Journal of Universal Computer Science, vol. 13, no. 4 (2007), 572-591
submitted: 16/2/07, accepted: 25/4/07, appeared: 28/4/07 © J.UCS

Random *k*-GD-SAT Model and its Phase Transition[1]

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Abstract: We present a new type of SAT problem called the k-GD-SAT, which generalizes k-SAT and GD-SAT. In k-GD-SAT, clause lengths have geometric distribution, controlled by a probability parameter p; for $p = 1$, a k-GD-SAT problem is a k-SAT problem. We report on the phase transition between satisfiability and unsatisfiability for randomly generated instances of k-GD-SAT. We provide theoretical analysis and experimental results suggesting that there is an intriguing relationship (linear in the parameter $1/p$) between crossover points for different parameters of k-GD-SAT. We also consider a relationship between crossover points for k-SAT and k-GD-SAT and provide links between these values.

Key Words: propositional satisfiability, random SAT problems, phase transition, NP-complete problems
Category: I.6.4, F.4.1, F.2.2

1 Introduction

The phenomenon of phase transition in SAT and in other NP-complete problems is one of the most intriguing problems linking logic and computer science. There is still no full understanding of this phenomenon. Phase transition in NP-complete problems has been deeply explored since the first results in the early 90's (see, for instance, [21]). Investigating phase transition in NP-complete problems gives insight into the famous NP=P problem. Most of the research in this area has been devoted to different variants of the SAT problem, especially k-SAT, a typical NP-complete problem. For different variants of the SAT problem, experimental data

[1] This work was supported by the Ministry of Science of the Republic of Serbia, grant 144030.

suggests that there is a *phase transition* (occurring around a *crossover point*) between satisfiable and unsatisfiable formulae: a transition, in the parameter ratio between the number of clauses and the number of variables, from almost all satisfiable formulae to almost all unsatisfiable formulae. Experimental data also suggests that the hardest instances of the SAT problem for all decision procedures are those around the crossover point.

In this paper we introduce a random k-GD-SAT model, a generalisation of the random k-SAT. We present experimental results confirming a phase transition for this problem and present results about the relationship between crossover points for random k-GD-SAT with different parameters. We also consider the relationship between crossover points for k-SAT and k-GD-SAT. We build this work on some results from [15] and extend them in several directions. In [15], a model GD-SAT is considered, which is a special case of k-GD-SAT (when $k = 2$). Experimental results for GD-SAT from [15] are extended to much larger values of $1/p$. The linearity of the crossover curve for GD-SAT was shown in [15], and in this paper we generalise it for k-GD-SAT and prove an upper bound. We also give possible relationships between crossover points for k-SAT and k-GD-SAT, properties that cannot be discussed in the context of the GD-SAT model. We know of no other work dealing with random SAT formulae with geometric distribution of clause lengths.

In this paper, we concentrate on the satisfiability functions for k-GD-SAT and on locating the crossover point. We do not analyze computational complexity issues or performance of SAT solvers on k-GD-SAT.

Overview of the paper: The rest of the paper is organized as follows: in Section 2 we give some background information about the SAT problem, phase transition, and SAT solvers. In Section 3 we introduce k-GD-SAT, provide experimental results for its phase transition, prove the upper bound for crossover points (§3.1); discuss 50% satisfiability points (that approximate crossover points) (§3.2); analyze their behavior for $N = 1$ (§3.3); for $N > 1$ (§3.4); and approximate the crossover points for k-GD-SAT on the basis of a hypothesis from the literature. In Section 4 we establish a relationship between crossover points for k-SAT and k-GD-SAT on the basis of a conjecture from the literature (§4.1) and on the basis of our experimental results (§4.2). In Section 5 we draw some final conclusions and discuss future work.

2 Background

In this section we give a brief overview of the SAT problem, phase transition, and SAT solvers.

2.1 SAT Problem

The boolean satisfiability problem (SAT) is the problem of deciding if there is a truth assignment under which a given propositional formula (in conjunctive normal form) evaluates to true. Cook showed that SAT is NP-complete [3]. This was the first known example of an NP-complete problem, and it is still regarded as the canonical NP-complete problem. Practical applications reinforce the importance of the SAT problem, since many difficult real-world problems in AI planning, circuit satisfiability, and software verification can be efficiently reformulated as instances of SAT. Therefore, good SAT solvers are of great importance and much research is devoted to finding efficient SAT algorithms.

