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On the Subrecursive Computability of Several Famous Constants

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Abstract: For any class $\mathcal{F}$ of total functions in the set $\mathbb{N}$ of the natural numbers, we define the notion of $\mathcal{F}$-computable real number. A real number α is called $\mathcal{F}$-computable if there exist one-argument functions f, g and h in $\mathcal{F}$ such that for any n in $\mathbb{N}$ the distance between the rational number $f(n) - g(n)$ over $h(n) + 1$ and the number α is not greater than the reciprocal of $n + 1$. Most concrete real numbers playing a role in analysis can be easily shown to be $\mathcal{E}^3$-computable (as usually, $\mathcal{E}^m$ denotes the m-th Grzegorczyk class). Although (as it is proved in Section 5 of this paper) there exist $\mathcal{E}^3$-computable real numbers that are not $\mathcal{E}^2$-computable, we prove that π, e and other remarkable real numbers are $\mathcal{E}^2$-computable (the number π proves to be even $\mathcal{L}$-computable, where $\mathcal{L}$ is the class of Skolem's lower elementary functions). However, only the rational numbers would turn out to be $\mathcal{E}^2$-computable according to a definition of $\mathcal{F}$-computability using 2^n instead of $n + 1$.

Key Words: computable real number, Grzegorczyk classes, second Grzegorczyk class, lower elementary functions, π, e, Liouville's number, Euler's constant.
Category: F.1.3, F.2.1, G.0, G.1.0

1 Introduction

Let $\mathcal{F}$ be a class of total functions in the set $\mathbb{N}$ of the natural numbers. We shall call an $\mathcal{F}$*-sequence* any infinite sequence $r_0, r_1, r_2, \ldots$ of rational numbers that has a representation in the form

$$r_n = \frac{f(n) - g(n)}{h(n) + 1}, \quad n = 0, 1, 2, 3, \ldots,$$

with one-argument functions f, g and h belonging to $\mathcal{F}$, and a real number α will be called $\mathcal{F}$*-computable* if there exists an $\mathcal{F}$-sequence $r_0, r_1, r_2, \ldots$ such that $|r_n - \alpha| \leq (n + 1)^{-1}$ for all n in $\mathbb{N}$.[1]

In the case when $\mathcal{F}$ is the class of the recursive functions, the $\mathcal{F}$-computable real numbers are exactly the computable ones, although 2^{-n} is usually used instead of $(n + 1)^{-1}$ in the definition of computability of a real number (cf.

[1] Under some assumptions about the class $\mathcal{F}$, the $\mathcal{F}$-sequences were called $\mathcal{F}$-expressible in [Skordev 2002], and the $\mathcal{E}^2$-sequences were called $\mathcal{E}^2$-computable in [Skordev 2008]. When $\mathcal{F}$ satisfies the assumptions made in [Skordev 2002], the present definition of $\mathcal{F}$-computability of a real number coincides with the one accepted there.

for instance [Ko 1991, Weihrauch 2000]). Namely the definition obtained from the present one by replacement of $(n+1)^{-1}$ with 2^{-n} will be equivalent to it, whenever the class $\mathcal{F}$ is closed under composition and contains some one-argument function that dominates $2^n - 1$. That is the case not only when $\mathcal{F}$ is the class of all recursive functions, but also when it is some Grzegorczyk class $\mathcal{E}^m$ with $m \geq 3$. However, the equivalence is lost, for example, in the case of $\mathcal{F} = \mathcal{E}^2$. Indeed, as seen from the results proved in [Skordev 2002], all real algebraic numbers are $\mathcal{E}^2$-computable in the sense of the present definition[2], whereas only the rational numbers would be $\mathcal{E}^2$-computable in the sense of the definition with 2^{-n}, as the third statement of the following proposition shows.

Proposition 1. *Let h be a one-argument function belonging to the class $\mathcal{E}^2$, and let $r_0, r_1, r_2, \dots$ be rational numbers such that $(h(n)+1)r_n$ is an integer for any $n \in \mathbb{N}$. Then:*

1. *There exists a polynomial $p(n)$ such that $p(n)|r_n| \geq 1$ holds, whenever $r_n \neq 0$.*
2. *There exists a polynomial $q(n)$ such that $q(n)|r_{n+1} - r_n| \geq 1$ holds, whenever $r_{n+1} \neq r_n$.*
3. *If α is a real number such that $|r_n - \alpha| \leq 2^{-n}$ for all n in $\mathbb{N}$, then α is a rational number.*

Proof. The statement 1 follows from the fact that $(h(n)+1)|r_n| \geq 1$, whenever $r_n \neq 0$, and the function h is dominated by some polynomial. The statement 2 can be derived from the statement 1 by taking $r_{n+1} - r_n$ in the role of r_n and using the fact that $(h(n)+1)(h(n+1)+1)(r_{n+1} - r_n)$ is also an integer for any $n \in \mathbb{N}$. To prove the statement 3, suppose α is a real number such that $|r_n - \alpha| \leq 2^{-n}$ for all n in $\mathbb{N}$. Since

$$|r_{n+1} - r_n| \leq |r_{n+1} - \alpha| + |r_n - \alpha| \leq 3 \cdot 2^{-n-1},$$

the polynomial $q(n)$ from the statement 2 will satisfy the inequality

$$3q(n) \geq 2^{n+1}$$

for all n such that $r_{n+1} \neq r_n$, and therefore only finitely many such n can exist.

