn space which differ by their intricacy and usefulness in particular situations. We examine some of them here, though we no longer bother to explicitly prove the existence of their underlying sets; with similar tricks as in Lemma 12 and Proposition 13 they can all be realized as certain subsets of $\mathbb{D}^\infty$. Of course, by Proposition 18, any of these variations yield the same space up to isometric isomorphism.

First, some minor adjustments. One might dislike the fact that we apply E_n-s successively since that means we never add points with n prescribed distances to points that appear later then in $\mathbb{U}^{\mathbb{D}}_{n-1}$, because of which we had to repeat distances in the proof of Lemma 15. An alternative construction of the countable Urysohn space which solves this is to first apply E_1, then E_1, E_2, then E_1, E_2, E_3 and so forth (instead of just E_1, then E_2, then E_3 etc.).

There is also no need to take $\mathbb{U}^{\mathbb{D}}_0 = \mathbb{1}$. One may verify that we obtain a countable Urysohn space in the end if we start with any inhabited countable $\mathbb{D}$-valued metric space; we merely chose the simplest one.

Under stronger assumptions, one might consider building the real Urysohn space directly, without completing a countable space. An attentive reader might have noticed that the only reason why we needed countability of $\mathbb{D}$ was to have $\mathbb{U}^{\mathbb{D}}$ countable, but otherwise the construction still works. So if we afford ourselves decidable equality on real numbers, we can build $\mathbb{U}^{\mathbb{R}}$. Even though this space satisfies the statement of Lemma 15 for general real distances, its problem is that it is not complete (in spite of that every $\mathbb{U}_n^{\mathbb{R}}$ is) — for the same reason why $\mathbb{R}^\infty = \left\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \exists\, m \in \mathbb{N} \,.\, \forall\, n \in \mathbb{N}_{\geq m} \,.\, (x_n = 0)\right\}$ is not (consider for example the Cauchy sequence $a_0 = * \in \mathbb{U}_0^{\mathbb{R}}$, $a_n = (a_{n-1}, 2^{-n})_{i\in\mathbb{N}_{<n}} \in \mathbb{U}_n^{\mathbb{R}}$). Consequently, while we can embed countable metric spaces into $\mathbb{U}^{\mathbb{R}}$, we cannot directly extend isometries to separable spaces unless we also complete $\mathbb{U}^{\mathbb{R}}$. And if we do consider the completion, without the study what additional points we have added, we might have lost the extension property. We will not go this way, however. Our interest in this paper is in constructivism, and constructively the decidable equality on the reals is a very strong assumption[9]. There is a constructive way, however, to avoid the necessity of decidable equality on $\mathbb{D}$.

Lemma 21. *If $\mathbf{X} = (X, d_X)$ is a countable $\mathbb{D}$-valued pseudometric space which satisfies the extension property, then its Kolmogorov quotient $\widetilde{\mathbf{X}} = (\widetilde{X}, d_{\widetilde{X}})$ is a Urysohn space.*

Proof. If X is countable, then so is $\widetilde{X}$ since it is its image by the Kolmogorov quotient map $q\colon X \to \widetilde{X}$. Furthermore, if F is finite with decidable equality, any map $F \to \widetilde{X}$ lifts along q to a map $F \to X$ (because q is surjective). Extend this map to a total isometry into X and compose it with q to obtain the isometry into $\widetilde{X}$. □

Construction 22 *Let $A \subseteq \mathbb{R}$ be an additive subgroup and a sublattice of $\mathbb{R}$. We are given an A-valued pseudometric space $\mathbf{X} = (X, d_X)$ and a natural number $n \geq 1$. We construct a new A-valued pseudometric space $\overline{E_n}(\mathbf{X})$ and an isometry $\overline{e_n}^{\mathbf{X}}\colon \mathbf{X} \hookrightarrow \overline{E_n}(\mathbf{X})$.*

Define

$$\overline{E_n}(\mathbf{X}) = \Big(\{(x_i, d_i)_{i\in\mathbb{N}_{<n}} \in (X \times \mathbb{D}_{\geq 0})^n \mid \text{tuple } (x_i, d_i)_{i\in\mathbb{N}_{<n}} \text{ is permissible}\}\,, d\Big)$$

where d is defined, for $a = (x_i, a_i)_{i\in\mathbb{N}_{<n}}, b = (y_j, b_j)_{j\in\mathbb{N}_{<n}} \in \overline{E_n}(\mathbf{X})$, as

$$\begin{aligned} d(a, b) \;=\; \min\{\; & \min\{a_i + d_X(x_i, y_j) + b_j \mid i, j \in \mathbb{N}_{<n}\}\,, \\ & \max\{d_X(x_i, y_i) + |a_i - b_i| \mid i \in \mathbb{N}_{<n}\}\;\}. \end{aligned}$$

