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Notions of Probabilistic Computability on Represented Spaces

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Abstract: We define and compare several probabilistic notions of computability for mappings from represented spaces (that are equipped with a measure or outer measure) into computable metric spaces. We thereby generalize definitions by [Ko 1991] and Parker (see [Parker 2003, Parker 2005, Parker 2006]), and furthermore introduce the new notion of *computability in the mean*. Some results employ a notion of *computable measure* that originates in definitions by [Weihrauch 1999] and [Schröder 2007]. In the spirit of the well-known Representation Theorem (see [Weihrauch 2000]), we establish dependencies between the probabilistic computability notions and classical properties of mappings. We furthermore present various results on the computability of vector-valued integration, composition of mappings, and images of measures. Finally, we discuss certain measurability issues arising in connection with our definitions.

Key Words: computable analysis, computable measures, probabilistic computation
Category: F.1.1, F.1.2, F.4.1

1 Introduction

1.1 Motivation

The considerations in this article are inspired by real-world situations like the following: An agent (i.e. a person, a machine or a combination of such) has the task to perform a measurement ξ of a physical magnitude. Then a 2^{-k}-approximation to the value $f(\xi)$ shall be computed, where $k \in \mathbb{N}$ is a given precision parameter and $f : X \to Y$ is a given function that maps the state space X of the magnitude into a metric space (Y, d). *When it comes to computations, the abilities of the agent shall be modeled by a Turing machine*; so the results of the measurement must be available in machine readable form, i.e. encoded as a string over some finite alphabet Σ. The space X will typically not be countable, so the value ξ must be encoded as an infinite string. We assume that there is a surjective partial mapping $\delta :\subseteq \Sigma^\omega \to X$, a so-called *representation* of X, and that the measuring device puts out a δ-name $p \in \mathrm{dom}(\delta)$ of ξ, i.e. $\delta(p) = \xi$.[1] *We do not model the details of this process, so we can make no assumptions about what particular δ-name of ξ will finally be extracted from the measurement.* The

[1] One should note here that the requirement of producing a δ-name from the outcome of the measurement might be problematic in practice because the magnitude might change over time.

δ-name is progressively written onto the input tape of a Turing machine.[2] The codomain Y of f is typically not countable either, but we assume that Y has a countable dense subset A, and that there is a partial mapping $\alpha :\subseteq \Sigma^* \to A$, a so called *notation* of A. The question is: Is there a TM that takes a δ-name p of some measured ξ as well as a precision parameter k as inputs and halts (after a finite number of steps) with a word w on its output tape such that $d(f(\xi), \alpha(w)) \leq 2^{-k}$?

There are functions f for which there does not exist any Turing machine that could perform the above task. This is the case, for example, if there is a name $p \in \mathrm{dom}(\delta)$ and a precision parameter $k \in \mathbb{N}$ such that no prefix of p already determines $f(\delta(p))$ up to precision 2^{-k}. But even for functions for which such a discontinuity does not occur, there is possibly no Turing machine for the above task, simply because there are "too many functions" and "too few Turing machines"; so far, however, no one has given an example of a function of the latter kind, that comes up naturally in an application.

Now, additionally, assume that there is a σ-algebra $\mathcal{S}$ and a probability measure P such that $(X, \mathcal{S}, P)$ is a probability space, and that the observed magnitude is distributed according to P. The presence of a probability distribution allows us to weaken the demands on the Turing machine above in several meaningful ways; in particular, we might only ask for a TM that

(I) behaves correctly on P-almost every value of ξ, or

(II) behaves correctly, except on a set whose probability is at most 2^{-k} for any desired k, or

(III) produces an approximation whose *expected* error is at most 2^{-k} for any desired k.

In the following, it will be our aim to develop the foundations of a representation-based computability theory for these three settings. Although probability measures are most interesting for applications, we will also consider more general measures and outer measures whenever meaningful.

The general theory of Turing machine computability via representations is developed in the textbook [Weihrauch 2000]; the present work is formulated to fit into this framework. We will recall some basic notions from computable analysis below, but refer to [Weihrauch 2000] for some more technical definitions.

We assume that the reader has a basic background on measure theory and descriptive set theory. All facts we use can be found in any introductory textbook; we occasionally refer to [Kallenberg 2002, Kechris 1995].

[2] Each character of the name is extracted from the measurement before or just when the TM queries the corresponding tape cell for the first time.

1.2 Overview of the present work

In Section 2, we recall some definitions and results about continuity and computability via representations. We recall the definitions of effective/computable topological spaces (and what it means for them to be computably regular and computably quasi-compact) and computable metric spaces. We introduce several (multi-)representations of Borel measures and prove a result on computable measures on computable metric spaces. We finally recall some less common notions from measure theory.

Section 3 contains precise definitions of the three weakened concepts of computability corresponding to items (I), (II) and (III) above; by considering mixed settings, we arrive at a total of five concepts. Each of these computability concepts is accompanied by a corresponding relative continuity concept; multi-representations of mappings that are relatively continuous in the respective sense will be introduced.

The focus of Section 4 is on working out relations between the just mentioned probabilistic forms of relative continuity and classical properties of the representations, spaces, measures and mappings.

In Section 5, we study the pairwise relations between the five concepts: we either give a strong counter-example showing that one concept does not imply the other, show that one concept always implies the other, or show that one concept implies the other under mild additional assumptions.

Section 6 contains some positive results on the computability of integration of probabilistically computable mappings. As the proofs are essentially the same, we will not restrict ourselves to real-valued integrands, but we will prove the results for vector-valued integrands.

In the final section, we take up three more or less unrelated natural questions. The first two are: "Is the composition of probabilistically computable mappings again probabilistically computable?" and "Is the image of a computable measure under a probabilistically computable mapping again computable?" The third question is about the measurability of a certain "local error function" and comes up naturally in Section 3.

1.3 Related work

The book [Ko 1991] deals with computability and complexity of real functions in a way that is consistent with [Weihrauch 2000]. For functions $f : [0,1] \to \mathbb{R}$ and the Lebesgue measure λ, a weakened notion of computability, that corresponds to item (II) above, is defined and studied in Chapter 5 of that book. Building on Ko's definitions, probabilistic computability notions for characteristic functions of subsets of $\mathbb{R}^n$ have been studied by Parker in [Parker 2003, Parker 2005, Parker 2006]; Parker's definitions correspond to concepts (I) and (II). The works of Ko and Parker can be said to have taken a "bottom-up" approach by paying most attention to Euclidean spaces; we are attempting to go "top-down" and consider very general definitions.

The (multi-)representations of Borel-measures to be introduced below generalize/modify definitions by [Weihrauch 1999] and [Schröder 2007]. The reader might also find the articles [Müller 1999] and [Schröder and Simpson 2006] of interest. Furthermore, we would also like to mention the article [Gács 2005], whose definition of a computable probability measure is equivalent to Schröder's for the special case of metric spaces.

[Wu and Weihrauch 2006] introduce *computable measure spaces*; this notion is further studied in e.g. [Wu and Ding 2005, Wu and Ding 2006]. The focus of those works, however, is on representations (and the induced computability) of measurable sets and measurable functions, while we are interested in computability on points in a represented space that is in addition equipped with a measure.

Furthermore, measure and integration have been treated from the viewpoints of constructive mathematics (see [Bishop and Bridges 1985]), domain theory (see [Edalat 1993, Edalat 1995]), and digital topology (see [Webster 2006]). It is beyond the scope of this article to work out the relations between these approaches and the present one.

The main motivation for the present work was to establish weakened computability notions that correspond to weakened notions of *solvability* (more precisely the "probabilistic setting" and the "average-case setting") studied in information-based complexity (see [Traub et al. 1988]). IBC is mainly concerned with numerical problems on function spaces and uses an algebraic (aka "real number"-) model of computation. We hope that our definitions and results will be useful for studying numerical problems in the Turing machine model. A first application in this direction is given in [Bosserhoff], where the author answer a question posed by [Traub and Werschulz 1999].

2 Preliminaries

2.1 Computable analysis via representations

Let Σ be a finite alphabet containing at least two symbols, and $W \in \{\Sigma^*, \Sigma^\omega\}$. A *naming system* for a non-empty set X is a surjective partial mapping $\delta :\subseteq W \to X$. If $W = \Sigma^*$, a naming system is called a *notation*; if $W = \Sigma^\omega$, a naming system is called a *representation.* If X_1 and X_2 are sets with naming systems $\delta_1 :\subseteq W_1 \to X_1$, $\delta_2 \subseteq: W_2 \to X_2$, and f is a mapping $X_1 \to X_2$, then a mapping $h :\subseteq W_1 \to W_2$ is called a (δ_1, δ_2)*-realization* for f, if for every $p \in \mathrm{dom}(\delta_1)$, one has $h(p) \in \mathrm{dom}(\delta_2)$ and $(\delta_2 \circ h)(p) = (f \circ \delta_1)(p)$. f is called (δ_1, δ_2)-continuous (-computable), if there exists a continuous (computable) (δ_1, δ_2)-realization for

f, where the latter continuity is evaluated relative to the discrete topology on Σ^* and the Cantor topology on Σ^ω.

A naming system δ of some set X is said to be continuously (computably) *reducible* to another naming system δ' of X, if the identity on X is (δ, δ')-continuous (-computable); we write $\delta \leq_t \delta'$ ($\delta \leq \delta'$). We write $\delta \equiv_t \delta'$ for $\delta \leq_t \delta' \wedge \delta' \leq_t \delta$ (and $\delta \equiv \delta'$ for $\delta \leq \delta' \wedge \delta' \leq \delta$).

It is sometimes convenient to represent the elements of X by names that do not necessarily contain enough information to identify the elements uniquely. One calls such a surjective multi-valued mapping $\delta :\subseteq W \rightrightarrows X$ a *multi-notation* or *multi-representation*, respectively. The notions defined above for single-valued naming systems have natural extensions for multi-valued naming systems; see e.g. [Schröder 2002a].

Below, we will frequently use canonical notations $\nu_{\mathbb{N}}$ of $\mathbb{N}$ and $\nu_{\mathbb{Q}}$ of $\mathbb{Q}$, as well as the representations ρ, ρ_C, $\rho_<$, $\rho_>$ of $\mathbb{R}$, and $\overline{\rho}_<$ of $\mathbb{R} \cup \{-\infty, \infty\}$ just as defined in [Weihrauch 2000]. We will work with the wrapping function $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1 a_2 \ldots a_n) := \mathtt{110}a_1\mathtt{0}a_2\mathtt{0}\ldots a_n\mathtt{011}$. We will also use standard devices to construct new naming systems from given ones; these are described in [Weihrauch 2000, Section 3.3]. For example, if δ is a naming system of X then $[\delta]^n$ shall be a representation of X^n, and $[\delta]^\omega$ shall be a representation of the set X^ω of sequences in X; if $Y \subseteq X$, then $\delta|^Y$ shall be the representation of Y that is obtained by restricting δ to $\delta^{-1}(Y)$. We additionally use the convention: If X is a set with a naming system δ, then put

$$[\delta]^{<\omega} := [\delta]^1 \vee [\delta]^2 \vee [\delta]^3 \vee \cdots,$$

(cf. [Weihrauch 2000, Definition 3.3.11.2]), i.e. $[\delta]^{<\omega}$ is a naming system of the disjoint union $\bigcup_{n\geq 1} X^n$.

If X is a set with a representation δ, we shall write

$$W(\delta, w) := \delta(w\Sigma^\omega \cap \mathrm{dom}(\delta))$$

for every $w \in \Sigma^*$. We denote by $\sigma(\delta^{-1})$ the smallest σ-algebra on X which contains all sets $W(\delta, w)$, $w \in \Sigma^*$.

We finally note that any topological space that allows a continuous representation is hereditarily Lindelöf, i.e. every open cover of any subspace contains a countable subcover.

2.2 Computable topological spaces

Below, we will frequently work with the notion of an *effective/computable topological space* and its *standard representation* (cf. [Weihrauch 2000, Section 3.2]):

Definition 2.1 (Effective/computable topological space). An *effective topological space* is a tuple (X, β, ϑ), where X is a nonempty set, β is a countable subbase of a T_0-topology on X, and ϑ is a notation of β. The *standard representation* δ associated with (X, β, ϑ) is defined by

$$(p \in \mathrm{dom}(\delta) \text{ and } \iota(w) \lhd p) \;\Rightarrow\; w \in \mathrm{dom}(\vartheta)$$
$$\delta(p) = x \;:\Longleftrightarrow\; \{U \in \beta \;:\; x \in U\} = \{\vartheta(w) \;:\; \iota(w) \lhd p\}$$

for all $w \in \Sigma^*$, $x \in X$ and $p \in \Sigma^\omega$. ($\lhd$ denotes the subword relation.) (X, β, ϑ) is a *computable topological space* if $\mathrm{dom}(\vartheta)$ is computably enumerable (c.e.).[3]

Definition 2.2. Let (X, β, ϑ) be an effective topological space. In a canonical way, one can define

- a notation $\vartheta^\cap$ of the set $\beta^\cap$ of all finite intersections of elements of β plus the empty set.

- a notation ϑ_{alg} of the algebra $\mathcal{A}(\beta)$ generated by β.

A representation $\vartheta_<$ of the hyperspace $\mathcal{O}(X)$ of open subsets of X shall then be defined by

$$\vartheta_<(p) = \bigcup_i U_i \;:\Longleftrightarrow\; [\vartheta^\cap]^\omega(p) = (U_i)_i.$$

The following two lemmas can be shown easily:

Lemma 2.3. *Let (X, β, ϑ) be an effective topological space. Then the following mappings are computable w.r.t. the canonical representations given in Definition 2.2: Finite intersection on $\beta^\cap$; complementation, finite union and finite intersection on $\mathcal{A}(\beta)$; finite and countable union and finite intersection on $\mathcal{O}(X)$; the embeddings $\beta \hookrightarrow \beta^\cap$, $\beta^\cap \hookrightarrow \mathcal{A}(\beta)$, $\beta^\cap \hookrightarrow \mathcal{O}(X)$.* □

Lemma 2.4. *Let (X, β, ϑ) be a computable topological space with standard representation δ. Put*

$$D := \{w \in \Sigma^* \;:\; \iota(v) \lhd w \Rightarrow v \in \mathrm{dom}(\vartheta)\}.$$

Then D is c.e., and for every $w \in D$ one has $W(\delta, w) = \bigcap_{\iota(v) \lhd w} \vartheta(v)$. The mapping $D \to \beta^\cap,\ w \mapsto W(\delta, w)$, is $(\mathrm{id}_{\Sigma^*}|^D, \vartheta^\cap)$-computable. □

Computably regular topological spaces have been defined in [Schröder 1998]; we use the characterization given in [Schröder 1998, Lemma 4.2] as a definition:

[3] This is a sightly weaker definition then the one found in [Weihrauch 2000]. We chose this definition because it allows a simpler formulation of Lemma 2.8(ii) (as compared to [Weihrauch 2000, Theorem 8.1.4.2]).

Definition 2.5 (Computably regular space). An effective topological space (X, β, ϑ) is *computably regular* if from every ϑ-name of every $V \in \beta$ one can $[\vartheta^{\cap}, \vartheta_{<}]^{\omega}$-compute a sequence $(V_n, U_n)_n$ in $\beta^{\cap} \times \mathcal{O}(X)$ such that

$$V = \bigcup_n V_n \qquad \text{and} \qquad (\forall\, n)\ [X \setminus V \subseteq U_n \text{ and } V_n \cap U_n = \emptyset].$$

Definition 2.6 (Computably quasi-compact space). Let (X, β, ϑ) be an effective topological space. Put

$$C := \{(U_n)_n \in (\beta^{\cap})^{\omega}\ :\ \bigcup_n U_n = X\}.$$

(X, β, ϑ) is *computably quasi-compact* if from every $[\vartheta^{\cap}]^{\omega}$-name of every $(U_n)_n \in C$ one can compute an $m \in \mathbb{N}$ such that $\bigcup_{n \leq m} U_n = X$.