In the k-SAT problem, all clauses have length k. It is known that k-SAT is NP–complete for $k > 2$. There is a polynomial decision procedure for the 2-SAT problem (i.e., 2-SAT $\in P$) [11].

The recent advances in research in this area and some of the current problems in different subareas are given in [18].

2.2 SAT Solvers and zChaff

There is a number of different SAT solvers (for a survey, see [13]). Most of the state-of-the-art complete SAT solvers are based on the branch and backtracking algorithm called the Davis-Logemann-Loveland algorithm [5] (sometimes called the DPLL algorithm for historical reasons). Some of the algorithms also use heuristic local search techniques, but this makes them incomplete (they are not certain to find a satisfying assignment if one exists). Many modern DLL-based solvers use a pruning technique called *learning*. This technique extracts and memorizes information from the previously searched space to prune the search in the future. Also, in order to improve the efficiency of the system, techniques such as preprocessing, sophisticated branching heuristics, and random restarts are used (for a survey, see [24]). There are currently many SAT packages available. One of them, the zChaff solver [22], employs efficient pruning techniques, is highly optimized, and achieves very good results in practice. For that reason we chose it as the SAT solver for our experiments. All experimental results presented in the rest of the paper were obtained by using zChaff SAT solver.

2.3 Worst Case and Average Case Performance for SAT Solvers

The theory of NP–completeness is based on worst–case complexity. However, the theory of average–case complexity can better explain the behavior of SAT solvers in practice. For this, a probability distribution on formulae for each input length is required. There are two families of "random formulae": one based on fixed clause lengths and the other based on random clause lengths.

The first average-case analysis of SAT was given by Goldberg [12] on random clause length formulae. Formulae from his random clause length model (called the *fixed density model*), over a set of N variables, are constructed in the following way: for each of the L clauses, include each of the $2N$ literals with probability p (where p and L may be functions of N). It follows from Goldberg's work that, for any value of p, DPLL solves these formulae in time $O(LN^2)$ on average. Later, Franco and Paull [6] showed that this was a consequence of a favorable choice of distribution, rather than favorable properties of DPLL: a constant number of guesses of random truth assignments will find one that satisfies an instance from this family with probability tending to 1 as N grows. Deterministic algorithms are now known which solve instances of the fixed density problem in polynomial time on average for all but a vanishingly small part of the parameter space. The formulae not yet known to be solvable efficiently in the average case occur roughly when the expected clause length is a little less than $\ln(L)$ [7].

Fixed clause length formulae over N variables are generated by selecting clauses uniformly at random from the set of all possible (and nontrivial) clauses of a given length k. We call this model the *random k*-SAT. Franco found that the fixed clause length formulae took exponential time on average for DPLL when finding all solutions [6]. The empirical performance of a version of DPLL on random 3-SAT was investigated in [21]. When L/N is small (less than 3) most instances are very quickly solved. When L/N is large (more than 6) instances are harder than those at small ratios, but only moderately. In the region between these ratios, the average difficulty is dramatically greater. Also between these ratios, the probability of satisfiability shifts smoothly from near 1 to near 0.

Worst-case analysis can be important for the problem of finding sets of hard instances of the SAT problem. Finding sets of hard instances of SAT is of interest for understanding the complexity of SAT, and for experimentally evaluating SAT solvers. For instance, cryptography hash functions can be used for generating both hard satisfiable and hard unsatisfiable propositional formulae [17]. An overview of techniques for generating hard instances of the SAT problem is given in [4].

As said, in this paper we will not concentrate on computational complexity issues or on performance of SAT solvers but on the satisfiability function and crossover points for the model that we propose.

2.4 Phase Transition and Crossover Points in Random SAT Problems

Experimental results suggest that there is a phase transition in SAT problems between satisfiability and unsatisfiability as the ratio of the number of clauses to the number of variables is increased [21]. It is conjectured that for different types of problem sets (based on specific distributions of clause lengths and distributions of literals within one clause) there are values c_0 of L/N, which we call *phase* *transition points* such that:

$$\lim_{N\to\infty} s(N,[cN]) = \begin{cases} 1, \text{ for } & c < c_0 \\ 0, \text{ for } & c > c_0 \end{cases},$$

where s is a *satisfiability function* that maps sets of propositional formulae (of L clauses over N variables) into the segment $[0,1]$ and corresponds to a percentage of satisfiable formulae. As mentioned in §2.3, experimental results also suggest that in all SAT problems there is a typical easy-hard-easy pattern as the ratio L/N is increased, while the most difficult SAT formulae for all decision procedures are those in the crossover region.