Remark. A weaker result in this direction can be obtained by using Liouville's approximation theorem. Its application proves the statement 3 of the above

[2] Under the assumption that $\mathcal{F}$ contains the successor, projection and product functions, as well as the function $\lambda mn.|m-n|$, and $\mathcal{F}$ is closed under composition and bounded μ-operation, it was proved in [Skordev 2002] that the $\mathcal{F}$-computable real numbers form a field containing the real roots of any non-constant polynomial with coefficients from this field.

proposition under the additional assumption that α is an algebraic number (the possibility of such an application of Liouville's theorem is implicitly indicated in footnote 2 of [Peshev and Skordev 2006]).

Since, as we already mentioned, all real algebraic numbers are $\mathcal{E}^2$-computable, it is natural to ask whether there exist $\mathcal{E}^2$-computable transcendental numbers.[3] A positive answer to this question was given in the paper [Skordev 2008], where, in particular, the numbers π and e were shown to be $\mathcal{E}^2$-computable. The present paper is a wholly revised and extended version of the most essential parts of [Skordev 2008]. A radical change is done in the proofs that the considered concrete real numbers are $\mathcal{E}^2$-computable. Namely some general statements about $\mathcal{F}$-computability of sums of series are proved now, and applications of these statements are done instead of the lengthy direct proofs given in [Skordev 2008]. The number π is shown to be even $\mathcal{L}$-computable, where $\mathcal{L}$ is the class of Skolem's lower elementary functions studied in [Skolem 1962].

2 $\mathcal{F}$-computable real-valued functions with natural arguments

We shall prepare now some tools for facilitating the proofs of $\mathcal{E}^2$-computability of certain real numbers. Throughout this section, a class $\mathcal{F}$ of total functions in $\mathbb{N}$ will be supposed to be given such that $\mathcal{F}$ contains the zero, successor, projection, addition and Kronecker delta functions, and it is closed under composition and bounded summation (any class $\mathcal{E}^m$ with $m \geq 2$ satisfies these assumptions, and the class $\mathcal{L}$ of the lower elementary functions is the smallest class satisfying them).

Let l be a natural number, and θ be a function from $\mathbb{N}^l$ into the set $\mathbb{R}$ of the real numbers. The function θ will be called *$\mathcal{F}$-computable* if there exist $l+1$-argument functions f, g and h belonging to $\mathcal{F}$ such that

$$\left|\frac{f(i_1,\ldots,i_l,n)-g(i_1,\ldots,i_l,n)}{h(i_1,\ldots,i_l,n)+1}-\theta(i_1,\ldots,i_l)\right| \leq \frac{1}{n+1}$$

for all $i_1,\ldots,i_l,n$ in $\mathbb{N}$.

Obviously all values of an $\mathcal{F}$-computable real-valued function with natural arguments are $\mathcal{F}$-computable real numbers, and a real-valued function without arguments is $\mathcal{F}$-computable iff its value at the empty tuple is $\mathcal{F}$-computable (thus the 0-argument $\mathcal{F}$-computable real-valued functions can be identified with the $\mathcal{F}$-computable real numbers). Clearly any substitution of functions from

[3] It is quite easy to see that π, e and many other concrete real numbers playing a part in analysis are $\mathcal{E}^3$-computable. However, there exist $\mathcal{E}^3$-computable real numbers which are not $\mathcal{E}^2$-computable (cf. Section 5).

the class $\mathcal{F}$ into an $\mathcal{F}$-computable real-valued function with natural arguments produces again an $\mathcal{F}$-computable real-valued function with natural arguments.

Since every infinite sequence of real numbers is actually a function from $\mathbb{N}$ into $\mathbb{R}$, the above definition introduces, in particular, the notion of $\mathcal{F}$-computability for such sequences. Obviously, each $\mathcal{F}$-sequence of rational numbers is $\mathcal{F}$-computable as a sequence of real numbers. In general, however, an infinite sequence of rational numbers can be $\mathcal{F}$-computable as a sequence of real numbers without being an $\mathcal{F}$-sequence. For instance, let $\mathcal{F}$ be a subclass of the class of the recursive functions. Then, by a result proved in [Skolem 1962], there exists a two-argument lower elementary function φ such that the set $\{n \in \mathbb{N} \mid \exists t(\varphi(n,t) = 0)\}$ is non-recursive. If we set r_n to be $(s+1)^{-1}$ with $s = \mu t(\varphi(n,t) = 0)$ for any n in the set in question, and to be 0 for all other n in $\mathbb{N}$, then the sequence of the rational numbers $r_0, r_1, r_2, \ldots$ will be $\mathcal{F}$-computable as a sequence of real numbers, but without being an $\mathcal{F}$-sequence.