2.3 Computable metric spaces

Definition 2.7. A triple (X, d, α) is a *computable metric space*, if (X, d) is a metric space and $\alpha :\subseteq \Sigma^* \to A$ is a notation of a dense subset A of X, such that $\mathrm{dom}(\alpha)$ is c.e. and the restriction of d to $A \times A$ is (α, α, ρ)-computable. The *Cauchy representation* $\delta_X :\subseteq \Sigma^\omega \to X$ associated with a computable metric space is defined by

$$\delta_X(p) = x :\Longleftrightarrow \begin{cases} \text{there are words } w_0, w_1, \ldots \in \mathrm{dom}(\alpha) \\ \ \text{such that } p = \iota(w_0)\iota(w_1)\ldots, \\ (\alpha(w_i)) \text{ converges to } x, \text{ and} \\ d(\alpha(w_i), \alpha(w_j)) \leq 2^{-i} \text{ for } i < j. \end{cases}$$

For more information on computable metric spaces see [Weihrauch 2000, Section 8.1].

For any metric space (X, d) define

$$(\forall\, x_0 \in X,\, \epsilon > 0)\ B(x_0, \epsilon) := \{x \in X\ :\ d(x_0, x) < \epsilon\}$$

and

$$(\forall\, x_0 \in X,\, \epsilon \geq 0)\ \overline{B}(x_0, \epsilon) := \{x \in X\ :\ d(x_0, x) \leq \epsilon\}$$

The following is shown easily:

Lemma 2.8. *Let (X, d, α) be a computable metric space. A computable topological space (X, β, ϑ) can be defined by putting*

$$\beta := \{B(a, r)\ :\ a \in \mathrm{range}(\alpha),\ r \in \mathbb{Q} \cap]0, \infty[\}$$

and

$$\vartheta\langle u, v\rangle := B(\alpha(u), \nu_{\mathbb{Q}}(v)).$$

If δ is the corresponding standard representation, then $\delta_X \equiv \delta$. (X, β, ϑ) is computably regular. □

2.4 Computable measures

[Weihrauch 1999] considers a representation of probability measures on the unit interval. [Schröder 2007] generalizes this definition to Borel probability measures on arbitrary admissibly represented topological spaces. On the one hand, we will only consider this definition for the special case of effective topological spaces with standard representation. On the other hand, we will extend the representation to unbounded measures. The latter can be done in a straight-forward manner, but then the representation will not be single-valued anymore. [4]

Assumption 2.9 In this subsection, we assume that (X, β, ϑ) is an effective topological space.

Definition 2.10. A multi-representation $\vartheta_{\mathcal{M}<}$ of the class $\mathcal{M}(X)$ of Borel measures on X is given by

$$\nu \in \vartheta_{\mathcal{M}<}(p) \ :\Longleftrightarrow\ [\vartheta_< \to \overline{\rho}_<](p) = \nu|_{\mathcal{O}(X)}.$$

Lemma 2.11. *The restriction of $\vartheta_{\mathcal{M}<}(p)$ to locally finite measures is single-valued.*

Proof. Let ν_1 and ν_2 be locally finite Borel measures on X with $\nu_1|_{\mathcal{O}(X)} = \nu_2|_{\mathcal{O}(X)}$. $\mathcal{O}(X)$ is a $\cap$-stable generator of $\mathcal{B}(X)$, and – by the local finiteness of ν_1, ν_2 and the Lindelöf property of X – contains an ascending sequence $(O_n)_n$ with $\bigcup_n O_n = X$ and $\nu_1(O_n) = \nu_2(O_n) < \infty$ for all n. It is well-known that this implies $\nu_1 = \nu_2$. □

By including information on the value $\nu([0,1])$, [Weihrauch 1999] also defines a representation of bounded Borel measures on $[0,1]$. We generalize this idea:

Definition 2.12. A representation of the class $\mathcal{M}_0(X)$ of finite Borel measures on X is given by

$$\vartheta^0_{\mathcal{M}<}\langle p, q\rangle = \nu \ :\Longleftrightarrow\ \vartheta_{\mathcal{M}<}(p) = \nu \text{ and } \rho_>(q) = \nu(X).$$

We will furthermore need the following strong representation:

Definition 2.13. A representation of the class $\mathcal{M}_0(X)$ of finite Borel measures on X is given by

$$\vartheta_{\mathcal{M}=}(p) = \nu \ :\Longleftrightarrow\ [\vartheta_{\mathrm{alg}} \to \rho](p) = \nu|_{\mathcal{A}(\beta)}.$$

[4] This is because the representation only contains information on the values of the measure on open sets. Unbounded measures, however, are not necessarily defined uniquely by these values.

It is easy to see that $\vartheta_{\mathcal{M}=} \leq \vartheta^0_{\mathcal{M}<}$. It has been pointed out by Weihrauch (for the case $X = [0,1]$), however, that $\vartheta_{\mathcal{M}=}$ has a number of undesirable properties (see [Weihrauch 1999, Theorem 2.7]). We will work with this representation anyway, because we do not see how to do without such strong information when it comes to computing the integration operator on non-metric spaces (see Theorem 6.2 below).

There are $\vartheta^0_{\mathcal{M}<}$-computable measures that are not $\vartheta_{\mathcal{M}=}$-computable[5]. From a $\vartheta^0_{\mathcal{M}<}$-name of a measure ν on a metric space, however, one can effectively find a basis $(U_n)_n$ of open sets and a sequence $(\widetilde{U}_n)_n$ of open sets such that $X \setminus U_n$ and $\widetilde{U}_n$ are ν-equivalent, as we shall now demonstrate. (A similar idea already appears in [Weihrauch 1999, Proof of Theorem 3.6].) This will be useful in the proofs of Theorems 5.11 and 6.3 below.

Assumption 2.14 We additionally assume that (X, β, ϑ) is the computable topological space derived from a computable metric space (X, d, α) (see Lemma 2.8).

Lemma 2.15. *Put*

$$Q := \{(a, s, t) \in \text{range}(\alpha) \times \mathbb{Q} \times \mathbb{Q} \ : \ 0 < s < t\}.$$

Then from every $\vartheta^0_{\mathcal{M}<}$-name of every $\nu \in \mathcal{M}_0(X)$ one can $[[\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]|^Q \to \rho]$-compute a mapping $g : Q \to \mathbb{R}$ such that

$$(\forall\, (a, s, t) \in Q)\ \big[s \leq g(a, s, t) \leq t \text{ and } \nu\big(\overline{B}(a, g(a, s, t)) \setminus B(a, g(a, s, t))\big) = 0\big].$$

Proof. Let ν be a $\vartheta^0_{\mathcal{M}<}$-encoded input measure. We demonstrate how to compute a suitable g. Let $(a, s, t) \in Q$ be an $[\alpha, \nu_{\mathbb{Q}}, \nu_{\mathbb{Q}}]$-encoded input tuple to g. Put

$$Q' := \{(s', t') \in \mathbb{Q} \times \mathbb{Q} \ : \ 0 < s' < t'\}$$

and

$$(\forall\, (s', t') \in Q')\ R(s', t') := \overline{B}(a, t') \setminus B(a, s').$$

From any $(s', t') \in Q'$ we can $\vartheta_<$-compute $X \setminus R(s', t')$, and hence we can $\rho_>$-compute $\nu(X) - \nu(X \setminus R(s', t')) = \nu(R(s', t'))$. For all $(s', t') \in Q'$, we have

$$\inf\{\nu(R(s'', t'')) \ : \ s'', t'' \in \mathbb{Q},\, s' \leq s'' < t'' \leq t'\} = 0$$

(because otherwise there would be a number $c > 0$ and a sequence

$$R(s''_0, t''_0), R(s''_1, t''_1), \ldots$$

[5] For example: Let $(x_n)_n$ be a computable sequence of non-negative rationals such that $c := \sum_n x_n < 1$ is not computable from the right. Now consider the measure ν defined by $\nu(A) := (1 - c)\chi_A(0) + \sum_n x_n \chi_A((n + 1)^{-1})$.

of pairwise disjoint subsets of $R(s', t')$ with $\nu(R(s''_i, t''_i)) > c$ for all $i \in \mathbb{N}$, which implies $\nu(R(s', t')) = \infty$.) We can compute a mapping $h : Q' \times \mathbb{N} \to \mathbb{Q} \times \mathbb{Q}$ such that

$$h((s', t'), k) = (s'', t'') \Longrightarrow [s' < s'' < t'' \leq t' \text{ and } \nu(R(s'', t'')) \leq 2^{-k}].$$

By repeated use of h, we can compute a sequence $(s_k, t_k)_k$ in $\mathbb{Q} \times \mathbb{Q}$ such that $s \leq s_0 < t_0 \leq t$ and

$$s_k \leq s_{k+1} < t_{k+1} \leq t_k, \qquad t_k - s_k \leq 2^{-k}, \qquad \nu(R(s_k, t_k)) \leq 2^{-k}$$

for all $k \in \mathbb{N}$. We can hence ρ-compute $\lim_{k\to\infty} r_k = \lim_{k\to\infty} s_k =: g(a, s, t)$. One has $s \leq g(a, s, t) \leq t$ and

$$\nu\big(\overline{B}(a, g(a, s, t)) \setminus B(a, g(a, s, t))\big) = \nu\big(\bigcap_k R(s_k, t_k)\big) = 0.$$

□

Corollary 2.16. *From every* $[\vartheta^0_{\mathcal{M}<}, [\vartheta^\cap]^\omega]$*-name of every* $(\nu, (V_n)_n) \in \mathcal{M}_0(X) \times (\beta^\cap)^\omega$ *one can* $[\vartheta_<, \vartheta_<, \nu_\mathbb{N}]^\omega$*-compute a sequence* $(U_m, \widetilde{U}_m, n_m)_m$ *in* $\mathcal{O}(X) \times \mathcal{O}(X) \times \mathbb{N}$ *such that*

$$\bigcup_m U_m = \bigcup_n V_n \qquad \textit{and} \qquad U_m \subseteq V_{n_m} \qquad \textit{and} \qquad \nu((X \setminus U_m) \triangle \widetilde{U}_m) = 0.$$

Proof. Let $\nu \in \mathcal{M}_0(X)$ and $(V_n)_n \in (\beta^\cap)^\omega$ be given in the specified representations. It is easy to see that the computable metric structure of the space allows us to $[\vartheta]^{\omega\times\omega}$-compute a double-sequence $(\widetilde{V}_{n,m})_{n,m} \in \beta^{\omega\times\omega}$ such that $\bigcup_m \widetilde{V}_{n,m} = V_n$. We can hence reduce the algorithm to a procedure that takes as input a ϑ-encoded element $B(a, r)$ of β and puts out a sequence $(U_m, \widetilde{U}_m)_m \in (\mathcal{O}(X) \times \mathcal{O}(X))^\omega$ such that $\bigcup_m U_m = B(a, r)$, $U_m \cap \widetilde{U}_m = \emptyset$, and $\nu(U_m) + \nu(\widetilde{U}_m) = \nu(X)$. So suppose we are given an $[\alpha, \nu_\mathbb{Q}]$-name of some (a, r). Apply the algorithm from Lemma 2.15 to ν; let g be the result. Put $r_n := g(a, r(1 - 2^{-(n+1)}), r)$ and choose $U_n = B(a, r_n)$ and $\widetilde{U}_n = X \setminus \overline{B}(a, r_n)$ (these sets can clearly be $\vartheta_<$-computed). □

Corollary 2.17. *Fix a* $\vartheta^0_{\mathcal{M}<}$*-computable measure* ν*. Then there exists a computable topological space* $(X, \beta^\nu, \vartheta^\nu)$ *such that*

1. ν *is* $\vartheta^\nu_{\mathcal{M}=}$*-computable,*
2. $\delta \equiv \delta^\nu$*, where* δ *and* δ^ν *are the standard representations of* (X, β, ϑ) *and* $(X, \beta^\nu, \vartheta^\nu)$*, respectively.*
3. $(X, \beta^\nu, \vartheta^\nu)$ *is computably regular.*

Proof. Apply the algorithm from Lemma 2.15 to ν; let g be the result. Let Q be as in Lemma 2.15 and choose

$$\beta^\nu := \{B(a, g(a, r, s)) \ : \ (a, r, s) \in Q\},$$
$$\vartheta^\nu(w) = B(a, g(a, r, s)) \ :\Longleftrightarrow \ [\alpha, \nu_\mathbb{Q}, \nu_\mathbb{Q}](w) = (a, r, s).$$

It is not hard to verify items 2. and 3. It remains to show that $\nu|_{\mathcal{A}(\beta^\nu)}$ is $(\vartheta^\nu_{\mathrm{alg}}, \rho)$-computable.

We consider the correspondence G on $\mathcal{A}(\beta^\nu) \times (\mathcal{O}(X) \times \mathcal{O}(X))$ with

$$(V, (U_1, U_2)) \in G \ :\Longleftrightarrow \ \nu(V \triangle U_1) = \nu((X \setminus V) \triangle U_2) = 0.$$

We can compute a $h : \mathrm{dom}(\vartheta^\nu) \to \mathrm{dom}([\vartheta_<, \vartheta_<])$ such that

$$(\forall\, (a, r, s) \in Q) \ \big[[\alpha, \nu_\mathbb{Q}, \nu_\mathbb{Q}](w) = (a, r, s)$$
$$\Longrightarrow \ [\vartheta_<, \vartheta_<](h(w)) = (B(a, g(a, r, s)), X \setminus \overline{B}(a, g(a, r, s)))\big].$$

Note that

$$(\forall\, w \in \mathrm{dom}(\vartheta^\nu)) \ (\vartheta^\nu(w), [\vartheta_<, \vartheta_<](h(w))) \in G.$$

The correspondence G has the following properties:

$$(V, (U_1, U_2)) \in G \Rightarrow (X \setminus V, (U_2, U_1)) \in G,$$
$$(V, (U_1, U_2)),\ (V', (U_1', U_2')) \in G \Rightarrow (V \cup V', (U_1 \cup U_1', U_2 \cap U_2')) \in G,$$
$$(V, (U_1, U_2)),\ (V', (U_1', U_2')) \in G \Rightarrow (V \cap V', (U_1 \cap U_1', U_2 \cup U_2')) \in G.$$

In view of these properties and Lemma 2.3, one can extend h to a computable $h' : \mathrm{dom}(\vartheta^\nu_{\mathrm{alg}}) \to \mathrm{dom}([\vartheta_<, \vartheta_<])$ with

$$(\forall\, w \in \mathrm{dom}(\vartheta^\nu_{\mathrm{alg}})) \ (\vartheta^\nu_{\mathrm{alg}}(w), [\vartheta_<, \vartheta_<](h'(w))) \in G.$$

Let a $\vartheta^\nu_{\mathrm{alg}}$-input $V \in \mathcal{A}(\beta^\nu)$ be given. Using h', we can $\vartheta_<$-compute sets $U_1, U_2 \in \mathcal{O}(X)$ such that $(V, (U_1, U_2)) \in G$. We can $\rho_<$-compute $\nu(U_1)$ and $\nu(U_2)$ by assumption. Because $\nu(U_1) = \nu(X) - \nu(U_2)$, we can also $\rho_>$-compute $\nu(U_1)$. Finally note that $\nu(V) = \nu(U_1)$ because $\nu(V \triangle U_1) = 0$. $\square$

2.5 From measure theory

2.5.1 Completion of a measure space

Let $(X, \mathcal{S}, \nu)$ be a measure space. A set $N \subseteq X$ is called *ν-null* if there is a set $B \in \mathcal{S}$ with $\nu(B) = 0$ and $N \subseteq B$. A property $P \subseteq X$ is said to *hold ν-almost everywhere (ν-a.e.)* if $X \setminus P$ is ν-null. The σ-algebra $\mathcal{S}_\nu$ generated by $\mathcal{S}$ and all ν-null sets is called the *completion of $\mathcal{S}$ w.r.t. ν*. $\mathcal{S}_\nu$ contains exactly the sets of the form $A \cup N$ with $A \in \mathcal{S}$ and N ν-null. We call the elements of $\mathcal{S}_\nu$ the *ν-measurable* sets. The measure ν extends to a measure $\overline{\nu}$ on $\mathcal{S}_\nu$ by putting $\overline{\nu}(A \cup N) = \nu(A)$. A measure space that is identical to its completion is called *complete*.