For a random k-SAT, experiments suggest that the phase transition occurs at[2] $c_3 \approx 4.17 \pm 0.05$, $c_4 \approx 9.75 \pm 0.05$, $c_5 \approx 20.9 \pm 0.1$, $c_6 \approx 43.2 \pm 0.2$ (c_k denotes a crossover point for k-SAT) [19]. Figure 1 (left) shows a satisfiability function experimentally approximated for 3-SAT, for 25, 50, 75, and 100 variables. Non-rigorous results based on techniques from statistical physics give the estimates $c_2 = 1$, $c_3 \approx 4.267$, $c_4 \approx 9.931$, $c_5 \approx 21.117$, $c_6 \approx 43.37$, $c_7 \approx 87.79$ [20]. In [2], there are rigorous bounds for c_k given: $2^k \ln 2 - k \leq c_k \leq 2^k \ln 2$ (see Figure 1 (right)). Although 2-SAT $\in P$, there is a phase transition in 2-SAT, as for k-SAT for $k > 2$. Goerdt proved that the crossover point for 2-SAT problem is 1 [11]. Friedgut proved that the transition region for k-SAT problems narrows as the number of variables increases [8]. However, this still does not prove that the crossover points for k-SAT exist.

In *random mixed* SAT [10], each clause is generated as in random k-SAT except that the length of clauses is chosen randomly according to a finite probability distribution ϕ on integers. For instance, if $\phi(2) = 1/3$ and $\phi(4) = 2/3$, clauses of length 2 appear with the probability 1/3 and clauses of length 4 with the probability 2/3 (this problem is then called $2,4,4$-SAT). For instance, the crossover point for $2,3$-SAT is estimated at 1.75, and for $2,4,4$-SAT at 2.74 [10].

In the $2+p$-SAT model [1], a formula with L clauses has (approximately) $(1-p)L$ clauses of the length 2 and pL clauses of the length 3 ($0 \leq p \leq 1$).[3] Hence, this model smoothly interpolates between 2-SAT and 3-SAT. For $p \leq 2/5$, the crossover point is at $1/(1-p)$, while for $p > 2/5$, the crossover point is between $\frac{24p}{(p+2)^2}$ and $\min(1/(1-p), r)$, where r is the solution of $(7.6)^{rp}(3/4)^r(2 - e^{-r(2/3-5p/21)}) = 1$.

In the fixed density model (or *the constant probability model* as called in [14]), given N variables and L clauses, each clause is generated so that it contains each of $2N$ different literals with probability p. For instance, for $p = 1.5N$, the crossover point is estimated to be around 2.8 [10].

[2] It was shown that allowing repeating variables in clauses does not influence the satisfiability function for formulae with a large number of variables [16].

[3] This model is closely related to the random mixed SAT and can be considered as its special case.

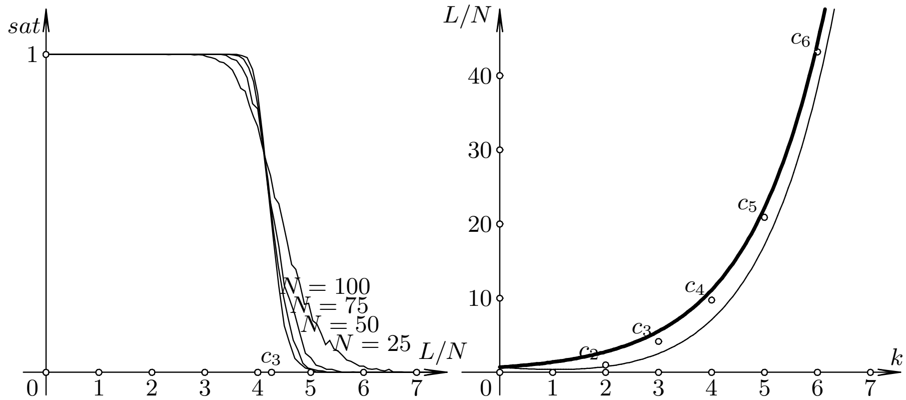

Figure 1: Experimentally approximated satisfiability function for 3-SAT problem, for $N = 25, 50, 75, 100$ (left); Experimental estimates for c_k and rigorous lower (thin) and upper (thick) bound (right)