The first statement in the next proposition shows that one can take $h(i_1, \ldots, i_l, n) = n$ in the definition of $\mathcal{F}$-computability of real-valued functions with natural arguments.[4]

Proposition 2. *Let l be a natural number, and θ be a function from $\mathbb{N}^l$ into $\mathbb{R}$. Then:*

1. *If θ is $\mathcal{F}$-computable, and c is a real number greater than $1/2$, then there exist $l+1$-argument functions f and g belonging to $\mathcal{F}$ such that*

$$\left| \frac{f(i_1, \ldots, i_l, n) - g(i_1, \ldots, i_l, n)}{n+1} - \theta(i_1, \ldots, i_l) \right| \leq \frac{c}{n+1} \tag{1}$$

   for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$.

2. *If for some $l+1$-argument functions f and g belonging to $\mathcal{F}$ and some real number c the inequality (1) holds for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$, then θ is $\mathcal{F}$-computable.*

Proof. For the proof of the statement 1, suppose θ is $\mathcal{F}$-computable, and c is a real number greater than $1/2$. One easily shows the existence of $l+1$-argument functions f_0, g_0 and h_0 belonging to $\mathcal{F}$ such that

$$\left| \frac{f_0(i_1, \ldots, i_l, n) - g_0(i_1, \ldots, i_l, n)}{h_0(i_1, \ldots, i_l, n) + 1} - \theta(i_1, \ldots, i_l) \right| \leq \frac{c - 1/2}{n+1}$$

[4] This holds, in particular, for 0-argument functions, thus giving a characterization of the $\mathcal{F}$-computable real numbers which is in the spirit of the definition in [Grzegorczyk 1955] for computability of real numbers (that definition, taken literally, defines computability only of non-negative real numbers, but its extension to arbitrary ones is easy).

for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$. There exists also a two-argument function A in $\mathcal{F}$ such that

$$\left| A(i,j) - \frac{i}{j+1} \right| \leq \frac{1}{2}$$

for all natural numbers i and j; for instance, we may set

$$A(i,j) = \left[\frac{i}{j+1} + \frac{1}{2} \right].$$

Now set

$$f(i_1, \ldots, i_l, n) = A((n+1)(f_0(i_1, \ldots, i_l, n) \dot{-} g_0(i_1, \ldots, i_l, n)), h_0(i_1, \ldots, i_l, n)),$$
$$g(i_1, \ldots, i_l, n) = A((n+1)(g_0(i_1, \ldots, i_l, n) \dot{-} f_0(i_1, \ldots, i_l, n)), h_0(i_1, \ldots, i_l, n)).$$

Clearly the functions f and g belong to $\mathcal{F}$. It is easy to see that

$$\left| f(i_1, \ldots, i_l, n) - g(i_1, \ldots, i_l, n) - (n+1)\frac{f_0(i_1, \ldots, i_l, n) - g_0(i_1, \ldots, i_l, n)}{h_0(i_1, \ldots, i_l, n) + 1} \right| \leq \frac{1}{2}$$

both in the case of $f_0(i_1, \ldots, i_l, n) \geq g_0(i_1, \ldots, i_l, n))$ and in the case of $f_0(i_1, \ldots, i_l, n) < g_0(i_1, \ldots, i_l, n))$. Therefore

$$\left| \frac{f(i_1, \ldots, i_l, n) - g(i_1, \ldots, i_l, n)}{n+1} - \frac{f_0(i_1, \ldots, i_l, n) - g_0(i_1, \ldots, i_l, n)}{h_0(i_1, \ldots, i_l, n) + 1} \right| \leq \frac{1/2}{n+1},$$

hence the inequality (1) holds. To prove the statement 2, suppose $f, g \in \mathcal{F}$, $c \in \mathbb{R}$, and the inequality (1) holds for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$. Then, taking a positive integer k such that $k \geq c$, we shall have

$$\left| \frac{f(i_1, \ldots, i_l, kn+k-1) - g(i_1, \ldots, i_l, kn+k-1)}{(kn+k-1)+1} - \theta(i_1, \ldots, i_l) \right| \leq \frac{1}{n+1}$$

for all $i_1, \ldots, i_l, n$ in $\mathbb{N}$.