Lemma 2.18. *Let $(X, \mathcal{S}, \nu)$ be a complete measure space and $(Y, \mathcal{S}')$ a measurable space. Let $f : X \to Y$ be a mapping such that $f|_{X \setminus N}$ is $(\mathcal{S} \cap (X \setminus N), \mathcal{S}')$-measurable for some ν-null set N. Then f is $(\mathcal{S}, \mathcal{S}')$-measurable.* □

2.5.2 Outer measures

An *outer measure* on a set X is a set function $\mu^* : 2^X \to [0, \infty]$ such that

$$\mu^*(\emptyset) = 0, \qquad A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B), \qquad \mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n).$$

A set $A \subseteq X$ is called *μ^*-measurable* if

$$(\forall\, E \subseteq X)\ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

The μ^*-measurable sets form a σ-algebra MEAS_{μ^*}. Restricting μ^* to MEAS_{μ^*} yields a complete measure space.

Let $(X, \mathcal{S}, \nu)$ be a measure space. The measure ν induces an outer measure ν^* via

$$\nu^*(A) := \inf\{\nu(B)\ :\ B \in \mathcal{S},\ A \subseteq B\}.$$

If ν is σ-finite, it turns out that $\mathrm{MEAS}_{\nu^*} = \mathcal{S}_\nu$, and that $\overline{\nu}$ and ν^* coincide on this σ-algebra. It is known that not every outer measure is induced by a measure.

The following two results are actually well-known but usually not stated for outer measures. We will use the second one in the proof of Proposition 4.2.

Lemma 2.19 (Cantelli Theorem). *Let X be a set with an outer measure μ^*. Then for every sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of X with $\sum_n \mu^*(A_n) < \infty$, we have $\mu^*\big(\limsup_n A_n\big) = 0$, where $\limsup_n A_n := \bigcap_n \bigcup_{k \geq n} A_k$.*

Proof. One has $\mu^*\big(\bigcap_n \bigcup_{k \geq n} A_k\big) \leq \mu^*\big(\bigcup_{k \geq m} A_k\big) \leq \sum_{k \geq m} \mu^*(A_k)$ for every $m \in \mathbb{N}$. □

For a topological space Y, let $\mathcal{B}(Y)$ denote the *Borel σ-algebra* on Y, i.e. the σ-algebra generated by the topology.

Lemma 2.20. *Let X be a set with an outer measure μ^*. If $(f_n)_{n \in \mathbb{N}}$ is a sequence of $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable mappings from X into a metric space (Y, d), and $f : X \to Y$ is an arbitrary mapping with*

$$(\forall\, n \in \mathbb{N})\ \mu^*([d(f_n, f) > 2^{-n}]) \leq 2^{-n}, \tag{1}$$

then[6] *f is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.*

[6] $[d(f_n, f) > 2^{-n}]$ denotes the set $\{x \in X\ :\ d(f_n(x), f(x)) > 2^{-n}\}$. In the following, similar expressions are to be interpreted accordingly.

Proof. Define $G := \{x \in X \ : \ f_n(x) \to f(x)\}$. By Cantelli's Theorem and (1), we have

$$\mu^*(X \setminus G) \leq \mu^*([(\forall n)(\exists k \geq n)\ d(f_k, f) > 2^{-k}]) = 0.$$

From the completeness of $(X, \mathrm{MEAS}_{\mu^*}, \mu^*)$, we especially have that G is μ^*-measurable. [Kallenberg 2002, Lemma 1.10(ii)] implies that $f|_G = \lim f_n|_G$ is $(\mathrm{MEAS}_{\mu^*} \cap G, \mathcal{B}(Y))$-measurable. The claim now follows from Lemma 2.18. □

2.5.3 Outer integrals

Let $(X, \mathcal{S}, \nu)$ be a measure space, and let $h : X \to [0, \infty]$ be an arbitrary function. We define the *outer integral of h w.r.t. ν* as

$$\int^* h\, d\nu := \inf\left\{\int g\, d\nu \ : \ g \text{ is } (\mathcal{S}, \mathcal{B}(\mathbb{R}))\text{-measurable, } h \leq g\right\}.$$

One easily verifies:

Lemma 2.21. *1. The outer integral is monotone, i.e. $h_1 \leq h_2 \Rightarrow \int^* h_1\, d\nu \leq \int^* h_2\, d\nu$.*

2. The outer integral is sublinear, i.e. $\int^*(h_1 + h_2)\, d\nu \leq \int^* h_1\, d\nu + \int^* h_2\, d\nu$ and $\int^* th\, d\nu = t\int^* h\, d\nu$ for all $t \in [0, \infty)$.

3. For every $A \subseteq X$, one has $\nu^*(A) = \int^* \chi_A\, d\nu$.

□

2.5.4 Outer regularity

Let X be a topological space and let $\mathcal{S}$ be a σ-algebra on X that includes $\mathcal{B}(X)$. A measure μ on $\mathcal{S}$ is called *outer-regular* if[7]

$$(\forall A \in \mathcal{S})\ \inf\{\mu(G \setminus A) \ : \ G \supseteq A,\, G \text{ open}\} = 0.$$

We will call an outer measure μ^* on 2^X *outer-regular* if

$$(\forall A \in \mathrm{MEAS}_{\mu^*})\ \inf\{\mu^*(G \setminus A) \ : \ G \supseteq A,\, G \text{ open}\} = 0.$$

It is well known that on metric spaces all finite Borel measures are outer-regular (see [Kallenberg 2002, Lemma 1.34]).

The following lemma will be needed in the proof of Theorem 4.8 below:

[7] In many textbooks, a measure μ is called outer-regular if it fulfills the weaker condition that $\mu(A) = \inf\{\mu(G) \ : \ G \supseteq A,\, G \text{ open}\}$ for all $A \in \mathcal{S}$. It will be crucial for some of the results below that outer regularity is understood in the strong sense!

Lemma 2.22. *Let X be a topological space, and let $\mathcal{S}$ be a σ-algebra on X that includes $\mathcal{B}(X)$. Let μ be an outer-regular measure on $\mathcal{S}$ and let $f : X \to [0, \infty]$ be a μ-integrable function. Then the measure ν on $\mathcal{S}$ defined by $\nu(A) := \int_A f \, d\mu$ is outer-regular.*

Proof. Let $A \in \mathcal{S}$ be arbitrary and consider a descending sequence $(G_n)_{n \in \mathbb{N}}$ of open sets such that $G_n \supseteq A$ and $\mu(G_n \setminus A) \to 0$. The set $C := \bigcap_n G_n \setminus A$ has measure 0 and so $\int_C f \, d\mu = 0$. Dominated Convergence now yields $\int_{G_n \setminus A} f \, d\mu \to 0$. □

3 Three probabilistic concepts of computability

Assumption 3.1 Throughout the remaining of this article, we denote by

- X, X_1 nonempty sets,
- δ, δ_1 naming systems of X, X_1, respectively,
- (Y, d, α) a computable metric space with Cauchy representation δ_Y,
- μ^* an outer measure on 2^X,
- $\mathcal{S}$ a σ-algebra on X,
- ν a measure on $(X, \mathcal{S})$,
- ν^* the outer measure induced by ν.

3.1 The local error

Definition 3.2. For any mapping $f : X \to Y$ and any $\phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ define the *local error*

$$e(f, \delta, \phi, \cdot) : X \to [0, \infty],$$
$$e(f, \delta, \phi, x) := \sup_{p \in \delta^{-1}\{x\}} d((\alpha \circ \phi)(p), f(x)).$$

The following observation will be useful below:

Lemma 3.3. *Consider the assumptions of Definition 3.2, and additionally, let $g :\subseteq W \to \mathrm{dom}(\delta)$ ($W \in \{\Sigma^*, \Sigma^\omega\}$) be a mapping such that $\delta \circ g$ is a naming system of X. Then*

$$(\forall x \in X) \; e(f, \delta \circ g, \phi \circ g, x) \le e(f, \delta, \phi, x)$$

□

3.2 Concept (I): Computability almost everywhere

Parker (see [Parker 2003, Parker 2005, Parker 2006]) introduces the concept of "decidability up to measure zero". The following is a rather straight-forward generalization:

Definition 3.4. 1. A mapping $f : X \to X_1$ is $(\delta, \delta_1)^{\nu}_{\text{AE}}$*-continuous (-computable)* if there is ν-null set $N \subseteq X$ such that $f|_{X \setminus N}$ is $(\delta|^{X \setminus N}, \delta_1)$-continuous (-computable).

2. A multi-representation of the class $\text{AE}_{\delta,\delta_1,\nu}$ of all $(\delta, \delta_1)^{\nu}_{\text{AE}}$-continuous mappings is given by

$$f \in [\delta \to \delta_1]^{\nu}_{\text{AE}}(p) \ :\Longleftrightarrow \text{there is a } \nu\text{-null set } N \text{ such that } [\delta \to \delta_1]_{X \setminus N}(p) = f|_{X \setminus N},$$

where $[\delta \to \delta_1]_{X \setminus N}$ is defined in [Weihrauch 2000, Definition 3.3.13].

3.3 Concept (II): Computable approximation

The definitions in this subsection generalize a definition of Ko (cf. [Ko 1991, Definition 5.10]).

For any mapping $\phi :\subseteq \mathbb{N} \times A \to B$ (for sets A, B) and any $n \in \mathbb{N}$, we shall denote by $\phi_n :\subseteq A \to B$ the mapping given by

$$\text{dom}(\phi_n) := \{a \in A \ : \ (n, a) \in \text{dom}(\phi)\} \qquad \text{and} \qquad \phi_n(a) := \phi(n, a).$$

Definition 3.5. Let $f : X \to Y$ be a mapping.

1. A mapping $\phi : \mathbb{N} \times \text{dom}(\delta) \to \text{dom}(\alpha)$ is a $(\delta, \alpha)^{\mu^*}_{\text{APP}}$*-realization* of f if

$$(\forall\, n \in \mathbb{N})\ \mu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}]) \leq 2^{-n}.$$

f is $(\delta, \alpha)^{\mu^*}_{\text{APP}}$*-continuous (-computable)* if it has a continuous (computable) $(\delta, \alpha)^{\mu^*}_{\text{APP}}$-realization.

2. A multi-representation of the class $\text{APP}_{\delta,\alpha,\mu^*}$ of all $(\delta, \alpha)^{\mu^*}_{\text{APP}}$-continuous mappings is given by

$$f \in [\delta \to \alpha]^{\mu^*}_{\text{APP}}(p) \ :\Longleftrightarrow [[\nu_{\mathbb{N}}, \text{id}_{\text{dom}(\delta)}] \to \text{id}_{\text{dom}(\alpha)}](p) \text{ is a } (\delta, \alpha)^{\mu^*}_{\text{APP}}\text{-realization of } f.$$

3. f is $(\delta, \alpha)^{\nu}_{\text{APP}}$*-continuous (-computable)* if f is $(\delta, \alpha)^{\nu^*}_{\text{APP}}$-continuous (-computable). Define $[\delta \to \alpha]^{\nu}_{\text{APP}} := [\delta \to \alpha]^{\nu^*}_{\text{APP}}$.

The definition just given requires $(\delta, \alpha)^{\mu^*}_{\text{APP}}$-realizations to be defined on all of $\mathbb{N} \times \text{dom}(\delta)$, i.e. a Turing machine that implements such a realization must halt on every (properly encoded) input from $\mathbb{N} \times \text{dom}(\delta)$ and put out an element of $\text{dom}(\alpha)$. Concerning this definition, we assent to the following statement of Parker (see [Parker 2003, p. 8]):

> "Why require a machine that always halts? Assuming we have a machine that sometimes gives incorrect output, the epistemological situation would seem no worse if in principle that machine could also fail to halt, but with probability zero."

This leads to a combination of concepts (I) and (II):

Definition 3.6. Let $f : X \to Y$ be a mapping.

1. f is $(\delta, \alpha)^{\mu^*}_{\text{APP/AE}}$-*continuous (-computable)* if there is a μ^*-null set $N \subseteq X$ such that $f|_{X \setminus N}$ is $(\delta|^{X\setminus N}, \alpha)^{\mu^*}_{\text{APP}}$-continuous (-computable).

2. A multi-representation of the class $\text{APP/AE}_{\delta,\alpha,\mu^*}$ of all $(\delta, \alpha)^{\mu^*}_{\text{APP/AE}}$-continuous mappings is given by
$$f \in [\delta \to \alpha]^{\mu^*}_{\text{APP/AE}}(p) \;:\Longleftrightarrow \text{there is a } \mu^*\text{-null set } N \text{ such that } f|_{X\setminus N} \in [\delta|^{X\setminus N} \to \alpha]^{\mu^*}_{\text{APP}}(p).$$

3. f is $(\delta, \alpha)^{\nu}_{\text{APP/AE}}$-*continuous (-computable)* if f is $(\delta, \alpha)^{\nu^*}_{\text{APP/AE}}$-continuous (-computable). Define $[\delta \to \alpha]^{\nu}_{\text{APP/AE}} := [\delta \to \alpha]^{\nu^*}_{\text{APP/AE}}$.

3.4 Concept (III): Computability in the mean

We now come to a notion that has been proposed in a talk by [Hertling 2005], but has apparently not been treated in the literature so far. We would like to call f "computable in the mean w.r.t. ν" if there is a computable $\Phi : \mathbb{N} \times \text{dom}(\delta) \to \text{dom}(\alpha)$ such that

$$(\forall\, n \in \mathbb{N}) \int e(f, \delta, \Phi_n, x)\, \nu(dx) \leq 2^{-n}. \tag{2}$$

But this is not a definition unless the integral is well-defined, i.e. unless we impose additional conditions on X, δ, f, Φ, and $\mathcal{S}$ which ensure that $e(f, \delta, \Phi_n, \cdot)$ is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable. We will discuss such conditions in Section 7.3. It is possible, however, to give a reasonable definition of "computable in the mean" that does not assume measurability of the local error. This is achieved in a natural way by replacing the integral in (2) by an outer integral:

Definition 3.7. Let $f : X \to Y$ be a mapping.

1. A mapping $\Phi : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ is a $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-*realization* of f if

$$(\forall\, n \in \mathbb{N}) \ \int^{*} e(f, \delta, \Phi_n, x)\, \nu(dx) \leq 2^{-n}.$$

   f is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-*continuous (-computable)* if it has a continuous (computable) $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization.