Observe that $\overline{E_n}(\mathbf{X})$ is a $\mathbb{D}$-pseudometric space. The verification of triangle inequality is a similar (albeit lengthier) exercise as in Construction 10; we supply only the following worthwhile detail.

$$\min\{a_i + d_X(x_i, z_k) + c_k \mid i, k \in \mathbb{N}_{<n}\} - \max\{d_X(y_j, z_j) + |b_j - c_j| \mid j \in \mathbb{N}_{<n}\} =$$

$$= \min\{a_i + d_X(x_i, z_k) + c_k - d_X(y_j, z_j) - |b_j - c_j| \mid i, j, k \in \mathbb{N}_{<n}\} \leq$$

[9] It implies the nonconstructive Lesser Principle of Omniscience (LPO): every sequence of natural numbers is either constantly zero or it contains a nonzero term.

$$\leq \min\{a_i + d_X(x_i, z_j) - d_X(y_j, z_j) + c_j - |b_j - c_j| \mid i, j \in \mathbb{N}_{<n}\} \leq$$
$$\leq \min\{a_i + d_X(x_i, y_j) + b_j \mid i, j \in \mathbb{N}_{<n}\}$$

For the isometry $\overline{e_n}^{\mathbf{X}} : \mathbf{X} \hookrightarrow \overline{E_n}(\mathbf{X})$, take the map $\overline{e_n}^{\mathbf{X}}(x) = (x, 0)_{i \in \mathbb{N}_{<n}}$. $\square$

The pseudometric in the above construction is an adaptation of the one given in Construction 10, with an added term which abolishes the need for separating cases.

Define now $\overline{\mathbb{U}^A}_0 = \mathbf{1}$ with its only possible (pseudo)metric, and inductively $\overline{\mathbb{U}^A}_n = \overline{E_n}(\overline{\mathbb{U}^A}_{n-1})$ for $n \in \mathbb{N}_{\geq 1}$. Let $\overline{\mathbb{U}^A}_\infty$ be the colimit, and finally, let $\overline{\mathbb{U}^A} = \widetilde{\overline{\mathbb{U}^A}_\infty}$ be its Kolmogorov quotient. At this point we mention a result which will be useful later on.

Lemma 23. *Let $A, B \subseteq \mathbb{R}$ be two additive subgroups and sublattices of $\mathbb{R}$, and let $A \subseteq B$.*

1. *For every $n \in \mathbb{N}$ we have $\overline{\mathbb{U}^A}_n \subseteq \overline{\mathbb{U}^B}_n$, as well as $\overline{\mathbb{U}^A}_\infty \subseteq \overline{\mathbb{U}^B}_\infty$ and $\overline{\mathbb{U}^A} \subseteq \overline{\mathbb{U}^B}$. All these inclusions are isometric embeddings.*
2. *If in addition A is dense in B (in the Euclidean metric), then these inclusions are also dense.*

Proof. 1. Obvious.

2. By induction on n for $\overline{\mathbb{U}^A}_n \subseteq \overline{\mathbb{U}^B}_n$, and then the density for the colimit and the Kolmogorov quotient follow.

$\square$

If we take $A = \mathbb{D}$, then $\overline{\mathbb{U}^{\mathbb{D}}}$ is another construction of the countable Urysohn space — verify the conditions for $\overline{\mathbb{U}^{\mathbb{D}}}_\infty$, and then apply Lemma 21. Even though this construction is more complicated than the original one, it has some nicer properties. To begin with, the structure is closer to what we might expect for our desired space — for example, while there is a natural bijection between $\mathbb{D}_{\geq 0}$ and $\mathbb{U}^{\mathbb{D}}_1$, it is not an isometry; but there is a natural isometric isomorphism between $\mathbb{D}_{\geq 0}$ and $\overline{\mathbb{U}^{\mathbb{D}}}_1$. And second, even though we do not obtain a countable Urysohn space in general, we can at least in principle use an arbitrary subgroup and sublattice A in the construction. Thus, we can build $\overline{\mathbb{U}^{\mathbb{R}}}$ with no problem this time; and while this is still not the real Urysohn space (since it is not complete), it will come in handy for Proposition 27.

This construction also brings us closer to the solution of the problem that extensions from similar points with similar distances are not close; but it is not yet good enough since preservation of closeness is limited to a particular level $\overline{\mathbb{U}^{\mathbb{D}}}_n$. Thus we consider yet another construction of the countable Urysohn space.