Proposition 3. *Let k be a natural number, θ be a $k+1$-argument real-valued function with natural arguments, and θ^{Σ} be the mapping of $\mathbb{N}^{k+1}$ into $\mathbb{R}$ defined by*

$$\theta^{\Sigma}(i_1, \ldots, i_k, t) = \sum_{s=0}^{t} \theta(i_1, \ldots, i_k, s).$$

Then θ^{Σ} is $\mathcal{F}$-computable iff θ is $\mathcal{F}$-computable.

Proof. Suppose θ is $\mathcal{F}$-computable. By Proposition 2, there exist $k+2$-argument functions f and g belonging to $\mathcal{F}$ such that

$$\left| \frac{f(i_1, \ldots, i_k, s, n) - g(i_1, \ldots, i_k, s, n)}{n+1} - \theta(i_1, \ldots, i_k, s) \right| \leq \frac{1}{n+1}$$

for all $i_1, \ldots, i_k, s, n$ in $\mathbb{N}$. We consider the functions

$$f^{\Sigma}(i_1, \ldots, i_k, t, n) = \sum_{s=0}^{t} f(i_1, \ldots, i_k, s, nt + n + t),$$
$$g^{\Sigma}(i_1, \ldots, i_k, t, n) = \sum_{s=0}^{t} g(i_1, \ldots, i_k, s, nt + n + t).$$

They also belong to the class $\mathcal{F}$, since this class is closed under bounded summation. In addition, for any $i_1, \ldots, i_k, s, n, t$ in $\mathbb{N}$ the number

$$\left| \frac{f(i_1, \ldots, i_k, s, tn + t + n) - g(i_1, \ldots, i_k, s, tn + t + n)}{tn + t + n + 1} - \theta(i_1, \ldots, i_k, s) \right|$$

does not exceed the reciprocal of $(t + 1)(n + 1)$, hence

$$\left| \frac{f^{\Sigma}(i_1, \ldots, i_k, t, n) - g^{\Sigma}(i_1, \ldots, i_k, t, n)}{nt + n + t + 1} - \theta^{\Sigma}(i_1, \ldots, i_k, t) \right| \leq \frac{1}{n + 1}$$

for all $i_1, \ldots, i_l, t, n$ in $\mathbb{N}$. Thus the $\mathcal{F}$-computability of θ implies the $\mathcal{F}$-computability of θ^{Σ}. The converse implication follows from the equality

$$\theta(i_1, \ldots, i_k, t) = \begin{cases} \theta^{\Sigma}(i_1, \ldots, i_k, t) & \text{if } t = 0 \\ \theta^{\Sigma}(i_1, \ldots, i_k, t) - \theta^{\Sigma}(i_1, \ldots, i_k, t - 1) & \text{otherwise.} \end{cases}$$

Theorem. *Let k be a natural number, θ be such an $\mathcal{F}$-computable $k+1$-argument real-valued function with natural arguments that the series*

$$\sum_{s=0}^{\infty} \theta(i_1, \ldots, i_k, s)$$

converges for all $i_1, \ldots, i_k$ in $\mathbb{N}$, and $\sigma(i_1, \ldots, i_k)$ be the sum of this series. Let there exist a $k + 1$-argument function p belonging to $\mathcal{F}$ and such that

$$\left| \sum_{s=t+1}^{\infty} \theta(i_1, \ldots, i_k, s) \right| \leq \frac{1}{n + 1}$$

for any natural numbers $i_1, \ldots, i_k, n$ and $t = p(i_1, \ldots, i_k, n)$. Then the function σ is also $\mathcal{F}$-computable.

Proof. Let θ^{Σ} be as in Proposition 3. Since θ^{Σ} is $\mathcal{F}$-computable, there exist $k + 2$-argument functions f_1, g_1 and h_1 belonging to $\mathcal{F}$ such that

$$\left| \frac{f_1(i_1, \ldots, i_k, t, n) - g_1(i_1, \ldots, i_k, t, n)}{h_1(i_1, \ldots, i_k, t, n) + 1} - \theta^{\Sigma}(i_1, \ldots, i_k, t) \right| \leq \frac{1}{n + 1}$$

for all $i_1, \ldots, i_k, t, n$ in $\mathbb{N}$. If we set

$$f(i_1, \ldots, i_k, n) = f_1(i_1, \ldots, i_k, p(i_1, \ldots, i_k, 2n + 1), 2n + 1),$$
$$g(i_1, \ldots, i_k, n) = g_1(i_1, \ldots, i_k, p(i_1, \ldots, i_k, 2n + 1), 2n + 1),$$
$$h(i_1, \ldots, i_k, n) = h_1(i_1, \ldots, i_k, p(i_1, \ldots, i_k, 2n + 1), 2n + 1),$$

then $f, g, h \in \mathcal{F}$ and

$$\left|\frac{f(i_1,\ldots,i_k,n)-g(i_1,\ldots,i_k,n)}{h(i_1,\ldots,i_k,n)+1}-\sigma(i_1,\ldots,i_k)\right| \leq$$

$$\left|\frac{f(i_1,\ldots,i_k,n)-g(i_1,\ldots,i_k,n)}{h(i_1,\ldots,i_k,n)+1}-\theta^{\Sigma}(i_1,\ldots,i_k,p(i_1,\ldots,i_k,2n+1))\right|+$$

$$\left|\theta^{\Sigma}(i_1,\ldots,i_k,p(i_1,\ldots,i_k,2n+1))-\sigma(i_1,\ldots,i_k)\right| \leq \frac{1}{2n+2}+\frac{1}{2n+2}=\frac{1}{n+1}$$

for all $i_1, \ldots, i_k, n$ in $\mathbb{N}$.