2. A multi-representation of the class $\mathrm{MEAN}_{\delta,\alpha,\nu}$ of all $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous mappings is given by

$$\begin{aligned} f \in [\delta \to \alpha]^{\nu}_{\mathrm{MEAN}}(p) \ :\Longleftrightarrow\ & [[\nu_{\mathbb{N}}, \mathrm{id}_{\mathrm{dom}(\delta)}] \to \mathrm{id}_{\mathrm{dom}(\alpha)}](p) \\ & \text{is a } (\delta, \alpha)^{\nu}_{\mathrm{MEAN}}\text{-realization of } f. \end{aligned}$$

3. f is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-*continuous (-computable)* if there is a ν-null set $N \subseteq X$ such that $f|_{X\setminus N}$ is $(\delta|^{X\setminus N}, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous (-computable).

4. A multi-representation of the class $\mathrm{MEAN/AE}_{\delta,\alpha,\nu}$ of all $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-continuous mappings is given by

$$\begin{aligned} f \in [\delta \to \alpha]^{\nu}_{\mathrm{MEAN/AE}}(p) \ :\Longleftrightarrow\ & \text{there is a } \nu\text{-null set } N \text{ such that} \\ & f|_{X\setminus N} \in [\delta|^{X\setminus N} \to \alpha]^{\nu}_{\mathrm{MEAN}}(p). \end{aligned}$$

The notion of MEAN-computability just defined has a property that one would expect any reasonable notion of "computability in the mean" to have: Recall the setting described in the introduction and suppose now that our agent is supplied with a sequence of independent identically distributed measurements of the physical magnitude and has the task to compute an approximation to f on each of them. If f is "computable in the mean", then there should be an approximation algorithm whose error is small if one considers the arithmetic mean over "a large number" of inputs.

Proposition 3.8. *Suppose that ν is a probability measure. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $(w_i)_i$ be a sequence of mappings $w_i : \Omega \to \mathrm{dom}(\delta)$ such that the mappings $\delta \circ w_i$ are independent ν-distributed random variables. Let $f : X \to Y$ be a mapping which has a $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization Φ. Then for every $n \in \mathbb{N}$ one has*

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{i<m} e_i \leq 2^{-n} \qquad P\text{-almost surely}$$

where $e_i := d((\alpha \circ \Phi_n)(w_i), (f \circ \delta)(w_i))$.

Proof. For all i, we have $e_i \leq e(f, \delta, \Phi_n, \delta(w_i))$. It follows from Definition 3.7 and the definition of the outer integral that there is a sequence $(g_k)_k$ of measurable functions $g_k : X \to [0,\infty]$ such that $e(f,\delta,\Phi_n,\cdot) \leq g_k$ and $\int g_k \, d\nu \leq 2^{-n} + 2^{-k}$ for every k. The Strong Law of Large Numbers (see [Kallenberg 2002, Theorem 4.23]) now yields that for every k

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{i<m} e_i \leq \lim_{m\to\infty} \frac{1}{m} \sum_{i<m} g_k(\delta(w_i)) = \int g_k \, d\nu \leq 2^{-n} + 2^{-k} \qquad P\text{-a.s.}$$

Intersecting over k yields the claim. □

We close this section with the following lemma which is a simple consequence of Lemma 3.3. Its analogue for plain computability can be found in [Weihrauch 2000, Exercise 3.3.13].

Lemma 3.9. *Suppose that δ' is another naming system of X, and $(\sim, \Box)$ is one of (APP, μ^*), $(\text{APP/AE}, \mu^*)$, (MEAN, ν), $(\text{MEAN/AE}, \nu)$.*

1. *If $\delta' \leq_t \delta$, then every $(\delta,\alpha)^{\Box}_{\sim}$-continuous mapping is $(\delta',\alpha)^{\Box}_{\sim}$-continuous and $[\delta \to \alpha]^{\Box}_{\sim} \leq_t [\delta' \to \alpha]^{\Box}_{\sim}$.*
2. *If $\delta' \leq \delta$, then $[\delta \to \alpha]^{\Box}_{\sim} \leq [\delta' \to \alpha]^{\Box}_{\sim}$.*
3. *If $\delta' \equiv \delta$, then $[\delta \to \alpha]^{\Box}_{\sim} \equiv [\delta' \to \alpha]^{\Box}_{\sim}$.*

□

4 Representation theorems

An important topic in TTE is the relation between (δ,δ_1)-continuity and classical continuity of a mapping $f : X \to X_1$. A key result is the Kreitz-Weihrauch Representation Theorem (see [Weihrauch 2000, Theorem 3.2.11]) which has later been generalized by [Schröder 2002b]: A representation of a topological space is called *admissible*, if it is continuous and every continuous representation of the same space is continuously reducible to it. If both δ and δ_1 are admissible, then the (δ,δ_1)-continuous mappings are exactly the sequentially continuous mappings. (Note that in most applications the topology of X is countably based, and then sequential continuity is equivalent to plain continuity.)

In the spirit of the Representation Theorem (RT), we now seek for connections between classical properties of a mapping and its probabilistic relative continuity as defined in the preceding section.

Proposition 4.1 (RT for AE-Continuity). *Assume that X and X_1 are endowed with topologies w.r.t. which δ and δ_1 are admissible. Then a mapping $f : X \to Y$ is $(\delta,\delta_1)^{\nu}_{\text{AE}}$-continuous iff there is a ν-null set N such that $f|_{X\setminus N}$ is sequentially continuous.*

Proof. By [Schröder 2002b, Subsection 4.1], $\delta|^{X\setminus N}$ is an admissible representation of $X \setminus N$ for any subset N of X. The claim hence follows from the Representation Theorem. □

Proposition 4.2. *Assume that $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$. Then every $(\delta,\alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous $f : X \to Y$ is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.*

Proof. It follows from Lemma 2.18 that it is sufficient to prove the claim for $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-continuous f. Let ϕ be a continuous $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f, and let $(a_m)_{m\in\mathbb{N}}$ be an enumeration of $\mathrm{dom}(\alpha)$. For every $n, m \in \mathbb{N}$ put $A_{n,m} := \phi_n^{-1}\{a_m\}$. Then every $A_{n,m}$ is open in $\mathrm{dom}(\delta)$, and $\mathrm{dom}(\delta) \subseteq \bigcup_m A_{n,m}$. The assumption $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$ implies that all sets $D_{n,m} := \delta(A_{n,m})$ are μ^*-measurable. Define

$$c(n,x) := \min\{m \in \mathbb{N} \ : \ x \in D_{n,m}\},$$
$$f_n(x) := \alpha(a_{c(n,x)}).$$

Then $f_n^{-1}\{a_m\} = D_{n,m} \setminus \bigcup_{k<m} D_{n,k}$ for every m, which yields that the f_n are $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable. $f_n(x)$ is the output of ϕ_n on a certain name of x; it hence follows from the definition of the local error that $d(f_n(x), f(x)) \leq e(f,\delta,\phi_n,x)$ for all $x \in X$, so $\mu^*([d(f,f_n) > 2^{-n}]) \leq 2^{-n}$ for every $n \in \mathbb{N}$. The claim now follows with Lemma 2.20. □

Proposition 4.3. *Suppose X is endowed with a topology w.r.t. which δ is continuous and μ^* is outer-regular. Then every $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable $f : X \to Y$ is $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-continuous.*

Proof. Let $(a_m)_{m\in\mathbb{N}}$ be an enumeration of $\mathrm{dom}(\alpha)$. For all $m, n \in \mathbb{N}$, put

$$A_{m,n} := f^{-1}(B(\alpha(a_m), 2^{-n})).$$

Note that $X = \bigcup_m A_{m,n}$. By the outer regularity of μ^*, there are open sets $G_{m,n}$ with $A_{m,n} \subseteq G_{m,n}$ and $\mu^*(G_{m,n} \setminus A_{m,n}) \leq 2^{-(n+m+1)}$. Now for every $n \in \mathbb{N}$, there is a continuous "selector" $c_n : \mathrm{dom}(\delta) \to \mathbb{N}$ such that $\delta(p) \in G_{c_n(p),n}$ for every $p \in \mathrm{dom}(\delta)$. Put $\phi(n,p) := a_{c_n(p)}$. It is easy to see that

$$[e(f,\delta,\phi_n,\cdot) > 2^{-n}] \subseteq \bigcup_{m\in\mathbb{N}} (G_{m,n} \setminus A_{m,n})$$

and that the set on the right hand side has μ^*-content at most 2^{-n}. □

Combining the last two propositions yields the following corollary, which should apply in most situations of practical interest:

Corollary 4.4 (RT for APP-Continuity). *Suppose that X is topological, δ is continuous, μ^* is outer-regular, and $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\mu^*}$. Then for every mapping $f : X \to Y$, the following statements are equivalent:*

1. *f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP}}$-continuous.*
2. *f is $(\delta, \alpha)^{\mu^*}_{\mathrm{APP/AE}}$-continuous.*
3. *f is $(\mathrm{MEAS}_{\mu^*}, \mathcal{B}(Y))$-measurable.*

□

The next result follows as simple combination of Proposition 4.2 and Proposition 5.1; although the latter will be proved only below, we think that the corollary should be stated already here:

Corollary 4.5. *Assume that $\sigma(\delta^{-1}) \subseteq \mathrm{MEAS}_{\nu^*}$. If $f : X \to Y$ is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN/AE}}$-continuous, then f is $(\mathrm{MEAS}_{\nu^*}, \mathcal{B}(Y))$-measurable.* □

We will see below (Proposition 5.4.2) that conditions such as those of Proposition 4.3 (δ continuous, ν outer-regular, f measurable) are not sufficient to ensure MEAN-continuity. The next natural step is to consider integrable f. This makes sense only if Y is a normed space.

Assumption 4.6 Throughout the remaining of this section, we additionally assume that

– Y is a normed space with norm $\|\cdot\|$, and d is the metric induced by the norm.

– $0 \in \mathrm{range}(\alpha)$.

– X is endowed with a topology.

Proposition 4.7. *Suppose that δ is open and ν^* is locally finite. If a mapping $f : X \to Y$ is $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-continuous, then $\|f\|$ is locally outer-integrable w.r.t. ν, i.e. for every $x \in X$ there is an open neighbourhood $G \subseteq X$ of x such that $\int^*_G \|f\| \, d\nu < \infty$.*

Proof. Let Φ be a continuous $(\delta, \alpha)^{\nu}_{\mathrm{MEAN}}$-realization of f. Let $x_0 \in X$ be arbitrary, and let p be an arbitrary δ-name of x_0. Φ_n is constantly equal to $\Phi_n(p)$ on an open (in $\mathrm{dom}(\delta)$) neighbourhood $U \subseteq \mathrm{dom}(\delta)$ of p. Put $a := (\alpha \circ \Phi_0)(p)$. By the definition of the local error, we have

$$(\forall\, x \in \delta(U))\ e(f, \delta, \Phi_0, x) \geq \|a - f(x)\|.$$

$\delta(U)$ is open, and by the local finiteness of ν^*, we can find an open neighbourhood $G \subseteq \delta(U)$ of x_0 such that $\nu^*(G) < \infty$. We finally have

$$\begin{aligned} 1 &\geq \int^* e(f, \delta, \Phi_0, x)\, \nu(dx) \geq \int^*_G e(f, \delta, \Phi_0, x)\, \nu(dx) \geq \int^*_G \|a - f(x)\|\, \nu(dx) \\ &\geq \int^*_G \|f\|\, d\nu - \int^*_G \|a\|\, d\nu \geq \int^*_G \|f\|\, d\nu - \nu^*(G)\|a\|. \end{aligned}$$

□

Theorem 4.8 *Suppose that δ is continuous, $\mathcal{B}(X) \subseteq \mathcal{S}$, ν is outer-regular, f is $(\mathcal{S}, \mathcal{B}(Y))$-measurable, and $\|f\|$ is locally ν-integrable. Then f is $(\delta, \alpha)^{\nu}_{\text{MEAN}}$-continuous.*

Proof. We first assume that $\|f\|$ is integrable over the whole space. Let $(a_m)_{m \in \mathbb{N}}$ be an enumeration of $\operatorname{dom}(\alpha)$. For all $m, n \in \mathbb{N}$ put

$$A_{m,n} := \begin{cases} f^{-1}(\ B(\alpha(a_m), \min\{2^{-n}, \|\alpha(a_m)\|/2\})\ ) & \text{if } \alpha(a_m) \neq 0 \\ f^{-1}\{0\} & \text{else.} \end{cases}$$

Note that $X = \bigcup_m A_{m,n}$. Put $C_{m,n} := A_{m,n} \setminus \bigcup_{k<m} A_{k,n}$ and

$$g_n := \sum_m \alpha(a_m) \chi_{C_{m,n}},$$

and note that (g_n) converges to f pointwise and that $\|f - g_n\| \leq \|f\|$. So (g_n) converges to f in $L^1(\nu)$ by Dominated Convergence. By transition to a subsequence, we can assume that $\int \|f - g_n\| \, d\nu < 2^{-(n+1)}$ for all $n \in \mathbb{N}$. The measures ν_n on $\mathcal{S}$ defined by

$$\nu_n(A) := \int_A \|g_n\| \, d\nu$$

are outer-regular by Lemma 2.22. So there are open sets $G_{m,n}$ with $G_{m,n} \supseteq C_{m,n}$ and $\nu(G_{m,n} \setminus C_{m,n}) \leq (2^{n+m+3} \cdot \max\{1, \|\alpha(a_m)\|\})^{-1}$ and $\nu_n(G_{m,n} \setminus C_{m,n}) \leq 2^{-(n+m+3)}$. Now for every $n \in \mathbb{N}$ there is a continuous $m_n : \operatorname{dom}(\delta) \to \mathbb{N}$ such that $\delta(p) \in C_{m_n(p),n}$ for every $p \in \operatorname{dom}(\delta)$. Put $\Phi_n(p) = a_{m_n(p)}$. We have

$$\begin{aligned} &\int^* e(f, \delta, \Phi_n, x) \, \nu(dx) \\ &\leq \int \|f - g_n\| \, d\nu + \int^* e(g_n, \delta, \Phi_n, x) \, \nu(dx) \\ &\leq 2^{-(n+1)} + \sum_m \int_{G_{m,n} \setminus C_{m,n}} \|\alpha(a_m) - g_n(x)\| \nu(dx) \\ &\leq 2^{-(n+1)} + \sum_m \nu(G_{m,n} \setminus C_{m,n}) \|\alpha(a_m)\| + \sum_m \nu_n(G_{m,n} \setminus C_{m,n}) \\ &\leq 2^{-n}. \end{aligned}$$

We have hence shown that f is $(\delta, \alpha)^{\nu}_{\text{MEAN}}$-continuous.