Construction 24 *We construct a set* $V^{\mathbb{D}}$*, a* $\mathbb{D}$*-valued pseudometric* $d\colon V^{\mathbb{D}} \times V^{\mathbb{D}} \to \mathbb{D}$ *and a map* $age\colon V^{\mathbb{D}} \to \mathbb{N}$*.*

Define the set $V^{\mathbb{D}}$ inductively by the following rule: for any finite family of elements $x_0, \ldots, x_{n-1} \in V^{\mathbb{D}}$ and the associated values $d_0, \ldots, d_{n-1} \in \mathbb{D}_{\geq 0}$, if the tuple $a = (x_i, d_i)_{i \in \mathbb{N}_{<n}}$ is permissible[10], then $a \in V^{\mathbb{D}}$. Thus we start with the empty tuple $() \in V^{\mathbb{D}}$, and then lump together all tuples of arbitrarily large positive length. For the sake of induction, we structure the set $V^{\mathbb{D}}$ with the map $age\colon V^{\mathbb{D}} \to \mathbb{N}$, inductively defined as follows: age of the empty tuple is 0 while for $a = (x_i, d_i)_{i \in \mathbb{N}_{<n}}$ where $n \in \mathbb{N}_{\geq 1}$,

$$age(a) = \max\{age(x_i) \mid i \in \mathbb{N}_{<n}\} + 1.$$

Finally, for $a = (x_i, a_i)_{i \in \mathbb{N}_{<n}}, b = (y_j, b_j)_{j \in \mathbb{N}_{<m}} \in V^{\mathbb{D}}$, define the map $d\colon V^{\mathbb{D}} \times V^{\mathbb{D}} \to \mathbb{D}$ inductively on $age(a) + age(b)$ as

$$d(a,b) = \max\Big\{ \max\{|d(x_i, b) - a_i| \mid i \in \mathbb{N}_{<n}\}, \max\{|d(a, y_j) - b_j| \mid j \in \mathbb{N}_{<m}\} \Big\}.$$

Here we adopt the convention that the maximum of the empty subset of $\mathbb{D}_{\geq 0}$ is 0, so in particular $d(a, ()) = \max\{|d(x_i, ()) - a_i| \mid i \in \mathbb{N}_{<n}\}$ and $d((), ()) = 0$.

Let $a = (x_i, a_i)_{i \in \mathbb{N}_{<n}}, b = (y_j, b_j)_{j \in \mathbb{N}_{<m}}, c = (z_k, c_k)_{k \in \mathbb{N}_{<l}} \in V^{\mathbb{D}}$. We prove the triangle inequality $d(a,b) + d(b,c) \geq d(a,c)$ inductively on $age(a) + age(b) + age(c)$. It is sufficient to write the calculations below for every $i \in \mathbb{N}_{<n}$, $k \in \mathbb{N}_{<l}$.

$$d(x_i, c) - a_i \leq d(x_i, b) - a_i + d(b, c)$$

$$a_i - d(x_i, c) \leq a_i - d(x_i, b) + d(b, c)$$

$$d(z_k, a) - c_k \leq d(z_k, b) - c_k + d(a, b)$$

$$c_k - d(z_k, a) \leq c_k - d(z_k, b) + d(a, b)$$

Symmetry of d is obvious. The condition that $d(a, a) = 0$ for every $a = (x_i, a_i)_{i \in \mathbb{N}_{<n}} \in V^{\mathbb{D}}$ is equivalent that for every $h \in \mathbb{N}_{<n}$ we have $d(a, x_h) = a_h$ which we verify after this construction in Proposition 25. Thus, $(V^{\mathbb{D}}, d)$ is a $\mathbb{D}$-valued pseudometric space. □

Proposition 25. *For any* $a = (x_i, a_i)_{i \in \mathbb{N}_{<n}} \in V^{\mathbb{D}}$*, we have* $d(a, x_h) = a_h$ *for all* $h \in \mathbb{N}_{<n}$*.*

Proof. We prove this by a double induction. We begin with the induction on $age(a)$. Of course, in the base case $age(a) = 0$, i.e. $a = ()$, there is nothing to prove.

[10] Notice that below we also inductively define the pseudometric, so we already know the distances $d(x_i, x_j)$.

For a general a, take any $h \in \mathbb{N}_{<n}$, and let $x_h = (y_j, b_j)_{j \in \mathbb{N}_{<m}}$. We have

$$d(a, x_h) = \max\Big\{ \max_i\{|d(x_i, x_h) - a_i|\}, \max_j\{|d(a, y_j) - b_j|\}\Big\}.$$

We claim $\max_i\{|d(x_i, x_h) - a_i|\} = a_h$. This is so because of permissibility of a,

$$d(x_i, x_h) - a_i \leq a_i + a_h - a_i = a_h,$$

$$a_i - d(x_i, x_h) \leq a_i - (a_i - a_h) = a_h,$$

and because $|d(x_h, x_h) - a_h| = a_h$. Therefore, we will have proved our statement once we verify that $|d(a, y_j) - b_j| \leq a_h$ for all $j \in \mathbb{N}_{<m}$.