Corollary. *Let θ be such an $\mathcal{F}$-computable real-valued function of one natural argument that the series*

$$\sum_{s=0}^{\infty}\theta(s)$$

converges, and α be the sum of this series. Let there exist a one-argument function p belonging to $\mathcal{F}$ and such that

$$\left|\sum_{s=t+1}^{\infty}\theta(s)\right| \leq \frac{1}{n+1}$$

for any natural number n and $t = p(n)$. Then the number α is also $\mathcal{F}$-computable.

3 $\mathcal{E}^2$-computability of the numbers π and e, of Liouville's number and of Euler's constant

3.1 $\mathcal{E}^2$-computability of the number π

The terms of the series in the well-known formula

$$\frac{\pi}{4}=\sum_{s=0}^{\infty}\frac{(-1)^s}{2s+1} \tag{2}$$

form an $\mathcal{L}$-sequence, since

$$(-1)^s = (s+1) \bmod 2 - s \bmod 2$$

for any $s \in \mathbb{N}$. In addition,

$$\left|\sum_{s=t+1}^{\infty}\frac{(-1)^s}{2s+1}\right| < \frac{1}{2t+3}$$

for any $t \in \mathbb{N}$. An application of the corollary from Section 2 immediately shows that $\pi/4$ is $\mathcal{L}$-computable, hence π is also $\mathcal{L}$-computable (thus it is $\mathcal{E}^2$-computable).

Due to the slow convergence of the series in the formula (2), this formula is not convenient for the numerical computation of π. There are formulas that are much more appropriate for this, e.g. Machin's formula

$$\frac{\pi}{4} = 4\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)5^{2s+1}} - \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)239^{2s+1}}. \tag{3}$$

The sums of the two series in (3) also turn out to be $\mathcal{E}^2$-computable. Of course any of the two series has a modulus of convergence of the sort required by the corollary in Section 2. Unfortunately, the sequences of their terms are not $\mathcal{E}^2$-recursive, as it is seen from Proposition 1. Nevertheless, the corollary is applicable to these series, since the sequences in question are still $\mathcal{E}^2$-computable (as sequences of real numbers). To prove their $\mathcal{E}^2$-computability, we may, for example, consider the three-argument function f_0 in $\mathbb{N}$ defined by

$$f_0(i,s,t) = \left[\frac{t}{(s+1)^i}\right].$$

This function belongs to $\mathcal{E}^2$, since

$$f_0(0,s,t) = t, \quad f_0(i+1,s,t) = \left[\frac{f_0(i,s,t)}{s+1}\right], \quad f_0(i,s,t) \le t$$

for all i, s, t in $\mathbb{N}$. In addition,

$$\left|\frac{f_0(i,s,n+1)}{n+1} - \frac{1}{(s+1)^i}\right| = \frac{1}{n+1}\left|f_0(i,s,n+1) - \frac{n+1}{(s+1)^i}\right| < \frac{1}{n+1},$$

for any i, s, n in $\mathbb{N}$, hence also

$$\left|\frac{(-1)^i f_0(2i+1,s,n+1)}{(2i+1)(n+1)} - \frac{(-1)^i}{(2i+1)(s+1)^{2i+1}}\right| < \frac{1}{n+1}.$$

Thus we may complete the proof by using the instances for values 4 and 238 of s of the above inequality and by representing $(-1)^i f_0(2i+1,s,n+1)$ as the difference $((i+1) \bmod 2) f_0(2i+1,s,n+1) - (i \bmod 2) f_0(2i+1,s,n+1)$.