Now assume that $\|f\|$ is only locally integrable. Remember that X is Lindelöf (because it allows a continuous representation). There hence is a countable open cover $(G_\ell)_\ell$ of X, such that $\|f\|$ is integrable on each G_ℓ. By the first part of the proof, each mapping $f|_{G_\ell}$ is $(\delta|^{G_\ell}, \alpha)^{\nu}_{\text{MEAN}}$-continuous; let $\Phi^{(\ell)}$ be the corresponding realization. Let $c : \operatorname{dom}(\delta) \to \mathbb{N}$ be a continuous selector such that

$\delta(p) \in G_{c(p)}$ for every $p \in \mathrm{dom}(\delta)$. Now put $\Phi(n,p) := \Phi^{(c(p))}(n + c(p) + 1, p)$. One then has the estimate:

$$\begin{aligned}\int^* e(f,\delta,\Phi_n,x)\,\nu(dx) &\leq \int^* \sup_\ell \chi_{G_\ell}(x) e(f,\delta,\Phi^{(\ell)}_{n+\ell+1},x)\,\nu(dx)\\ &\leq \sum_\ell \int^*_{G_\ell} e(f,\delta,\Phi^{(\ell)}_{n+\ell+1},x)\,\nu(dx)\\ &\leq 2^{-n}.\end{aligned}$$

□

The following corollary should apply in most situations of practical interest:

Corollary 4.9 (RT for MEAN-Continuity). *Suppose that the topology of X is countably-based* T_0*, δ is admissible, $\mathcal{B}(X) \subseteq \mathcal{S}$, ν is locally finite and outer-regular, and $f : X \to Y$ is $(\mathcal{S}, \mathcal{B}(Y))$-measurable. Then f is $(\delta,\alpha)^\nu_{\mathrm{MEAN}}$-continuous iff $\|f\|$ is locally integrable w.r.t. ν.*

Proof. The "if" direction follows directly by Theorem 4.8. For the "only if" direction, first recall that an admissible representation of a countably-based T_0-space is continuously equivalent to an open standard representation of that space (see [Schröder 2002b, Section 2.2]). By Lemma 3.9, it is hence sufficient to prove the assertion for open δ. It then follows directly from Proposition 4.7. □

5 Mutual relations between the probabilistic computability notions

5.1 Simple reductions and strong counter-examples

We will now clarify the mutual relations between the concepts defined above. The first proposition sums up the cases in which there is a computable reduction of one multi-representation to the other. Then we give some strong counter-examples – i.e. examples involving functions from $[0,1]$ to $\mathbb{R}$ and the Lebesgue measure – for other cases. The remaining cases are treated in the next subsection.

Proposition 5.1. *1.* $[\delta \to \delta_1] \leq [\delta \to \delta_1]^\nu_{\mathrm{AE}}$.

2. $[\delta \to \delta_Y]^\nu_{\mathrm{AE}} \leq [\delta \to \alpha]^\nu_{\mathrm{APP/AE}}$.

3. $[\delta \to \alpha]^\nu_{\mathrm{MEAN}} \leq [\delta \to \alpha]^\nu_{\mathrm{APP}}$.

4. $[\delta \to \alpha]^\nu_{\mathrm{MEAN/AE}} \leq [\delta \to \alpha]^\nu_{\mathrm{APP/AE}}$.

Proof. 1. and 2. are obvious. 4. is a corollary of 3. We prove 3.: Lemma 2.21 yields that for every $h : X \to [0, \infty]$ and every $\epsilon > 0$, one has

$$\int^* h\,d\nu \geq \int^* \epsilon \cdot \chi_{[h>\epsilon]} d\nu = \epsilon \cdot \nu^*([h > \epsilon]).$$

From this it immediately follows: If Φ is a $(\delta, \alpha)^\nu_{\mathrm{MEAN}}$-realization of some f, then a $(\delta, \alpha)^\nu_{\mathrm{APP}}$-realization ϕ of the same f is given by $\phi(n, p) := \Phi(2n, p)$; this yields a computable reduction. □

Lemma 5.2. *Suppose that Y is a normed space, the mapping $a \mapsto \|a\|$ is (α, ρ)-computable, and ν is finite. Consider the set*

$$B := \{(f, N)\ :\ f : X \to Y \text{ is } (\delta, \alpha)^\nu_{\mathrm{APP}}\text{-continuous, } N \in \mathbb{N}, \text{ and } \|f\| \leq N\}.$$

The mapping $(f, N) \mapsto f$ is $([[\delta \to \alpha]^\nu_{\mathrm{APP}}, \nu_{\mathbb{N}}]|^B, [\delta \to \alpha]^\nu_{\mathrm{MEAN}})$-computable.

Proof. We need to demonstrate how to compute a $(\delta, \alpha)^\nu_{\mathrm{MEAN}}$-realization Φ of some f from a $(\delta, \alpha)^\nu_{\mathrm{APP}}$-realization ϕ of f and an integer bound $N \geq \|f\|$. We can assume $N > 0$. Fix an $a_0 \in \mathrm{dom}(\alpha)$ such that $\|\alpha(a_0)\| \leq N$. We can compute a $\phi' : \mathrm{dom}(\phi) \to \mathrm{dom}(\alpha)$ such that for all $(n, p) \in \mathrm{dom}(\phi)$, one has

$$\begin{gathered}\phi'(n, p) \in \{\phi(n, p), a_0\},\\ \|(\alpha \circ \phi)(n, p)\| \geq 3N + 1 \implies \phi'(n, p) = a_0,\\ \|(\alpha \circ \phi)(n, p)\| \leq 3N \implies \phi'(n, p) = \phi(n, p).\end{gathered}$$

By distinguishing the cases

$$\text{(i) } \|(\alpha \circ \phi)(n, p)\| \leq 3N, \qquad \text{(ii) } 3N < \|(\alpha \circ \phi)(n, p)\| < 3N + 1,$$
$$\text{(iii) } 3N + 1 \leq \|(\alpha \circ \phi)(n, p)\|,$$

one finds that

$$(\forall\, a \in \overline{B}(0, N))\ \|(\alpha \circ \phi')(n, p) - a\| \leq \min\{\|(\alpha \circ \phi)(n, p) - a\|, 4N + 1\};$$

hence one has $e(f, \delta, \phi'_n, \cdot) \leq \min\{e(f, \delta, \phi_n, \cdot), 4N + 1\}$. This yields

$$\begin{aligned}\int^* e(f, \delta, \phi'_n, \cdot)\, d\nu &\leq \nu^*([e(f, \delta, \phi_n, \cdot) > 2^{-n}])(4N + 1) + 2^{-n}\nu(X)\\ &\leq 2^{-n}(4N + 1 + \nu(X))\end{aligned}$$

for all $n \in \mathbb{N}$. A suitable Φ is hence given by

$$\Phi(n, p) := \phi'(n + \lceil \log(4N + 1 + \nu(X)) \rceil, p).$$

□

Proposition 5.3. *There is a set $S \subseteq [0,1]$ such that χ_S is $(\rho, \nu_{\mathbb{Q}})^\lambda_{\mathrm{MEAN}}$-computable but not $(\rho, \rho)^\lambda_{\mathrm{AE}}$-continuous.*

Proof. Parker (see [Parker 2003, Theorem IV]) considers a positive-measure Cantor set $S \subseteq [0,1]$ and proves that χ_S is $(\rho, \nu_{\mathbb{Q}})^\lambda_{\mathrm{APP}}$-computable but not $(\rho, \rho)^\lambda_{\mathrm{AE}}$-continuous (although he does not use these terms). By the previous lemma, χ_S is even $(\rho, \nu_{\mathbb{Q}})^\lambda_{\mathrm{MEAN}}$-computable. □

Proposition 5.4. *1. There is a function $f : [0,1] \to \mathbb{R}$ which is $(\rho|^{[0,1]}, \rho)^\lambda_{\mathrm{AE}}$- and $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^\lambda_{\mathrm{MEAN/AE}}$-computable but not $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^\lambda_{\mathrm{MEAN}}$-continuous.*

2. There is a function $f : [0,1] \to \mathbb{R}$ which is $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^\lambda_{\mathrm{APP}}$-computable but not $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^\lambda_{\mathrm{MEAN/AE}}$-continuous.

Proof. Recall that $\rho|^{[0,1]}$ is an open representation of $[0,1]$. We can hence apply Proposition 4.7.

For item 1., simply consider $f(x) := x^{-1} \cdot \chi_{(0,1]}(x)$, which clearly is computable and MEAN-computable on $(0,1]$, but not locally integrable in 0.

For item 2., we need a more elaborate example: For every $a \in [0,1], n \in \mathbb{N}$, define

$$f_{a,n}(x) := (x-a)^{-1} \chi_{(a,a+2^{-n}] \cap [0,1]}(x).$$

Let $(a_n)_{n \in \mathbb{N}}$ be a computable dense sequence of rationals in $[0,1]$. Choose $\widetilde{f} := \sup_{n \in \mathbb{N}} f_{a_n,n}$. $\widetilde{f}$ is a measurable function into $\overline{\mathbb{R}}$, that is not integrable on any open subset of $[0,1]$, because any such open subset must contain an interval of the form $[a_n, a_n + \epsilon] =: I$ and one already has $\int_I f_{a_n,n}\, d\lambda = \infty$. Obviously, $\widetilde{f}(x) = \infty$ implies that x is contained in infinitely many of the $(a, a+2^{-n}]$, and hence Cantelli's Theorem yields $\lambda([\widetilde{f} = \infty]) = 0$. So, the function $f := \widetilde{f} \cdot \chi_{[\widetilde{f} \neq \infty]}$ is into $\mathbb{R}$ and is still measurable and nowhere integrable. Clearly, $f|_{X \setminus N}$ is still nowhere integrable for any ν-null set N. So f is not $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^\lambda_{\mathrm{MEAN/AE}}$-continuous. On the other hand, it is not hard to see that f is $(\rho|^{[0,1]}, \nu_{\mathbb{Q}})^\lambda_{\mathrm{APP}}$-computable. □

5.2 Reductions that require certain effectivity assumptions

Only the following relations have not been covered yet: AE $\rightsquigarrow$ MEAN/AE, AE $\rightsquigarrow$ APP, APP/AE $\rightsquigarrow$ APP, MEAN/AE $\rightsquigarrow$ APP. For these, computable reductions do not exist in general, but under a number of additional assumptions, which should, however, be fulfilled in most situations of practical interest.

The question whether AE-computability implies MEAN/AE-computability leads to the question whether plain computability implies MEAN-computability. Note that if ν is a measure on $[0,1]$ which is not locally finite, $c \in \mathbb{R} \setminus \mathbb{Q}$ is computable, and $f(x) := c$ for all $x \in [0,1]$, then f is (ρ, ρ)-computable but not $(\rho, \nu_{\mathbb{Q}})^\nu_{\mathrm{MEAN}}$-continuous. Buf if ν is locally finite (in an effective sense), a reduction can be proved:

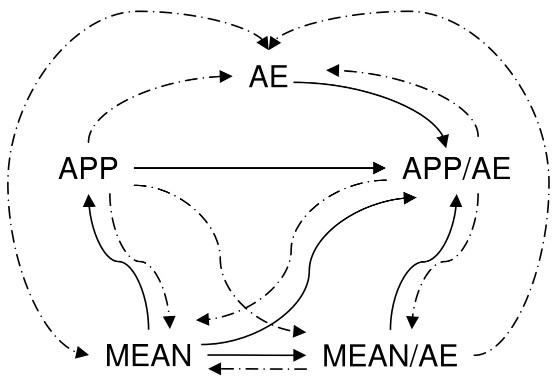

Figure 1: The graphic subsumes the results of Section 5.1. A solid arrow indicates computable reduction, a dashed arrow indicates a strong counter-example.

Definition 5.5. Suppose that X is topological and θ is a representation of the hyperspace $\mathcal{O}(X)$ of open subsets of X. Then μ^* is *effectively locally finite* w.r.t. θ if there is a $[\theta, \nu_{\mathbb{N}}]^\omega$-computable sequence $(U_r, M_r)_r$ in $\mathcal{O}(X) \times \mathbb{N}$ such that $X = \bigcup_r U_r$ and $\mu^*(U_r) \leq M_r$ for all $r \in \mathbb{N}$.

Definition 5.6. Suppose that X is topological, δ is continuous, and θ is a representation of the hyperspace $\mathcal{O}(X)$ of open subsets of X. δ and θ are said to be *compatible*, if the relation $\{(x, U) \in X \times \mathcal{O}(X) \ : \ x \in U\}$ is (δ, θ)-c.e.

Proposition 5.7. *Suppose that X is topological, δ is continuous, and θ is a compatible representation of $\mathcal{O}(X)$. Further suppose that ν^* is effectively locally finite w.r.t. θ. Then*

1. $[\delta \to \delta_Y] \leq [\delta \to \alpha]^\nu_{\text{MEAN}}$.
2. $[\delta \to \delta_Y]^\nu_{\text{AE}} \leq [\delta \to \alpha]^\nu_{\text{MEAN/AE}}$.

Proof. Item 2. follows from item 1. We prove 1.: We need to demonstrate how to compute a $(\delta, \alpha)^\nu_{\text{MEAN}}$-realization Φ of some f from a mapping $\phi : \mathbb{N} \times \text{dom}(\delta) \to \text{dom}(\alpha)$ with $e(f, \delta, \phi_n, x) \leq 2^{-n}$ for all $x \in X$, $n \in \mathbb{N}$. So let an input pair $(n, p) \in \mathbb{N} \times \text{dom}(\delta)$ be given. As ν^* is effectively locally finite, there is a sequence $(U_r, M_r)_r$ as in Definition 5.5. As δ and θ are compatible, we can effectively determine a $c(p) \in \mathbb{N}$ such that $\delta(p) \in U_{c(p)}$. Put

$$\Phi(n, p) := \phi(n + \lceil \log M_{c(p)} \rceil + 1, p).$$

The correctness of this procedure follows from the estimate

$$\int^* e(f,\delta,\Phi_n,x)\,\nu(dx) \le \int^* \sup_r \chi_{U_r}(x) e(f,\delta,\phi_{n+\lceil \log M_r\rceil+1},x)\,\nu(dx)$$
$$\le \sum_r M_r 2^{-(n+\lceil \log M_r\rceil+1)} \le 2^{-n}.$$

□

We will next look for assumptions that imply computable reducibility from APP/AE to APP (and hence from AE to APP and from MEAN/AE to APP). The next lemma is intended as preparation for the proof of Theorem 5.9.[8]

Lemma 5.8. *Suppose that X is topological, δ is continuous, and θ is a compatible representation of $\mathcal{O}(X)$. Furthermore, suppose that*

(i) there is a $[\theta]^\omega$-computable sequence $(U_r)_r$ in $\mathcal{O}(X)$ such that $X = \bigcup_r U_r$ and

(ii) from any prefix-free sequence $(w_\ell)_\ell$ in Σ^* with

$$\mu^*\left(X \setminus \bigcup_\ell W(\delta, w_\ell)\right) = 0$$

and any $r,k \in \mathbb{N}$, one can $[\theta]^\omega$-compute a sequence $(V_\ell)_\ell$ in $\mathcal{O}(X)$ and θ-compute a set $\widetilde{V} \in \mathcal{O}(X)$, such that

$$U_r \subseteq \bigcup_\ell V_\ell \cup \widetilde{V}$$

and $\mu^*(L) \le 2^{-k}$ where

$$L := U_r \cap \left(\widetilde{V} \cup \bigcup_\ell (V_\ell \setminus W(\delta, w_\ell))\right).$$

Then $[\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}} \le [\delta \to \alpha]^{\mu^*}_{\mathrm{APP}}$.

Proof. We need to demonstrate how to compute a $[\delta \to \alpha]^{\mu^*}_{\mathrm{APP}}$-realization ϕ of some f from a $[\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}}$-realization ϕ' of f. So suppose we are given an input pair $(n,p) \in \mathbb{N} \times \mathrm{dom}(\delta)$. We simulate ϕ' on all $(m,q) \in \mathbb{N} \times \Sigma^\omega$; whenever ϕ' converges, we check whether the output is in $\mathrm{dom}(\alpha)$. This way we compute a double-sequence $(w_{m,\ell}, a_{m,\ell})_{m,\ell}$ in $\Sigma^* \times \mathrm{dom}(\alpha)$ such that the following holds for all m: the sequence $(w_{m,\ell})_\ell$ is prefix-free, $\bigcup_\ell w_{m,\ell}\Sigma^\omega \supseteq \delta^{-1}(X \setminus N)$ (where N is as in Definition 3.6), and $\phi'(m,q) = a_{m,\ell}$ whenever $\delta(q) \in X \setminus N$, $q \in w_{m,\ell}\Sigma^\omega$.