The basic strategy is to write what $d(a, y_j)$ is, observe that we can again deal with the first maximum directly using permissibility, and then focus on the second one in which we again write out the distance and so on. This ends once we reach the empty tuple since then the second maximum is zero.

Here is the formal proof by induction. First, describe all predecessors of x_h inductively as follows: $y_{j_0,\ldots,j_r} = (y_{j_0,\ldots,j_r,j_{r+1}}, b_{j_0,\ldots,j_r,j_{r+1}})_{j_{r+1} \in \mathbb{N}_{<m_{j_0,\ldots,j_r}}}$.

The crux is in the following *claim*:

– Let $j_0, j_1, \ldots, j_l \in \mathbb{N}$. Assume that for all $j_{l+1} \in \mathbb{N}_{<m_{j_0,\ldots,j_l}}$ we have

$$d(a, y_{j_0,\ldots,j_{l+1}}) \leq a_h + b_{j_0} + b_{j_0,j_1} + \ldots + b_{j_0,\ldots,j_l} + b_{j_0,\ldots,j_{l+1}}.$$

Then $d(a, y_{j_0,\ldots,j_l}) \leq a_h + b_{j_0} + b_{j_0,j_1} + \ldots + b_{j_0,\ldots,j_l}$.

Proof. Write out

$$d(a, y_{j_0,\ldots,j_l}) =$$
$$= \max\Big\{ \max_i\{|d(x_i, y_{j_0,\ldots,j_l}) - a_i|\}, \max_{j_{l+1}}\{|d(a, y_{j_0,\ldots,j_{l+1}}) - b_{j_0,\ldots,j_{l+1}}|\}\Big\}.$$

Take any $i \in \mathbb{N}_{<n}$, recall permissibility and the original induction hypothesis.

$$d(x_i, y_{j_0,\ldots,j_l}) - a_i \leq$$
$$\leq d(x_i, x_h) + d(x_h, y_{j_0}) + d(y_{j_0}, y_{j_0,j_1}) + \ldots + d(y_{j_0,\ldots,j_{l-1}}, y_{j_0,\ldots,j_l}) - a_i \leq$$
$$\leq a_h + b_{j_0} + b_{j_0,j_1} + \ldots + b_{j_0,\ldots,j_l}$$

$$a_i - d(x_i, y_{j_0,\ldots,j_l}) \leq$$
$$\leq a_i - d(x_i, x_h) + d(x_h, y_{j_0}) + d(y_{j_0}, y_{j_0,j_1}) + \ldots + d(y_{j_0,\ldots,j_{l-1}}, y_{j_0,\ldots,j_l}) \leq$$
$$\leq a_h + b_{j_0} + b_{j_0,j_1} + \ldots + b_{j_0,\ldots,j_l}$$

Now take any $j_{l+1} \in \mathbb{N}_{<m_{j_0,\ldots,j_l}}$.

$$b_{j_0,\ldots,j_{l+1}} - d(a, y_{j_0,\ldots,j_{l+1}}) = b_{j_0,\ldots,j_{l+1}} -$$

$$- \max\Big\{ \max_i\{|d(x_i, y_{j_0,\dots,j_{l+1}}) - a_i|\}, \max_{j_{l+2}}\{|d(a, y_{j_0,\dots,j_{l+2}}) - b_{j_0,\dots,j_{l+2}}|\}\Big\} \leq$$
$$\leq b_{j_0,\dots,j_{l+1}} - \Big(d(x_h, y_{j_0,\dots,j_{l+1}}) - a_h\Big) =$$
$$= d(y_{j_0,\dots,j_l}, y_{j_0,\dots,j_{l+1}}) - d(x_h, y_{j_0,\dots,j_{l+1}}) + a_h \leq$$
$$\leq d(x_h, y_{j_0,\dots,j_l}) + a_h \leq$$
$$\leq d(x_h, y_{j_0}) + d(y_{j_0}, y_{j_0,j_1}) + \ldots + d(y_{j_0,\dots,j_{l-1}}, y_{j_0,\dots,j_l}) + a_h =$$
$$= a_h + b_{j_0} + b_{j_0,j_1} + \ldots + b_{j_0,\dots,j_l}$$

The final inequality $d(a, y_{j_0,\dots,j_{l+1}}) - b_{j_0,\dots,j_{l+1}} \leq a_h + b_{j_0} + b_{j_0,j_1} + \ldots + b_{j_0,\dots,j_l}$ holds by assumption.