The $\mathcal{E}^2$-computability of the sequences of the terms of the two series in (3) can be proved also as follows. One considers the three-argument function g_0 in $\mathbb{N}$ defined by

$$g_0(i,s,t) = \min((s+1)^i, t+1). \tag{4}$$

This function belongs to $\mathcal{E}^2$ since

$$g_0(0,s,t) = t+1, \quad g_0(i+1,s,t) = \min(g_0(i,s,t)(s+1), t+1), \quad g_0(i,s,t) \le t+1$$

for all i, s, t in $\mathbb{N}$. On the other hand,

$$\left|\frac{1}{g_0(i,s,n)} - \frac{1}{(s+1)^i}\right| < \frac{1}{n+1}$$

for any i, s, n in $\mathbb{N}$, hence also

$$\left|\frac{(-1)^i}{(2i+1)g_0(2i+1,s,n)} - \frac{(-1)^i}{(2i+1)(s+1)^{2i+1}}\right| < \frac{1}{n+1}.$$

Remark. The $\mathcal{E}^2$-computability of f_0 can be derived also from the $\mathcal{E}^2$-computability of g_0, since

$$f_0(i,s,t) = \left[\frac{t}{g_0(i,s,t)}\right]$$

for all i, s, t in $\mathbb{N}$.

3.2 $\mathcal{E}^2$-computability of the number e

To prove the $\mathcal{E}^2$-computability of the number e, we shall use the equality

$$e = \sum_{i=0}^{\infty} \frac{1}{i!} \tag{5}$$

by showing that the series in it is $\mathcal{E}^2$-convergent, and the sequence of its terms is $\mathcal{E}^2$-computable (although, as seen from Proposition 1, this sequence is not $\mathcal{E}^2$-recursive). The $\mathcal{E}^2$-convergence of this series follows from the fact that

$$\sum_{i=n+1}^{\infty} \frac{1}{i!} < \frac{1}{n!n} \leq \frac{1}{n}$$

for any positive integer n. To prove the $\mathcal{E}^2$-computability of the sequence of the terms of the series, we shall use the following two-argument function in $\mathbb{N}$:

$$f_1(i,t) = \left[\frac{t}{i!}\right].$$

This function belongs to $\mathcal{E}^2$ since

$$f_1(0,t) = t, \quad f_1(i+1,t) = \left[\frac{f_1(i,t)}{i+1}\right], \quad f_1(i,t) \leq t.$$

for all i, t in $\mathbb{N}$. In addition,

$$\left|\frac{f_1(i,n+1)}{n+1} - \frac{1}{i!}\right| = \frac{1}{n+1}\left|f_1(i,n+1) - \frac{n+1}{i!}\right| < \frac{1}{n+1}$$

for any i, n in $\mathbb{N}$.

The $\mathcal{E}^2$-computability of the sequences of the terms of the series in (5) can be proved also as follows. One considers the two-argument function g_1 in $\mathbb{N}$ defined by

$$g_1(i,t) = \min(i!, t+1). \tag{6}$$

This function belongs to $\mathcal{E}^2$ since

$$g_1(0,t) = t+1, \quad g_1(i+1,t) = \min(g_1(i,t)(i+1), t+1), \quad g_1(i,t) \le t+1$$

for all i, t in $\mathbb{N}$. On the other hand,

$$\left| \frac{1}{g_1(i,n)} - \frac{1}{i!} \right| < \frac{1}{n+1}$$

for any i, n in $\mathbb{N}$.

Remark. The $\mathcal{E}^2$-computability of f_1 can be derived also from the $\mathcal{E}^2$-computability of g_1, since

$$f_1(i,t) = \left[\frac{t}{g_1(i,t)} \right]$$

for all i, t in $\mathbb{N}$.

The proof in [Skordev 2008] of the $\mathcal{E}^2$-computability of the number e can be briefly described as follows. Let $r_0, r_1, r_2, r_3, \ldots$ be the sequence of the partial sums of the series in (5), i.e.

$$r_n = \sum_{i=0}^{n} \frac{1}{i!}$$

for any $n \in \mathbb{N}$. Although this sequence of rational numbers is not $\mathcal{E}^2$-recursive, there exists a monotonically increasing sequence $k_0, k_1, k_2, \ldots$ of natural numbers such that

$$|r_{k_n} - e| < \frac{1}{n+1}$$

for any $n \in \mathbb{N}$, and the sequence $r_{k_0}, r_{k_1}, r_{k_3}, \ldots$ is $\mathcal{E}^2$-recursive. Clearly the idea of the present proof is rather different from the so described one.

3.3 $\mathcal{E}^2$-computability of Liouville's number

As well-known, the first examples of transcendental real numbers were constructed by Liouville. The most famous of them is the sum of the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{10^{i!}}.$$

This number is called now Liouville's number or Liouville's constant. It is sometimes denoted by L, and we shall adopt this notation here. We shall prove that L is $\mathcal{E}^2$-computable. For technical convenience, we shall actually prove the equivalent statement that $L + 1/10$ is $\mathcal{E}^2$-computable. Since

$$L + 1/10 = \sum_{i=0}^{\infty} \frac{1}{10^{i!}}, \tag{7}$$

we shall proceed by proving the $\mathcal{E}^2$-convergence of the series in the above equality and the $\mathcal{E}^2$-computability of the sequence of its terms. The $\mathcal{E}^2$-convergence follows from the inequalities

$$\sum_{i=n+1}^{\infty} \frac{1}{10^{i!}} < \frac{1}{10^{n!n}} \leq \frac{1}{n+1}.$$