[8] But Lemma 5.8 might also be interesting in its own right, because the assumptions it makes are somewhat weaker than needed for the proof of Theorem 5.9.

As ϕ' is a $[\delta \to \alpha]^{\mu^*}_{\mathrm{APP/AE}}$-realization of f, we have $\mu^*(H_m) \leq 2^{-m}$ for every m where

$$H_m := \bigcup_\ell ([d(f, \alpha(a_{m,\ell})) > 2^{-m}] \cap W(\delta, w_{m,\ell})).$$

We can now apply assumption (ii) to each sequence $(w_{m,\ell})_\ell$ and compute sequences $(V_{m,\ell,r,k})_{m,\ell,r,k}$ and $(\widetilde{V}_{m,r,k})_{m,r,k}$ such that

$$U_r \subseteq \bigcup_\ell V_{m,\ell,r,k} \cup \widetilde{V}_{m,r,k}$$

and $\mu^*(L_{m,r,k}) \leq 2^{-k}$, where

$$L_{m,r,k} := U_r \cap \left(\widetilde{V}_{m,r,k} \cup \bigcup_\ell (V_{m,\ell,r,k} \setminus W(\delta, w_{m,\ell})) \right).$$

Now first find an r_0 such that $\delta(p) \in U_{r_0}$, then put $m_0 := n+1$, $k_0 := n + r_0 + 2$ and effectively determine a set

$$A \in \{V_{m_0,\ell,r_0,k_0}\}_\ell \cup \{\widetilde{V}_{m_0,r_0,k_0}\}$$

with $\delta(p) \in A$. In case that A is $\widetilde{V}_{m_0,r_0,k_0}$, put out an arbitrary $a \in \mathrm{dom}(\alpha)$; in case that A is V_{m_0,ℓ,r_0,k_0} for some ℓ, put out $a_{m_0,\ell}$.

We have to verify that the ϕ computed by this algorithm is correct. From the construction it follows that if $d((\alpha \circ \phi)(p,n), (f \circ \delta)(p)) > 2^{-n}$ for some $p \in \mathrm{dom}(\delta), n \in \mathbb{N}$, then this must be because of one of the following:

- $\delta(p) \in N$,
- $\delta(p) \in H_{n+1} \setminus N$ (i.e. $\delta(p)$ is in the set where ϕ'_{n+1} does not work well),
- there is an $r \in \mathbb{N}$ such that $\delta(p) \in L_{n+1,r,n+r+2} \setminus (N \cup H_{n+1})$ (i.e. ϕ'_{n+1} would work well on p, but ϕ_{n+1} possibly differs from it here).

We can hence estimate:

$$\begin{aligned}\mu^*([e(f, \delta, \phi_n, \cdot)] > 2^{-n}) &\leq \mu^*(H_{n+1}) + \sum_r \mu^*(L_{n+1,r,n+r+2}) \\ &\leq 2^{-(n+1)} + \sum_r 2^{-(n+r+2)} = 2^{-n}.\end{aligned}$$

□

Theorem 5.9 *Let (X, β, ϑ) be a computably regular computable topological space with standard representation δ. Let ν be a $\vartheta_{\mathcal{M}<}$-computable Borel measure on X with the additional property:*

$$\nu|_\beta \text{ takes only finite values and is } (\vartheta, \rho_>)\text{-computable.} \tag{3}$$

Then $[\delta \to \alpha]^\nu_{\mathrm{APP/AE}} \leq [\delta \to \alpha]^\nu_{\mathrm{APP}}$.

We start with an auxiliary lemma:

Lemma 5.10. *Under the assumptions of Theorem 5.9, we can $\vartheta_<$-compute from every $[\vartheta, \nu_{\mathbb{N}}]$-name of every $(V, k) \in \beta \times \mathbb{N}$ a $U \in \mathcal{O}(X)$ such that $X \setminus V \subseteq U$ and $\nu(V \cap U) \leq 2^{-k}$.*

Proof. From the ϑ-input V one can $[\vartheta^\cap, \vartheta_<]^\omega$-compute a sequence $(V_n, U_n)_n$ as in Definition 2.5. As the number $\nu(V)$ can be $\rho_>$-computed and the numbers $\nu(V_0 \cup \ldots \cup V_m)$ can be $\rho_<$-computed for every m, we can effectively find some m such that

$$\nu(V \cap U_0 \cap \cdots \cap U_m) \leq \nu(V \setminus (V_0 \cup \ldots \cup V_m)) = \nu(V) - \nu(V_0 \cup \ldots \cup V_m) \leq 2^{-k}.$$

So put out $U_0 \cap \cdots \cap U_m$. □

Proof of Theorem 5.9. It is sufficient to check that assumptions of Lemma 5.8 are fulfilled for $\mu^* = \nu^*$ and $\theta = \vartheta_<$. It is easy to check that that δ and $\vartheta_<$ are compatible. Let us turn to assumptions (i) and (ii) from the lemma: Let $(u_r)_r$ be a computable enumeration of $\mathrm{dom}(\vartheta)$; choose $U_r := \vartheta(u_r)$. Now suppose we are given a sequence $(w_\ell)_\ell$ and r, k as in assumption (ii). If D is defined as in Lemma 2.4, we can compute a sequence $(w'_\ell)_\ell$ such that $\{w'_\ell\}_\ell = \{w_\ell\}_\ell \cap D$. For $w \notin D$, one has $W(\delta, w) = \emptyset$, so $(w'_\ell)_\ell$ still has the property $\nu\left(X \setminus \bigcup_\ell W(\delta, w'_\ell)\right) = 0$. Let us w.l.o.g. assume that $\{w_\ell\}_\ell \subseteq D$. By the second assertion of Lemma 2.4, we can $[\vartheta^\cap]^\omega$-compute the sequence $(W(\delta, w_\ell))_\ell$. We have $\nu\left(U_r \setminus \bigcup_\ell W(\delta, w'_\ell)\right) = 0$, and hence, in view of the computability of ν and (3), we can effectively find a number $s \in \mathbb{N}$ such that

$$\nu\big(U_r \setminus \bigcup_{\ell \leq s} W(\delta, w_\ell)\big) = \nu(U_r) - \nu\big(U_r \cap \bigcup_{\ell \leq s} W(\delta, w_\ell)\big) \leq 2^{-(k+1)}.$$

Choose

$$V_\ell := \begin{cases} W(\delta, w_\ell) & \text{for } \ell \leq s \\ \emptyset & \text{for } \ell > s. \end{cases}$$

Resolving the definition of $\vartheta^\cap$, we have a $[[\vartheta]^{<\omega}]^s$-computable tuple

$$((V_{1,1}, \ldots, V_{1,t(1)}), \ldots, (V_{s,1}, \ldots, V_{s,t(s)}))$$

such that $W(\delta, w_\ell) = V_{\ell,1} \cap \cdots \cap V_{\ell,t(\ell)}$ for all $\ell \leq s$. For all $\ell \leq s$ and $i \leq t(\ell)$, apply the auxiliary lemma to the pair $(V_{\ell,i}, \lceil \log s + \log t(\ell) \rceil + k + 1)$ and let

$$((\widetilde{V}_{1,1}, \ldots, \widetilde{V}_{1,t(1)}), \ldots, (\widetilde{V}_{s,1}, \ldots, \widetilde{V}_{s,t(s)}))$$

be the tuple $[[\vartheta_<]^{<\omega}]^s$-computed that way. Choose

$$\widetilde{V} := \bigcap_{\ell \leq s} \bigcup_{i \leq t(\ell)} \widetilde{V}_{\ell,i}$$

and note that we can $\vartheta_<$-compute $\widetilde{V}$. One easily verifies that $X \subseteq \bigcup_\ell V_\ell \cup \widetilde{V}$, and hence the first part of assumption (iv) is fulfilled. The second part is fulfilled because

$$U_r \cap \big(\widetilde{V} \cup \bigcup_\ell (V_\ell \setminus W(\delta, w_\ell))\big) = U_r \cap \widetilde{V}$$

and

$$\begin{aligned}
\nu(U_r \cap \widetilde{V}) &\le \nu\Big(\big(\bigcup_{\ell \le s} W(\delta, w_\ell)\big) \cap \widetilde{V} \Big) + 2^{-k+1} \\
&= \nu\Big(\big(\bigcup_{\ell \le s} \bigcap_{i \le t(\ell)} V_{\ell,i}\big) \cap \big(\bigcap_{\ell \le s} \bigcup_{i \le t(\ell)} \widetilde{V}_{\ell,i}\big) \Big) + 2^{-k+1} \\
&\le \sum_{\ell \le s} \nu\Big(\big(\bigcap_{i \le t(\ell)} V_{\ell,i}\big) \cap \big(\bigcup_{i \le t(\ell)} \widetilde{V}_{\ell,i}\big) \Big) + 2^{-k+1} \\
&\le \sum_{\ell \le s} \sum_{i \le t(\ell)} \nu(V_{\ell,i} \cap \widetilde{V}_{\ell,i}) + 2^{-k+1} \\
&\le 2^{-k}.
\end{aligned}$$

□

Theorem 5.9 is in fact a generalization of result of Parker (cf. [Parker 2003, Theorem II]), who proves that the characteristic function of a subset of Euclidean space is APP-computable if it is AE-computable with respect to Lebesgue measure. Parker's proof already contains the central ideas of our the proof of Theorem 5.9.

We have the following corollary for finite measures on metric spaces:

Theorem 5.11 *Suppose that (X, β, ϑ) is the computable topological space derived from a computable metric space (see Lemma 2.8), and let δ be its standard representation. Also suppose that ν is a $\vartheta^0_{\mathcal{M}<}$-computable measure. Then $[\delta \to \alpha]^\nu_{\text{APP/AE}} \le [\delta \to \alpha]^\nu_{\text{APP}}$.*

Proof. Apply Corollary 2.17 to (X, β, ϑ), and note that the resulting computable topological space $(X, \beta^\nu, \vartheta^\nu)$ (with standard representation δ^ν) fulfills the assumptions of Theorem 5.9, hence $[\delta^\nu \to \alpha]^\nu_{\text{APP/AE}} \le [\delta^\nu \to \alpha]^\nu_{\text{APP}}$. δ and δ^ν are equivalent, hence Lemma 3.9 yields the claim. □

6 Computability of vector-valued integration

Assumption 6.1 Throughout this section we assume that

- ν is finite,
- Y is a normed space over $\mathbb{R}$,

– the norm on Y is (δ_Y, ρ)-computable,

– vector addition is $(\delta_Y, \delta_Y, \delta_Y)$-computable,

– scalar multiplication is $(\rho, \delta_Y, \delta_Y)$-computable.

The following definitions and basic facts are taken from [Vakhania et al. 1987, Section II.3.1]: Let Y^* denote the topological dual of Y, and let $\mathcal{C}(Y)$ be the *cylindrical σ-algebra* on Y, i.e. the coarsest σ-algebra w.r.t. which all elements of Y^* are measurable. Suppose that $f : X \to Y$ is an $(\mathcal{S}, \mathcal{C}(Y))$-measurable mapping. We say that f is of *weak order p* (for $0 < p < \infty$) if $\int |g \circ f|^p \, d\nu < \infty$ for every $g \in Y^*$. If f is of weak order one, then we call an element y_f of Y *(Pettis) integral of f w.r.t. ν* if

$$(\forall\, g \in Y^*) \ \int g \circ f \, d\nu = g(y_f).$$

If there is an integral of f, then it is unique and we denote it by $\mathbb{E}(f; \nu)$. The mappings for which the integral exists form a vector space on which $\mathbb{E}(\cdot; \nu)$ is linear. For real-valued mappings, the Pettis integral is equal to the usual integral. Now suppose that $f : X \to Y$ is $(\mathcal{S}, \mathcal{B}(Y))$-measurable. We say that f is of *strong order p* (for $0 < p < \infty$) if $\int \|f\|^p \, d\nu < \infty$. Every mapping of strong order p is of weak order p. If f is of strong order one and $\mathbb{E}(f; \nu)$ exists, then $\|\mathbb{E}(f; \nu)\| \leq \mathbb{E}(\|f\|; \nu)$. For the existence of $\mathbb{E}(f; \nu)$, it is sufficient that f is of strong order one and Y is complete.

Under what circumstances and from what input is $\mathbb{E}(f; \nu)$ δ_Y-computable? Consider for example $\nu = \gamma$ with γ being the standard Gaussian distribution on $\mathbb{R}$. It is an easy exercise to make up a γ-integrable $(\rho, \nu_{\mathbb{Q}})^{\gamma}_{\mathrm{MEAN}}$-computable function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}(f; \gamma)$ is not $\rho_>$-computable and hence not a computable real.

This example makes clear that integrals cannot be computed from MEAN-names in general, not even for computable probability measures on the real line. The next theorem, however, shows that integration becomes computable under the additional assumption of the computable quasi-compactness of X, or if certain stronger information on the input mapping is provided. The corresponding integration algorithms will be uniform in both the mapping and the measure.

Theorem 6.2 *Let (X, β, ϑ) be a computable topological space, and let δ be its standard representation. Put*

$$L := \{(\nu, f) \ : \nu \in \mathcal{M}_0(X),\ f \text{ is } (\delta, \alpha)^{\nu}_{\mathrm{MEAN}}\text{-continuous and } (\mathcal{S}, \mathcal{B}(Y))\text{-measurable, } \mathbb{E}(f; \nu) \text{ exists}\}.$$

Let Λ be the multi-representation of L defined by

$$(\nu, f) \in \Lambda\langle p, q\rangle \ :\Longleftrightarrow \ \vartheta_{\mathcal{M}=}(p) = \nu \text{ and } f \in [\delta \to \alpha]^{\nu}_{\mathrm{MEAN}}(q).$$

1. *If X is computably quasi-compact, then $(\nu, f) \mapsto \mathbb{E}(f;\nu)$ is (Λ, δ_Y)-computable.*

2. *Define the set*
$$B := \{((\nu, f), b) \in L \times \mathbb{N} \ : \ \|f\| \le b\}.$$
Then $((\nu, f), b) \mapsto \mathbb{E}(f;\nu)$ is $([\Lambda, \nu_{\mathbb{N}}]|^B, \delta_Y)$-computable.

3. *Define the set*
$$I := \{((\nu, f), c) \in L \times \mathbb{R} \ : \ \mathbb{E}(\|f\|;\nu) = c\}.$$
Then $((\nu, f), c) \mapsto \mathbb{E}(f;\nu)$ is $([\Lambda, \rho_>]|^I, \delta_Y)$-computable.