Notice that this *claim* serves not only as the inductive step but also as the base of induction since when we reach the empty tuple, the condition is vacuous. In the end we obtain $d(a, y_{j_0}) \leq a_h + b_{j_0}$. The other inequalities required to prove $|d(a, y_j) - b_j| \leq a_h$ for all $j \in \mathbb{N}_{<m}$ can be verified the same way as those in the proof of *claim*. □

We may verify that $V^{\mathbb{D}}$ is countable, and clearly $(V^{\mathbb{D}}, d)$ satisfies the extension property, so its Kolmogorov quotient is a countable Urysohn space by Lemma 21. Moreover, we finally have the approximation property that similar distances and points yield similar extensions.

Lemma 26. *Let $u, v \in V^{\mathbb{D}}$ where $u = (x_i, a_i)_{i \in \mathbb{N}_{<n}}, v = (y_i, b_i)_{i \in \mathbb{N}_{<n}}$ are permissible tuples of the same length $n \in \mathbb{N}$. For any $\epsilon \in \mathbb{R}$, if $d(x_i, y_i) \leq \epsilon$ and $|a_i - b_i| \leq \epsilon$ for all $i \in \mathbb{N}_{<n}$, then $d(u, v) \leq 2\epsilon$.*

Proof. The following four calculations prove the statement.

$$d(x_i, v) - a_i \leq d(x_i, y_i) + d(y_i, v) - a_i = d(x_i, y_i) + b_i - a_i \leq 2\epsilon$$
$$a_i - d(x_i, v) \leq a_i - d(y_i, v) + d(x_i, y_i) = a_i - b_i + d(x_i, y_i) \leq 2\epsilon$$
$$d(u, y_i) - b_i \leq d(u, x_i) + d(x_i, y_i) - b_i = d(x_i, y_i) + a_i - b_i \leq 2\epsilon$$
$$b_i - d(u, y_i) \leq b_i - d(u, x_i) + d(x_i, y_i) = b_i - a_i + d(x_i, y_i) \leq 2\epsilon$$

□

As already mentioned, in the original construction of the countable Urysohn space the distances between tuples are constrained by the triangle inequalities involving tuples and their components. We took the maximal values allowable by these constraints, and it turned out that all triangle inequalities were satisfied; however, we did not have the approximation property. So we could consider the other simplest choice and took the minimal allowable distances, but unfortunately not all triangle inequalities are satisfied in that case. Our last construction of the countable Urysohn space presents the solution to the problem of finding allowable distances which are close enough to minimal ones so that we have the approximation property, and the triangle inequalities still hold.

4 Real Urysohn Space

4.1 Construction

We now define the metric space $\mathbb{U}$ as the completion of the countable Urysohn space. Regardless how we constructed the countable Urysohn space, its completion will be the same up to isometric isomorphism, so in proofs we will take whichever construction suits us best at the time.

However, in order to avoid the need for countable choice, we use the notion of completion as defined and presented by Fred Richman in [Ric08][11]. Let $\mathbf{X} = (X, d_X)$ be a metric space. A family $S = \{S_q \subseteq X \mid q \in \mathbb{Q}_{>0}\}$ of inhabited subsets of X, indexed by positive rational numbers, is called a *regular family* when for all $p, q \in \mathbb{Q}_{>0}$, $x \in S_p$, $y \in S_q$ the inequality $d_X(x, y) \leq p + q$ holds[12]. Two regular families S, T are *equivalent* when $d_X(x, y) \leq p + q$ for all $x \in S_p$, $y \in T_q$ where $p, q \in \mathbb{Q}_{>0}$. The completion $\widehat{\mathbf{X}} = (\widehat{X}, d_{\widehat{X}})$ is then defined as follows: $\widehat{X}$ is the set of all equivalence classes of regular families of $\mathbf{X}$, and the distance between two classes $[S], [T]$ is

$$d_{\widehat{X}}([S], [T]) =$$

$$= \inf \Big\{ q \in \mathbb{Q}_{>0} \;\Big|\; \forall\, \epsilon \in \mathbb{Q}_{>0} \,.\, \exists\, a, b, c \in \mathbb{Q}_{>0} \,.\, \exists\, s \in S_a \,.\, \exists\, t \in T_b \,.$$
$$(a + b + c < q + \epsilon \text{ and } d_X(s, t) \leq c) \Big\}.$$

It turns out that the infimum of this set always exists as a real number, and that $d_{\widehat{X}}$ is in fact a well-defined metric. The natural dense isometry $\mathbf{X} \to \widehat{\mathbf{X}}$ takes $x \in X$ to the equivalence class of the family of which every member is $\{x\}$.