To prove the $\mathcal{E}^2$-computability of the sequence of the terms, we consider the function

$$g_2(i,t) = \min(10^{i!}, t+1).$$

It is easy to check that

$$g_2(i,t) = g_0(9, g_1(i,t)),$$

where g_0 and g_1 are the functions defined by (4) and (6), respectively. Therefore $g_2 \in \mathcal{E}^2$. On the other hand,

$$\left|\frac{1}{g_2(i,n)} - \frac{1}{10^{i!}}\right| < \frac{1}{n+1}$$

for any i, n in $\mathbb{N}$.

Another way to prove the $\mathcal{E}^2$-computability of the sequence of the terms of the series in (7) is by considering the function

$$f_2(i,t) = \left[\frac{t}{10^{i!}}\right].$$

This function also belongs to $\mathcal{E}^2$ since

$$f_2(i,t) = \left[\frac{t}{g_2(i,t)}\right]$$

for all i, t in $\mathbb{N}$, and

$$\left|\frac{f_2(i,n+1)}{n+1} - \frac{1}{10^{i!}}\right| = \frac{1}{n+1}\left|f_2(i,n+1) - \frac{n+1}{10^{i!}}\right| < \frac{1}{n+1}$$

for any i, n in $\mathbb{N}$.

3.4 $\mathcal{E}^2$-computability of Euler's constant

To prove that Euler's constant γ is $\mathcal{E}^2$-computable, we shall use its representation

$$\gamma = \sum_{i=0}^{\infty} \left(\frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right) \right), \tag{8}$$

as well as the fact that for any $i \in \mathbb{N}$ we have the equality

$$\frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right) = \sum_{j=0}^{\infty} u(i,j), \tag{9}$$

where

$$u(i,j) = \frac{(-1)^j}{(j+2)(i+1)^{j+2}}.$$

The series in (9) is $\mathcal{E}^2$-convergent thanks to the inequality

$$\left| \sum_{j=n+1}^{\infty} u(i,j) \right| < \frac{1}{n+3}.$$

The function u is $\mathcal{E}^2$-computable, since for all $i, j, k \in \mathbb{N}$ the inequality

$$\left| \frac{(-1)^j}{(j+2)g_0(j+2,i,k)} - u(i,j) \right| < \frac{1}{2(k+1)}$$

holds, where g_0 is the function defined by (4). Therefore the sum of the series is an $\mathcal{E}^2$-computable function of i. Thus the $\mathcal{E}^2$-computability of Euler's constant will be proved if we prove the $\mathcal{E}^2$-convergence of the series in (8). To do this, we note that, by the equality (9), we have the inequalities

$$0 < \frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right) < \frac{1}{2(i+1)^2}$$

for any $i \in \mathbb{N}$, hence

$$0 < \sum_{i=n+1}^{\infty} \left(\frac{1}{i+1} - \ln\left(1 + \frac{1}{i+1}\right) \right) < \sum_{i=n+1}^{\infty} \frac{1}{2(i+1)^2} < \frac{1}{2(n+1)}$$

for all $n \in \mathbb{N}$.

4 Some comments

Although our proofs concern only four concrete real numbers, the methods used in the proofs or similar ones can be applied in many other cases. It seems that

$\mathcal{E}^2$-computability of real numbers is present much more often than one could expect.

One of the four considered numbers turned out to be $\mathcal{L}$-computable. It seems that the other of them are also $\mathcal{L}$-computable, but the proof of this requires a greater amount of technical work.

Several characterizations of the class $\mathcal{E}^2$ are known that are in the terms of computational complexity, for instance the characterization from [Ritchie 1963] according to which a function belongs to $\mathcal{E}^2$ iff it can be computed on a linear tape bounded Turing machine in the case of binary encoding of inputs and outputs. As the referee of the preliminary version [Skordev 2008] of the paper indicated, such characterizations could be useful for comparison with already known results and for further studies, and, in particular, the characterization from [Ritchie 1963] allows relating complexity of real functions as in [Ko 1991, Weihrauch 2000] to $\mathcal{E}^2$-computability.