Proof. The proofs for item 1., 2. and 3. start the same: Let ν be the $\vartheta_{\mathcal{M}=}$-encoded input measure, and let f be a $[\delta \to \alpha]^\nu_{\text{MEAN}}$-encoded input mapping, i.e. we are given a $(\delta, \alpha)^\nu_{\text{MEAN}}$-realization Φ of f. It is sufficient to demonstrate how to δ_Y-compute a 2^{-k}-approximation to $\mathbb{E}(f;\nu)$ for $k = 0, 1, 2, \ldots$. So fix an arbitrary k (it will be clear that the construction is uniform in k). By simulation of Φ_{k+2}, we can compute a sequence $(w_\ell, a_\ell)_\ell$ in $\Sigma^* \times \Sigma^*$ such that $(w_\ell)_\ell$ is prefix free and

$$(\forall\, p \in \mathrm{dom}(\delta))(\forall\, a \in \mathrm{dom}(\alpha)) \quad \big[\Phi_{k+2}(p) = a \iff (\exists\, \ell \in \mathbb{N}) \ \big(p \in w_\ell \Sigma^\omega \text{ and } a = a_\ell\big)\big]. \tag{4}$$

Let D be as in Lemma 2.4. We can compute a sequence $(w'_\ell, a'_\ell)_\ell$ such that $\{(w'_\ell, a'_\ell)\}_\ell = \{(w_\ell, a_\ell)\}_\ell \cap (D \times \mathrm{dom}(\alpha))$; this sequence still fulfills (4). We shall w.l.o.g. assume $\{(w_\ell, a_\ell)\}_\ell \subseteq D \times \mathrm{dom}(\alpha)$. Put $A_\ell := W(\delta, w_\ell) \setminus \bigcup_{j<\ell} W(\delta, w_j)$. Note that $(A_\ell)_\ell$ can be $[\vartheta_{\text{alg}}]^\omega$-computed, and hence we can $[\rho]^\omega$-compute the sequence $(\nu(A_\ell))_\ell$. Put $v_\ell := \alpha(a_\ell)$ for all $\ell \in \mathbb{N}$, and

$$s(x) := \sum_\ell \chi_{A_\ell}(x) \cdot v_\ell.$$

For convenience, we set $\mathbb{E}(\cdot) := \mathbb{E}(\cdot;\nu)$. One has

$$\begin{aligned}
\mathbb{E}(\|f - s\|) &= \int \Big\|f(x) - \sum_\ell \chi_{A_\ell}(x) \cdot v_\ell\Big\| \,\nu(dx) \\
&= \int \sup_\ell \chi_{A_\ell}(x) \|f(x) - v_\ell\| \,\nu(dx) \\
&\le \int \sup_\ell \chi_{W(\delta, w_\ell)}(x) \cdot \|f(x) - \alpha(a_\ell)\| \,\nu(dx) \\
&= \int^* e(f, \delta, \Phi_{k+2}, x) \,\nu(dx) \\
&\le 2^{-(k+2)}.
\end{aligned} \tag{5}$$

For every $m \in \mathbb{N}$ put $B_m := \bigcup_{\ell \leq m} A_\ell = \bigcup_{\ell \leq m} W(\delta, w_\ell)$ and

$$y_m := \sum_{\ell \leq m} \nu(A_\ell) v_\ell, \qquad s_m(x) := \chi_{B_m}(x) \cdot s(x) = \sum_{\ell \leq m} \chi_{A_\ell}(x) \cdot v_\ell.$$

One immediately verifies that the sequence $(y_m)_m$ can be $[\delta_Y]^\omega$-computed, and that $\mathbb{E}(s_m) = y_m$. Combining this with (5) yields:

$$\begin{aligned} \|\mathbb{E}(f) - y_m\| &\leq \mathbb{E}(\|f - s_m\|) = \mathbb{E}(\chi_{X \setminus B_m} \cdot \|f\|) + \mathbb{E}(\chi_{B_m} \cdot \|f - s\|) \\ &\leq \mathbb{E}(\chi_{X \setminus B_m} \cdot \|f\|) + 2^{-(k+2)}. \end{aligned}$$

So it is sufficient to compute an m such that $\mathbb{E}(\chi_{X \setminus B_m} \cdot \|f\|) \leq 2^{-(k+1)} + 2^{-(k+2)}$.

For item 1.: By the computable quasi-compactness of X, we can compute an m such that $B_m = X$.

For item 2.: We can effectively find an m such that $\nu(X \setminus B_m) \leq b^{-1}(2^{-(k+1)} + 2^{-(k+2)})$.

For item 3.: From (5), it follows that

$$2^{-(k+2)} \geq \mathbb{E}(\|f\|) - \mathbb{E}(\|s\|) = \mathbb{E}(\|f\|) - \lim_{m \to \infty} \mathbb{E}(\chi_{B_m} \cdot \|s\|),$$

where $\lim_{m \to \infty} \mathbb{E}(\chi_{B_m} \cdot \|s\|) = \mathbb{E}(\|s\|)$ holds by Monotone Convergence. The sequence $(\mathbb{E}(\chi_{B_m} \cdot \|s\|))_m$ can be $[\rho]^\omega$-computed, because

$$\mathbb{E}(\chi_{B_m} \cdot \|s\|) = \mathbb{E}(\|s_m\|) = \sum_{\ell \leq m} \nu(A_\ell) \|v_\ell\|.$$

As we are given a $\rho_>$-name of $\mathbb{E}(\|f\|)$, we can effectively find an m such that $2^{-(k+1)} \geq \mathbb{E}(\|f\|) - \mathbb{E}(\|s_m\|)$. This estimate and (5) finally yield

$$\begin{aligned} \mathbb{E}(\chi_{X \setminus B_m} \cdot \|f\|) &= \big(\mathbb{E}(\|f\|) - \mathbb{E}(\chi_{B_m} \cdot \|s\|)\big) + \big(\mathbb{E}(\chi_{B_m} \cdot \|s\|) - \mathbb{E}(\chi_{B_m} \cdot \|f\|)\big) \\ &\leq 2^{-(k+1)} + 2^{-(k+2)}. \end{aligned}$$

□

Corollary 6.3. *If the space (X, β, ϑ) from the previous theorem is derived from a computable metric space as in Lemma 2.8, then the theorem still holds true if Λ is defined in the following (weaker) way:*

$$(\nu, f) \in \Lambda\langle p, q \rangle \;:\Longleftrightarrow\; \vartheta^0_{\mathcal{M}<}(p) = \nu \text{ and } f \in [\delta \to \alpha]^\nu_{\text{MEAN}}(q).$$

Proof. The proof of the previous theorem still goes through if the definitions of the sequences $(A_\ell)_\ell$ and $(v_\ell)_\ell$ are changed in the following way: First apply the algorithm from Corollary 2.16 to $(\nu, (W(\delta, w_\ell))_\ell)$; let $(U_m, \widetilde{U}_m, \ell_m)_m$ be the result. Define $A_m := U_m \cap \bigcap_{j<m} \widetilde{U}_j$, and define $v_m := \alpha(a_{\ell_m})$. Essentially two things have to be noted: (i) all estimates in (5) still hold true, and (ii) $(\nu(A_\ell))_\ell$

can be $[\rho]^\omega$-computed, because it can obviously be $[\rho_<]^\omega$-computed, and it can be $[\rho_>]^\omega$-computed by the identity

$$\begin{aligned}\nu(A_m) &= \nu\big(U_m \cap \bigcap_{j<m} \widetilde{U}_j\big) = \nu(X) - \nu\big((X\setminus U_m) \cup \bigcup_{j<m} (X \setminus \widetilde{U}_j)\big)\\ &= \nu(X) - \nu\big(\widetilde{U}_m \cup \bigcup_{j<m} U_j\big).\end{aligned}$$

□

7 Miscellaneous

7.1 Composition

We will now prove two theorems on APP-computability of compositions of mappings. The first result is a partial answer to the natural question whether the composition of two APP-computable mappings is still APP-computable. The second result is (a uniform version of) the observation that APP-computability is preserved under composition with computable mappings with a computable modulus of uniform continuity; this will be useful in the following subsection.

Assumption 7.1 In this subsection we assume that (Z, d', α') is a computable metric space with Cauchy representation δ_Z.

Theorem 7.2 *Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If f is $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-computable and g is $(\delta_Y,\alpha')^{\mu^*\circ f^{-1}}_{\mathrm{APP}}$-computable, then $g\circ f$ is $(\delta,\alpha')^{\mu^*}_{\mathrm{APP/AE}}$-computable.*

Proof. Let ϕ be a $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f. Consider the mapping

$$a : \mathbb{N}\times \mathrm{dom}(\delta) \to \Sigma^\omega, \quad a(n,p) := \iota(\phi(n+1,p))\iota(\phi(n+2,p))\iota(\phi(n+3,p))\cdots.$$

For all $n \in \mathbb{N}$, put

$$R_n := \{x \in X \ : \ (\exists\, p \in \delta^{-1}\{x\})\ a(n,p) \notin \delta_Y^{-1}\{f(x)\}\},$$

and note that $\mu^*(R_n) \le 2^{-n}$, because

$$R_n \subseteq \bigcup_k [e(f,\delta,\phi_{n+k+1},\cdot) > 2^{-(k+1)}] \subseteq \bigcup_k [e(f,\delta,\phi_{n+k+1},\cdot) > 2^{-(n+k+1)}],$$

and the set on the right hand side has μ^*-content at most $\sum_k 2^{-(n+k+1)} = 2^{-n}$ by assumption.

Now let ϕ' be a $(\delta_Y,\alpha')^{\mu^*\circ f^{-1}}_{\mathrm{APP}}$-realization of g. Consider the following procedure: “On input $(n,p)\in\mathbb{N}\times\mathrm{dom}(\delta)$, run a dovetailed process that simulates the computation of a machine for ϕ' on all inputs $(n+1, a(n+m+2,p))$, $m \geq 0$. Each time one of these threads of simulation halts, try to verify that its output is in the domain of α', and once this succeeds, halt and put it out." Put $N := \bigcup_n \bigcap_m R_{n+m+2}$ and note that $\mu^*(N) = 0$. For given (n,p), the procedure just described will surely halt, if $a(n+m+2,p) \in \mathrm{dom}(\delta_Y)$ for at least one m. Hence, if the procedure does not halt, then $\delta(p) \in \bigcap_m R_{n+m+2}$. So the procedure defines a computable mapping $\widetilde{\phi} : \mathbb{N} \times \mathrm{dom}(\delta|^{X\setminus N}) \to \mathrm{dom}(\alpha')$. It is sufficient to show that $\widetilde{\phi}$ is a $(\delta|^{X\setminus N}, \alpha')^{\mu^*\circ f^{-1}}_{\mathrm{APP}}$-realization of $g \circ f|_{X\setminus N}$.

If for some $n \in \mathbb{N}$, $x \in X$, we have that both the conditions

$$(\forall\, p \in \delta^{-1}\{x\})(\forall\, m \in \mathbb{N})\ a(n+m+2,p) \in \delta_Y^{-1}(f(x))$$

and

$$(\forall\, q \in \delta_Y^{-1}\{f(x)\})\ d((\alpha' \circ \phi')(n+1,q), (g \circ f)(x)) \leq 2^{-(n+1)}$$

are fufilled, then it follows from the construction of our procedure for $\widetilde{\phi}$ that

$$(\forall\, p \in \delta^{-1}\{x\})\ d((\alpha' \circ \widetilde{\phi})(n,p), (g \circ f)(x)) \leq 2^{-(n+1)} \leq 2^{-n}.$$

This implies

$$[e(g\circ f|_{X\setminus N}, \delta|^{X\setminus N}, \widetilde{\phi}_n, \cdot) > 2^{-n}] \subseteq \bigcup_m R_{n+m+2} \cup f^{-1}[e(g, \delta_Y, \phi'_{n+1}, \cdot) > 2^{-(n+1)}].$$

Now note that $\mu^*\left(\bigcup_m R_{n+m+2}\right) \leq 2^{-(n+1)}$ by construction, and

$$(\mu^* \circ f^{-1})[e(g, \delta_Y, \phi'_{n+1}, \cdot) > 2^{-(n+1)}] \leq 2^{-(n+1)}$$

by assumption. □

Definition 7.3. A representation $[\delta_Y \to \delta_Z]_{\mathrm{uni}}$ of the space $C(Y,Z)_{\mathrm{uni}}$ of all uniformly continuous mappings from Y to Z is given by

$$[\delta_Y \to \delta_Z]_{\mathrm{uni}}\langle p, q\rangle = f \ :\Longleftrightarrow [\delta_Y \to \delta_Z](p) = f \text{ and } [\nu_{\mathbb{N}} \to \nu_{\mathbb{N}}](q) \text{ is a modulus of continuity of } f \text{ on } X$$

(cf. [Weihrauch 2000, Definition 6.2.6.2]).

Proposition 7.4. *The mapping*

$$\mathrm{APP}_{\delta,\alpha,\mu^*} \times C(Y,Z)_{\mathrm{uni}} \to \mathrm{APP}_{\delta,\alpha',\mu^*} \qquad (f,g) \mapsto g \circ f,$$

is $([\delta \to \alpha]^{\mu^*}_{\mathrm{APP}}, [\delta_Y \to \delta_Z]_{\mathrm{uni}}, [\delta \to \alpha']^{\mu^*}_{\mathrm{APP}})$*-computable.*

Proof. Let ϕ be the given $(\delta,\alpha)^{\mu^*}_{\mathrm{APP}}$-realization of f, and let $m : \mathbb{N} \to \mathbb{N}$ be the given modulus of uniform continuity of g. On input $(n,p) \in \mathbb{N}\times\mathrm{dom}(\delta)$, compute and put out an α'-name of a $2^{-(n+1)}$-approximation of $g(\phi(\max\{n, m(n+1)\}, p))$. It is easy to see that the mapping $\phi' : \mathbb{N} \times \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha')$ computed this way is a $(\delta,\alpha')^{\mu^*}_{\mathrm{APP}}$-realization of $g \circ f$. □

7.2 Images of measures

Is the image of a computable measure under an APP-computable mapping again computable? The next theorem gives sufficient conditions for a positive answer:

Theorem 7.5 *Let (X,β,ϑ) be a computable topological space, and let δ be its standard representation. Let (Y,β',ϑ') be the computable topological space derived from (Y,d,α) as in Lemma 2.8. The mapping*

$$\mathrm{APP}_{\delta,\alpha,\nu} \to \mathcal{M}_0(Y), \qquad f \mapsto \nu \circ f^{-1},$$

is $([\delta \to \alpha]^{\nu}_{\mathrm{APP}}, \vartheta'^{0}_{\mathcal{M}<})$-computable if one of the following holds true:

(i) ν is $\vartheta_{\mathcal{M}=}$-computable.

(ii) (X,β,ϑ) is derived from a computable metric space and ν is $\vartheta^{0}_{\mathcal{M}<}$-computable.

Proof. It follows from the definition of $\vartheta'^{0}_{\mathcal{M}<}$ that it is sufficient to demonstrate how to uniformly $\rho_<$-compute the $\nu \circ f^{-1}$-content of any given set

$$V = \bigcup_{i=1}^{m} \bigcap_{j=1}^{k(i)} B(a_{i,j}, \epsilon_{i,j})$$

(with $a_{i,j} \in \mathrm{range}(\alpha)$, $\epsilon_{i,j} \in \mathbb{Q} \cap]0,\infty[$) from a $[\delta \to \alpha]^{\nu}_{\mathrm{APP}}$-name on f. For all $1 \le i \le m$, $1 \le j \le k(i)$, and $n \in \mathbb{N}$, we can $[\delta_Y \to \rho_C]_{\mathrm{uni}}$-compute the function

$$g_{i,j,n}(x) := \max\{0, \min\{1, 2^n(\epsilon_{i,j} - d(a_{i,j}, x))\}\}.$$

The sequence $(g_{i,j,n})_n$ converges monotonously to the characteristic function of $B(a_{i,j}, \epsilon_{i,j})$. For each $n \in \mathbb{N}$ we can $[\delta_Y \to \rho_C]_{\mathrm{uni}}$-compute the function

$$g_n(x) := \max_{1 \le i \le m} \min_{1 \le j \le k(i)} g_{i,j,n}(x).$$

We have $0 \le g_n \le 1$ and[9] $g_n \nearrow \chi_V$, and hence $\int g_n \circ f \, d\nu = \int g_n \, d(\nu \circ f^{-1}) \nearrow (\nu \circ f^{-1})(V)$. It is sufficient to demonstrate how to $[\rho]^{\omega}$-compute the sequence $(\int g_n \circ f \, d\nu)_n$. It follows from Proposition 7.4 that we can $[[\delta \to \nu_{\mathbb{Q}}]^{\nu}_{\mathrm{APP}}]^{\omega}$-compute the sequence $(g_n \circ f)_n$. The sequence is uniformly bounded by 1, so we can even $[[\delta \to \nu_{\mathbb{Q}}]^{\nu}_{\mathrm{MEAN}}]^{\omega}$-compute it by Lemma 5.2. The sequence of integrals can now be computed by Theorem 6.2 (in case item (i) above holds true) or Corollary 6.3 (in case item (ii) above holds true). $\square$

[9] $\nearrow$ denotes the relation "converges pointwise monotonously from below to".