We can generalize this construction by substituting $\mathbb{Q}$ with any $\mathbb{D}$ and still obtain an equivalent notion. Nor do we require the nondegeneracy of the metric; the construction works for pseudometric spaces in general, and the completions of the pseudometric space and its Kolmogorov quotient are isometrically isomorphic.

To finalize the construction, we justify the lack of the tag $\mathbb{D}$ in the symbol $\mathbb{U}$.

Proposition 27. *Up to isometric isomorphism, the space $\mathbb{U}$ does not depend on the choice of $\mathbb{D}$ used for the construction of the countable Urysohn space.*

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[11] Actually, Richman defines the notion of completion for spaces more general than metric which, as he notes, are essentially “metric spaces with values in the upper reals” (where *upper real numbers* are the set of upper Dedekind cuts). However, he proves that when we start with an actual metric space, its completion is again a metric space.

[12] Intuitively, a regular family is a family of sets which approximate some “ideal point”, with S_q the set of points which are at most q away from it. This is a direct generalization of regular Cauchy sequences where approximation at each stage is just a single point. In the presence of the Axiom of Countable Choice we can also go the other way by choosing an element from each $S_{2^{-n}}$, thus obtaining a Cauchy sequence which “converges to the same ideal point”.

Proof. Recall Lemma 23 and the discussion below it — for any $\mathbb{D}$, the space $\overline{\mathbb{U}^{\mathbb{D}}}$ is dense in $\overline{\mathbb{U}^{\mathbb{R}}}$, so they have isometrically isomorphic completions. □

4.2 Urysohn Properties

We now prove that $\mathbb{U}$ satisfies the Urysohn properties. It is certainly complete and separable since it is the completion of a countable space.

Lemma 28. *For any $n \in \mathbb{N}$, a finite metric space $\bar{\mathbf{F}} = (\{p_0, \dots, p_{n-1}, p_n\}, d_{\bar{F}})$, its finite subspace $\mathbf{F} = (\{p_0, \dots, p_{n-1}\}, d_F)$ and an isometry $f \colon \mathbf{F} \to \mathbb{U}$, there exists a canonical choice of an isometry $\bar{f} \colon \bar{\mathbf{F}} \to \mathbb{U}$ which extends f.*

Proof. In this proof we consider $\mathbb{U} = (U, d_U)$ as the completion of $(V^{\mathbb{D}}, d)$ from Construction 24, and we index regular families by elements of $\mathbb{D}_{>0}$.

If $n = 0$, let $\bar{f}(p_0) = ()$, and we are done. In the remainder of the proof, suppose $n \geq 1$. Take any $q \in \mathbb{D}_{>0}$. Define $S_q \subseteq V^{\mathbb{D}}$ as

$$S_q = \left\{ (x_i, d_i)_{i \in \mathbb{N}_{<n}} \in V^{\mathbb{D}} \;\middle|\; \forall\, i \in \mathbb{N}_{<n} \,.\, \left(d_U(f(p_i), x_i) \leq \tfrac{q}{2} \text{ and } |d_{\bar{F}}(p_n, p_i) - d_i| \leq \tfrac{q}{2} \right) \right\}.$$

In order to make the proof easier to read, we structure it into steps.

– <u>*S_q is an inhabited set.*</u>

Let $\lambda = \frac{q}{8n} > 0$ and choose, for every $i \in \mathbb{N}_{<n}$, $d_i \in (d_{\bar{F}}(p_n, p_i) + [3\lambda, 4\lambda]) \cap \mathbb{D}$ and $x'_i \in V^{\mathbb{D}}$, such that $d_U(f(p_i), x'_i) \leq \lambda$ (this can be done because of density). Denote $d''_{i,j} = d(x'_i, x'_j) + 3\lambda$ for short. Define $x_0, \dots, x_{n-1} \in V^{\mathbb{D}}$ inductively: if we already know what $x_0, \dots, x_{h-1}$ are, let

$$x_h = \left(x_0, d''_{h,0}, x_1, d''_{h,1}, \dots, x_{h-1}, d''_{h,h-1}, x'_h, \max\left\{ |d''_{h,j} - d(x'_h, x_j)| \;\middle|\; j \in \mathbb{N}_{<h} \right\} \right);$$

in particular $x_0 = (x'_0, 0)$. The calculations below confirm these are indeed permissible tuples:

$$|d''_{h,i} - d''_{h,j}| = |d(x'_h, x'_i) - d(x'_h, x'_j)| \leq d(x'_i, x'_j) \leq d(x_i, x_j),$$

$$d''_{h,i} + d''_{h,j} = d(x'_h, x'_i) + d(x'_h, x'_j) + 6\lambda \geq d(x'_i, x'_j) + 6\lambda \geq d(x_i, x_j),$$