5 Existence of $\mathcal{E}^3$-computable real numbers which are not $\mathcal{E}^2$-computable

It is shown in [Skordev 2002] (cf. footnote 9 there) that for any integer m greater than 2 there exist $\mathcal{E}^{m+1}$-computable real numbers which are not $\mathcal{E}^m$-computable. The proof from [Skordev 2002] cannot be used in the case of $m = 2$. We shall give now another proof that covers also the case of $m = 2$. Namely we shall make use of the fact that for any natural number $m \geq 2$ there exists a two-argument function in $\mathcal{E}^{m+1}$ which is universal for the one-argument functions in $\mathcal{E}^m$, i.e. each one-argument functions belonging to $\mathcal{E}^m$ can be obtained from the two-argument function in question by replacement of the first argument with some natural number.[5]

Let $m \in \mathbb{N}$, $m \geq 2$, and let h be a two-argument function from $\mathcal{E}^{m+1}$ which is universal for the one-argument functions in $\mathcal{E}^m$. We define a one-argument function g in $\mathbb{N}$ as follows: $g(0) = 0$, and, for any $k \in \mathbb{N}$,

$$g(k+1) = \begin{cases} 3g(k) & \text{if } 6g(k) + 3 \leq h(k, 2 \cdot 3^{k+1} - 1) \\ 3g(k) + 2 & \text{otherwise} \end{cases}$$

(thus $g(k+1) - 3g(k) \in \{0, 2\}$ for all $k \in \mathbb{N}$). Making use of the inequality

[5] In [Grzegorczyk 1953] a proof of this is sketched for $m > 2$. As Lars Kristiansen indicated, the truth of the statement for the case of $m = 2$ follows straightforwardly from what is written in section 6 of [Ritchie 1963], and the statement in question can be derived also from the equality $LINSPACE = \mathcal{E}^2_*$, the inclusion $ESPACE \subset \mathcal{E}^3_*$ and the fact that $ESPACE$ contains a universal function for $LINSPACE$.

$g(k) \leq 3^k - 1$, one sees that $g \in \mathcal{E}^{m+1}$. Now let

$$\alpha = \sum_{k=0}^{\infty} \frac{g(k+1) - 3g(k)}{3^{k+1}}.$$

For any natural number k, the sum of the first k terms of the above series is equal to $g(k)/3^k$, and, making use of this, we see that

$$0 \leq \alpha - \frac{g(k)}{3^k} \leq \frac{1}{3^k}$$

for all $k \in \mathbb{N}$, hence the real number α is $\mathcal{E}^{m+1}$-computable. We shall show that α is not $\mathcal{E}^m$-computable. Suppose the contrary. Then, by the case $l = 0$ of Proposition 2, one-argument functions f and g belonging to $\mathcal{E}^m$ exist such that

$$\left| \frac{f(n) - g(n)}{n+1} - \alpha \right| < \frac{1}{n+1}$$

for all $n \in \mathbb{N}$. The function $|f(n) - g(n)|$ also belongs to $\mathcal{E}^m$, and

$$\left| \frac{|f(n) - g(n)|}{n+1} - \alpha \right| < \frac{1}{n+1}$$

also holds for all $n \in \mathbb{N}$, since $\alpha \geq 0$. Let k be a natural number such that $|f(n) - g(n)| = h(k,n)$ for all $n \in \mathbb{N}$. Then

$$\left| \frac{h(k,n)}{n+1} - \alpha \right| < \frac{1}{n+1}$$

for all $n \in \mathbb{N}$. In particular, we shall have

$$\left| \frac{h(k, 2 \cdot 3^{k+1} - 1)}{2 \cdot 3^{k+1}} - \alpha \right| < \frac{1}{2 \cdot 3^{k+1}}\,.$$

We shall now consider separately the case, when $6g(k) + 3 \leq h(k, 2 \cdot 3^{k+1} - 1)$, and the case, when $6g(k) + 3 > h(k, 2 \cdot 3^{k+1} - 1)$. We shall get a contradiction in both of them. In the first of these cases, we have

$$\begin{aligned} \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k} &\leq \frac{h(k, 2 \cdot 3^{k+1} - 1)}{2 \cdot 3^{k+1}} < \alpha + \frac{1}{2 \cdot 3^{k+1}} \\ &\leq \frac{g(k+1)}{3^{k+1}} + \frac{1}{3^{k+1}} + \frac{1}{2 \cdot 3^{k+1}} = \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k}\,, \end{aligned}$$

and this is impossible. In the second of the cases, we have

$$\begin{aligned} \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k} &> \frac{h(k, 2 \cdot 3^{k+1} - 1)}{2 \cdot 3^{k+1}} > \alpha - \frac{1}{2 \cdot 3^{k+1}} \\ &\geq \frac{g(k+1)}{3^{k+1}} - \frac{1}{2 \cdot 3^{k+1}} = \frac{g(k)}{3^k} + \frac{1}{2 \cdot 3^k}\,, \end{aligned}$$

and this is again impossible.

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An observation made by Peter Peshev in 2005 stimulated the author to look for extending his knowledge about $\mathcal{E}^2$-computability in analysis. The observation was that almost all constructions described in [Rosenbloom 1945] can actually be accomplished by means of operators which are not only $\mathcal{E}^3$-computable, but even $\mathcal{E}^2$-computable (sufficiency of the $\mathcal{E}^3$-computable operators was the initially expected result of Peshev's study).

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