7.3 Measurability of the local error

In Definitions 3.5.3 and 3.7, we used the outer measure ν^* and the outer integral $\int^* d\nu$, respectively. The reason for using those instead of ν itself and the proper integral $\int d\nu$ is, that one cannot be sure that the local error is always $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable. Although we were able to develop the theory of APP- and MEAN-computability without the requirement that the local error is measurable, we consider it an interesting question, under what conditions it actually is measurable. It seems to be reasonable to put the question in the following form: When does the measurability of f imply the measurability of $e(f, \delta, \Phi, \cdot)$?

Proposition 7.6. *Suppose that $\sigma(\delta^{-1}) \subseteq \mathcal{S}$. Let $\Phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\rho)$ be continuous, and let $f : X \to Y$ be $(\mathcal{S}, \mathcal{B}(Y))$-measurable. Then $e(f, \delta, \Phi, \cdot)$ is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable.*

Proof. There is a prefix-free set $\{w_\ell\}_\ell \subseteq \Sigma^*$ such that $\mathrm{dom}(\delta) \subseteq \bigcup_\ell w_\ell \Sigma^\omega$ and Φ is constantly equal to some $a_\ell \in \mathrm{dom}(\alpha)$ on each set $w_\ell \Sigma^\omega \cap \mathrm{dom}(\delta)$. One then has

$$e(f, \delta, \Phi, x) = \sup_\ell \chi_{W(\delta, w_\ell)}(x) d((\alpha \circ \Phi)(p), f(x)).$$

$e(f, \delta, \Phi, \cdot)$ is a countable supremum of measurable functions and hence itself measurable. □

Lemma 7.7. *Suppose that $\sigma(\delta^{-1}) \nsubseteq \mathcal{S}$ and that Y contains at least two distinct points. Then there is a constant mapping $f : X \to Y$ and a computable $\Phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$ such that $e(f, \delta, \Phi, .)$ is not $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable.*

Proof. There must be at least two distinct points $\alpha(a_0), \alpha(a_1)$ in $\mathrm{range}(\alpha)$. Choose $f \equiv \alpha(a_0)$. There must be a $w \in \Sigma^*$ such that $W(\delta, w) \notin \mathcal{S}$. Define Φ by

$$\Phi(p) := \begin{cases} a_1 & \text{if } p \in w\Sigma^\omega \\ a_0 & \text{else.} \end{cases}$$

We then have $e(f, \delta, \Phi, \cdot)^{-1}((0, \infty)) = W(\delta, w)$. □

We combine the last two results:

Corollary 7.8. *Suppose that Y contains at least two distinct points. Then the following two statements are equivalent:*

1. *For every $(\mathcal{S}, \mathcal{B}(Y))$-measurable $f : X \to Y$ and every continuous $\Phi : \mathrm{dom}(\delta) \to \mathrm{dom}(\alpha)$, we have that $e(f, \delta, \Phi, \cdot)$ is $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable.*
2. *$\sigma(\delta^{-1}) \subseteq \mathcal{S}$.*

□

We have found that $\sigma(\delta^{-1}) \subseteq \mathcal{S}$ is the crucial condition. This condition has already appeared before in a different context (see Proposition 4.2 and Corollary 4.5). It is time to ask for conditions under which it is fulfilled. We have already seen (Lemma 2.4) that we have $\sigma(\delta^{-1}) \subseteq \mathcal{B}(X)$ if δ is the standard representation associated with an effective topological structure on X. For the Cauchy representation of a computable metric space, there are no complications, either:

Proposition 7.9. *Let (X, d', α') be a computable metric space with Cauchy representation δ_X. Then $\sigma(\delta_X^{-1}) \subseteq \mathcal{B}(X)$.*

Proof. Let $w \in \Sigma^*$ be arbitrary. Let us first suppose that w has the form

$$\iota(w_0)\iota(w_1)\ldots\iota(w_k). \tag{6}$$

Either $W(\delta_X, w) = \emptyset$ (and is hence measurable), or w can be extended to an element of $\mathrm{dom}(\delta_X)$. In the latter case, we know that $w_0, \ldots, w_k \in \mathrm{dom}(\alpha')$ and $d'(\alpha'(w_i), \alpha(w_j)) \leq 2^{-i}$ for $0 \leq i < j \leq k$. It is easy to see that

$$W(\delta, w) = \overline{\bigcap_{0 \leq i \leq k} \{a \in \mathrm{range}(\alpha') : d'(\alpha'(w_i), a) \leq 2^{-i}\}}$$

and hence is a closed set. If w is not necessarily of the form (6), we still have

$$W(\delta, w) = \bigcup \{W(\delta_X, wv) \ : \ wv \text{ is of the form (6)}\}$$

So $W(\delta, w)$ is an at most countable union of closed sets and hence Borel. □

A different different type of sufficient condition for $\sigma(\delta^{-1}) \subseteq \mathcal{S}$ is presented in the following:

Proposition 7.10. *Suppose $(X, \mathcal{A})$ is a standard Borel space*[10]*, μ is a σ-finite measure on $(X, \mathcal{A})$, D is a Borel subset of Σ^ω ,and $\delta : D \to X$ is a Borel measurable representation of X. Put $\mathcal{S} = \mathcal{A}_\mu$. Then $\sigma(\delta^{-1}) \subseteq \mathcal{S}$.*

Proof. By [Kechris 1995, Corollary 13.4], all Borel subsets of a standard Borel space are again standard Borel, hence all $w\Sigma^\omega \cap D$ are. From [Kechris 1995, Exercise 14.6] we have that Borel images of standard Borel spaces are analytic (see [Kechris 1995, Definition 14.1]); so all sets of the form $\delta(w\Sigma^\omega \cap D)$ are analytic. Finally, [Kechris 1995, Theorem 21.10] asserts that every analytic subset of a standard Borel space is universally measurable, which means μ-measurable with respect to any σ-finite Borel measure μ. □

[10] A measurable space is called a (standard) *Borel space* if it is isomorphic to $(Y, \mathcal{B}(Y))$ for some Polish space Y. A *Polish space* is a separable completely metrizable topological space.

A natural question is whether

$$\left[\sigma(\delta_1^{-1}) \subseteq \mathcal{S} \text{ and } \delta_1 \equiv \delta_2\right] \implies \sigma(\delta_2^{-1}) \subseteq \mathcal{S}.$$

Surprisingly, the answer is "No" (even for admissible real number representations), as can be seen by combining Proposition 7.9 with the next proposition:

Proposition 7.11. *Suppose that (X, d', α') is a perfect*[11] *and Polish computable metric space. Let δ_X denote its Cauchy representation. There is a representation δ of X such that $\mathrm{dom}(\delta) \in \mathcal{B}(\Sigma^\omega)$, $\delta \equiv \delta_X$, and $\sigma(\delta^{-1}) \not\subseteq \mathcal{B}(X)$.*

Proof. Let $\mathcal{N}$ be the Baire space and let $\delta_\mathcal{N}$ be its representation as defined in [Weihrauch 2000, Definition 3.1.2.8]; one easily verifies that

- $\delta_\mathcal{N}$ is a homeomorphism between $\mathrm{dom}(\delta_\mathcal{N})$ and $\mathcal{N}$,
- $\delta_\mathcal{N}^{-1} : \mathcal{N} \to \Sigma^\omega$ is $(\delta_\mathcal{N}, \mathrm{id}_{\Sigma^\omega})$-computable,
- $\mathrm{dom}(\delta_\mathcal{N}) \in \mathcal{B}(\Sigma^\omega)$.

It is also clear that the projection $\pi_{1,2} : \mathcal{N}^3 \to \mathcal{N}^2$ onto the first two coordinates as well as the standard homeomorphic tuplings

$$\underbrace{\langle \cdot, \cdot, \ldots, \cdot, \cdot \rangle}_{n} : \mathcal{N}^n \to \mathcal{N},\ n \geq 1,$$

are $([\delta_\mathcal{N}]^3, [\delta_\mathcal{N}]^2)$- and $([\delta_\mathcal{N}]^n, \delta_\mathcal{N})$-computable, respectively.

From [Kechris 1995, Proof of Theorem 14.2], we have that there is a closed set $\mathcal{F} \subseteq \mathcal{N}^3$ such that $\pi_{1,2}(\mathcal{F})$ is not Borel. The homeomorphous image $F := (\delta_\mathcal{N}^{-1} \circ \langle \cdot, \cdot, \cdot \rangle)(\mathcal{F})$ is closed in $\mathrm{dom}(\delta_\mathcal{N})$ and is hence Borel.

By a straightforward effectivization of [Kechris 1995, Theorem 6.2], there is an $(\mathrm{id}_{\Sigma^\omega}, \delta_X)$-computable injective mapping $\iota : \Sigma^\omega \to X$. We have that the composition

$$\mathcal{N}^2 \overset{\langle \cdot, \cdot \rangle}{\to} \mathcal{N} \overset{(\delta_\mathcal{N})^{-1}}{\to} \Sigma^\omega \overset{\iota}{\to} X,$$

which we shall call H, is injective and $([\delta_\mathcal{N}]^2, \delta_X)$-computable. We also have that $A := H(\pi_{1,2}(\mathcal{F}))$ is non-Borel in X (because A is the continuous injective image of a non-Borel set). We also have that

$$\widetilde{\delta} := H \circ \pi_{1,2} \circ \langle \cdot, \cdot, \cdot \rangle^{-1} \circ \delta_\mathcal{N} : F \to A$$

is a representation of A with Borel-domain and $\widetilde{\delta} \leq \delta_X|^A$.

It is easy to verify that $\mathrm{dom}(\delta_X) =: D$ itself is Borel. So we can define δ by $\mathrm{dom}(\delta) = \mathtt{0}D \cup \mathtt{1}F$, $\delta(\mathtt{0}p) = \delta_X(p)$ and $\delta(\mathtt{1}p) = \widetilde{\delta}(p)$. So of course δ has Borel domain, $\delta_X \equiv \delta$, and $W(\delta, \mathtt{1}) = A$ is not Borel. $\square$

[11] A topological space is called *perfect* if it does not have any isolated points.

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References

[Bishop and Bridges 1985] Bishop, E. and Bridges, D.: *Constructive Analysis.* Springer, Berlin, 1985.

[Bosserhoff] Bosserhoff, V.: Are unbounded linear operators computable on the average for Gaussian measures? *J. Complexity*, accepted for publication.

[Edalat 1993] Edalat, A.: Dynamical systems, measures and fractals via domain theory. In Burn, G.L., Gay, S.J. and Ryan, M., editors, *Theory and Formal Methods*, Workshops in Computing, pages 82-99. Springer, 1993.

[Edalat 1995] Edalat, A.: Domain theory and integration. *Theoret. Comput. Sci.*, 151(1):163-193, 1995.

[Gács 2005] Gács, P.: Uniform test of algorithmic randomness over a general space. *Theoret. Comput. Sci.*, 341:91-137, 2005.

[Hertling 2005] Hertling, P.: On the solvability of ill-posed problems in two models of computation. Talk presented at the Foundations of Computational Mathematics conference, Santander, Spain, July 2005.

[Kallenberg 2002] Kallenberg, O.: *Foundations of Modern Probability.* Springer, New York, second edition, 2002.

[Kechris 1995] Kechris, A.S.: *Classical Descriptive Set Theory.* Springer, New York, 1995.

[Ko 1991] Ko, K.-I: *Complexity Theory of Real Functions.* Birkhäuser, Boston, 1991.

[Müller 1999] Müller, N.Th.: Computability on random variables. *Theoret. Comput. Sci.*, 219:287-299, 1999.

[Parker 2003] Parker, M.W.: Undecidability in $\mathbb{R}^n$: Riddled basins, the KAM tori, and the stability of the solar system. *Philos. Sci.*, 70(2):359-382, 2003.

[Parker 2005] Parker, M.W.: *Undecidable long-term behavior in classical physics: Foundations, results, and interpretation.* PhD thesis, University of Chicago, 2005.

[Parker 2006] Parker, M.W.: Three concepts of decidability for general subsets of uncountable spaces. *Theoret. Comput. Sci.*, 351:2-13, 2006.

[Schröder 1998] Schröder, M.: Effective metrization on regular spaces. In Ko, K.-I, Nerode, A., Pour-El, M.B., Weihrauch, K. and Wiedermann, J., editors, *Computability and Complexity in Analysis (Proceedings of CCA Workshop, Brno, Czech Republic, August 1998)*, volume 235 of *Informatik Berichte*, pages 63-80, 1998.

[Schröder 2002a] Schröder, M.: Effectivity in spaces with admissible multirepresentations. *Math. Logic Quart.*, 48:78-90, 2002.

[Schröder 2002b] Schröder, M.: Extended admissibility. *Theoret. Comput. Sci.*, 284:519-538, 2002.

[Schröder 2007] Schröder, M.: Admissible representations for probability measures. *Math. Logic Quart.*, 53:431-445, 2007.

[Schröder and Simpson 2006] Schröder, M. and Simpson, A.: Representing probability measures using probabilistic processes. *J. Complexity*, 22(6):768-782, 2006.

[Traub et al. 1988] Traub, J.F., Wasilkowski, G.W. and Woźniakowski, H.: *Information-based complexity.* Academic Press, New York, 1988.

[Traub and Werschulz 1999] Traub, J.F. and Werschulz, A.G.: *Complexity and Information.* Cambridge University Press, New York, 1999.

[Vakhania et al. 1987] Vakhania, N.N., Tarieladze, V.I. and Chobanian, S.A.: *Probability Distributions on Banach Spaces.* Kluwer Academic, Dordrecht/Norwell, MA, 1987.

[Webster 2006] Webster, J.: Finite approximation of measure and integration. *Ann. Pure Appl. Logic*, 137:439-449, 2006.

[Weihrauch 1999] Weihrauch, K.: Computability on the probability measures on the Borel sets of the unit interval. *Theoret. Comput. Sci.*, 219(1-2):421-437, 1999.

[Weihrauch 2000] Weihrauch, K.: *Computable Analysis: An Introduction.* Springer, Berlin, 2000.

[Wu and Ding 2005] Wu, Y. and Ding, D.: Computability of measurable sets via effective metrics. *Math. Logic Quart.*, 51(6):543-559, 2005.

[Wu and Ding 2006] Wu, Y. and Ding, D.: Computability of measurable sets via effective topologies. *Arch. Math. Logic*, 45:365-379, 2006.

[Wu and Weihrauch 2006] Wu, Y. and Weihrauch, K.: A computable version of the Daniell-Stone theorem on integration and linear functionals. *Theoret. Comput. Sci.*, 359:28-42, 2006.