$$d''_{h,i} - \max\left\{ |d''_{h,j} - d(x'_h, x_j)| \;\middle|\; j \in \mathbb{N}_{<h} \right\} \leq d''_{h,i} - |d''_{h,i} - d(x'_h, x_i)| \leq d(x'_h, x_i),$$

$$d''_{h,j} - d(x'_h, x_j) - d''_{h,i} \leq d(x_i, x_j) - d(x'_h, x_j) \leq d(x'_h, x_i),$$

$$d(x'_h, x_j) - d''_{h,j} - d''_{h,i} \leq d(x'_h, x_j) - d(x_i, x_j) \leq d(x'_h, x_i),$$

$$\max\left\{ |d''_{h,j} - d(x'_h, x_j)| \;\middle|\; j \in \mathbb{N}_{<h} \right\} + d''_{h,i} \geq d(x'_h, x_i) - d''_{h,i} + d''_{h,i} = d(x'_h, x_i).$$

We claim that $d(x_i, x'_i) \leq 3i\lambda$ for all $i \in \mathbb{N}_{<n}$. This clearly holds for $i = 0$. By induction,

$$d(x_i, x'_i) = \max\left\{ |d''_{i,j} - d(x'_i, x_j)| \;\middle|\; j \in \mathbb{N}_{<i} \right\} =$$

$$= \max\left\{|d(x'_i, x'_j) + 3\lambda - d(x'_i, x_j)| \;\middle|\; j \in \mathbb{N}_{<i}\right\} \le$$

$$\le \max\left\{|d(x'_i, x'_j) - d(x'_i, x_j)| + 3\lambda \;\middle|\; j \in \mathbb{N}_{<i}\right\} \le$$

$$\le \max\left\{d(x_j, x'_j) \;\middle|\; j \in \mathbb{N}_{<i}\right\} + 3\lambda \le 3(i-1)\lambda + 3\lambda = 3i\lambda.$$

Therefore $d_U(f(p_i), x_i) \le d_U(f(p_i), x'_i) + d(x'_i, x_i) \le \lambda + 3i\lambda \le 3n\lambda \le \frac{q}{2}$. Also $|d_{\bar{F}}(p_n, p_i) - d_i| \le 4\lambda \le 4n\lambda = \frac{q}{2}$. So $(x_i, d_i)_{i\in\mathbb{N}_{<n}} \in S_q$.

– <u>$S = \{S_q \mid q \in \mathbb{D}_{>0}\}$ *is a regular family.*</u>
  Take any $p, q \in \mathbb{D}_{>0}$ and any $a = (x_i, a_i)_{i\in\mathbb{N}_{<n}} \in S_p$, $b = (y_i, b_i)_{i\in\mathbb{N}_{<n}} \in S_q$. Then for any $i \in \mathbb{N}_{<n}$,

$$d(x_i, y_i) \le d_U(x_i, f(p_i)) + d_U(f(p_i), y_i) \le \tfrac{p+q}{2},$$

$$|a_i - b_i| \le |a_i - d_{\bar{F}}(p_n, p_i)| + |d_{\bar{F}}(p_n, p_i) - b_i| \le \tfrac{p+q}{2}$$

  whence $d(a, b) \le p + q$ by Lemma 26.

We now wish to prove $d_U([S], f(p_i)) = d_{\bar{F}}(p_n, p_i)$ for all $i \in \mathbb{N}_{<n}$. So fix $i \in \mathbb{N}_{<n}$. First we need a regular family T, such that $f(p_i) = [T]$. As explained in [Ric08], every element of a completion has a canonical representative[13] which in our case means $T_q = \left\{t \in V^{\mathbb{D}} \;\middle|\; d_U(f(p_i), t) \le q\right\}$. Denote $e = d_{\bar{F}}(p_n, p_i)$ and

$$Z = \Big\{q \in \mathbb{D}_{>0} \;\Big|\; \forall\, \epsilon \in \mathbb{D}_{>0} \,.\, \exists\, a, b, c \in \mathbb{D}_{>0} \,.\, \exists\, s \in S_a \,.\, \exists\, t \in T_b \,.$$

$$(a + b + c < q + \epsilon \text{ and } d(s, t) \le c)\Big\};$$

then $d_U([S], f(p_i)) = \inf Z$. To prove our claim, it is sufficient to show $e \in Z$ and $\forall\, q \in Z \,.\, e \le q$.

– <u>$|d(s,t) - e| \le a + b$ *for all* $a, b \in \mathbb{D}_{>0}$ *and* $s \in S_a$, $t \in T_b$.</u>
  Write $s = (x_j, d_j)_{j\in\mathbb{N}_{<n}}$; then

$$d(s, t) \le d(s, x_i) + d_U(x_i, f(p_i)) + d_U(f(p_i), t) =$$

$$= d_i + d_U(f(p_i), x_i) + d_U(f(p_i), t) \le d_{\bar{F}}(p_n, p_i) + \tfrac{a}{2} + \tfrac{a}{2} + b = e + a + b,$$

$$e \le d_i + \tfrac{a}{2} = d(s, x_i) + \tfrac{a}{2} \le d(s, t) + d_U(t, f(p_i)) + d_U(f(p_i), x_i) + \tfrac{a}{2} \le$$

$$\le d(s, t) + b + \tfrac{a}{2} + \tfrac{a}{2} = d(s, t) + a + b.$$

[13] For every element of the completion there exists the largest regular family which represents it. Explicitly, if $i\colon (X, d_X) \to (\widehat{X}, d_{\widehat{X}})$ is the natural isometry and $y \in \widehat{X}$, then $y = [S]$ where $S_q = \left\{x \in X \;\middle|\; d_{\widehat{X}}(y, i(x)) \le q\right\}$.

– $\underline{e \in Z}$

Let $\epsilon \in \mathbb{D}_{>0}$. Choose any $a, b \in \left]0, \frac{\epsilon}{4}\right[\cap \mathbb{D}$ and $c \in \Big(e + \left]a+b, \frac{\epsilon}{2}\right[\Big) \cap \mathbb{D}$. Choose arbitrary $s \in S_a$, $t \in T_b$. We have $a + b + c < \frac{\epsilon}{4} + \frac{\epsilon}{4} + e + \frac{\epsilon}{2} = e + \epsilon$ and $d(s,t) \leq e + a + b < c$ which proves $e \in Z$.

– $\underline{\forall\, q \in Z \,.\, e \leq q}$

Let $q \in Z$, meaning that for every $\epsilon \in \mathbb{D}_{>0}$ we have appropriate a, b, c, s, t. In particular, $c \geq d(s,t) \geq e - a - b$, so $a + b + c \geq e$. But then, for every $\epsilon \in \mathbb{D}_{>0}$, we have $q + \epsilon > e$ which means $q \geq e$.

Finally, we define $\bar{f} \colon \bar{\mathbf{F}} \to \mathbb{U}$ as

$$\bar{f}(p_i) = \begin{cases} f(p_i) & \text{if } i \in \mathbb{N}_{<n}, \\ [S] & \text{if } i = n. \end{cases}$$

The equality $d_U([S], f(p_i)) = d_{\bar{F}}(p_n, p_i)$ proves both that $\bar{f}$ is well defined (since $p_n = p_i \implies d_{\bar{F}}(p_n, p_i) = 0 \implies d_U([S], f(p_i)) = 0 \implies [S] = f(p_i)$) and that it is an isometry. Clearly it is an extension of f. □

Theorem 29 (Extension). *Let* $\mathbf{S} = (S, d_S)$ *be a separable metric space,* $\mathbf{F} = (F, d_F)$ *a finite subspace of* $\mathbf{S}$*, and* $f \colon \mathbf{F} \to \mathbb{U}$ *an isometry. Then there exists an isometry* $\bar{f} \colon \mathbf{S} \to \mathbb{U}$ *which extends* f*.*

Proof. By Lemma 9 we may without loss of generality assume S is inhabited. Let $s \colon \mathbb{N} \to S$ be a sequence with a dense image. Denote $F = \{p_0, \ldots, p_{n-1}\}$.

Define a new sequence $s' \colon \mathbb{N} \to S$ by

$$s'_k = \begin{cases} p_k & \text{if } k \in \mathbb{N}_{<n}, \\ s_{k-n} & \text{if } k \in \mathbb{N}_{\geq n}. \end{cases}$$

Clearly s' also has a dense image in S. Denote it by $C \subseteq S$, and the subspace it determines by $\mathbf{C} = (C, d_C)$.

Using induction we can extend the isometry f from $\mathbf{F}$ to $\mathbf{C}$: if the extension is already defined for $s'_0, \ldots, s'_{m-1}$, use Lemma 28 to define it for s'_m. Because $\mathbb{U}$ is complete, this extends to an isometry $\bar{f} \colon \mathbf{S} \to \mathbb{U}$ by Corollary 7. □

Corollary 30 (Universality). *Every separable metric space isometrically embeds into* $\mathbb{U}$*.*

Proof. In Theorem 29, take $F = \emptyset$ and for f the unique map $\emptyset \to \mathbb{U}$. □

As future work it remains to verify whether uniqueness of the Urysohn space can be proved in our restrictive setting (but notice that this is equivalent to showing that any Urysohn space contains a countable Urysohn space as a dense subspace).

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