The relation between different crossover points is conjectured in [10]: if $\phi(k)$ is a distribution on clause lengths, c_ϕ the crossover point for that SAT model, and c_k $(k = 2, 3, \ldots)$ the crossover points for k-SAT, then it holds that[4]:

$$\frac{1}{c_\phi} = \sum_{i=2}^{\infty} \frac{\phi(i)}{c_i} \tag{1}$$

3 Random *k*-GD-SAT Model

We consider a family of random SAT problems based on geometric distribution of clause lengths, denoted by k-GD-SAT. In this model, the generation of clauses over the set of N variables, for the probability parameter p $(0 < p \leq 1)$, is specified by the stochastic context–free grammar given in Table 1.[5] Clauses are generated independently of each other.

By the given stochastic grammar for k-GD-SAT, only clauses of length equal or greater than k can be generated. Lengths of clauses in the k-GD-SAT model have a geometric distribution; the probability of a clause of length l is $p(1 - p)^{l-k}$, for $l \geq k$, and is equal to 0, for $l < k$. According to the properties of

[4] This estimate, for the SAT problem, is a more refined version of the generic estimate given in [9].

[5] A stochastic context–free grammar is a context–free grammar with a stochastic component that attaches a probability to each of the production rules and controls its use.

#	Rule	Probability
1.	$\langle clause\rangle := \langle clause\rangle \vee \langle literal\rangle$	$1-p$
2.	$\langle clause\rangle := \underbrace{\langle literal\rangle \vee \langle literal\rangle \vee \ldots \vee \langle literal\rangle}_{k}$	p
3.	$\langle literal\rangle := \langle variable\rangle \mid \neg\langle variable\rangle$	0.5
4.	$\langle variable\rangle := v_1 \mid v_2 \mid \ \ldots \ \mid v_N$	$1/N$

Table 1: Stochastic grammar for generating k-GD-SAT clauses

geometric distribution, the most probable clause length in k-GD-SAT is k (with the probability p), while the expected clause length can be shown to be equal to $k-1+1/p$. For $p=1$, the k-GD-SAT model is exactly the random k-SAT model. For $p=1$, the 2-GD-SAT model is exactly the 2-SAT model and, hence, it belongs to the class P. For any fixed p such that $p<1$, k-GD-SAT is NP-complete. As p decreases, 2-GD-SAT problems smoothly interpolate between 2-SAT and NP-complete 2-GD-SAT problems. This makes k-GD-SAT convenient for exploring a computational cost for directly linked P and NP-complete problems (in a similar manner as in $2+p$-SAT). Since it uses the probability parameter p, k-GD-SAT has some similarities with the fixed density model, but they have substantially different distributions of clause lengths and different crossover points.

Our experiments show that there is a phase transition between satisfiability and unsatisfiability in k-GD-SAT,[6] for a range of values of k and of the probability parameter p. Figure 2 shows satisfiability function for random k-GD-SAT, for $k=2$, $p=1/2$, and for $N=25,50,75,100$. Percentage of satisfiable formulae is shown (as usual) against parameter L/N; for each N, in each L/N point, there were 1000 formulae randomly generated.

Hereafter, we will let $c(k,p)$ denote a crossover point for k-GD-SAT with the probability parameter p. The following analysis, both theoretical and experimental, suggests that there is a linear relationship, in parameter $1/p$, between crossover points $c(k,p)$.

3.1 Upper Bounds for Crossover Points

Upper bounds for crossover points for the k-SAT problem can be established following the approach from [2]. We consider the k-SAT formulae with L clauses, over N variables. Let us fix any truth assignment (out of 2^N) and observe that a random k-SAT clause is satisfied by it with probability $1-2^{-k}$. Since constraints are chosen independently, the expected number of satisfying truth assignments

[6] All experimental data and programs used to obtain them are available upon request from the first author.

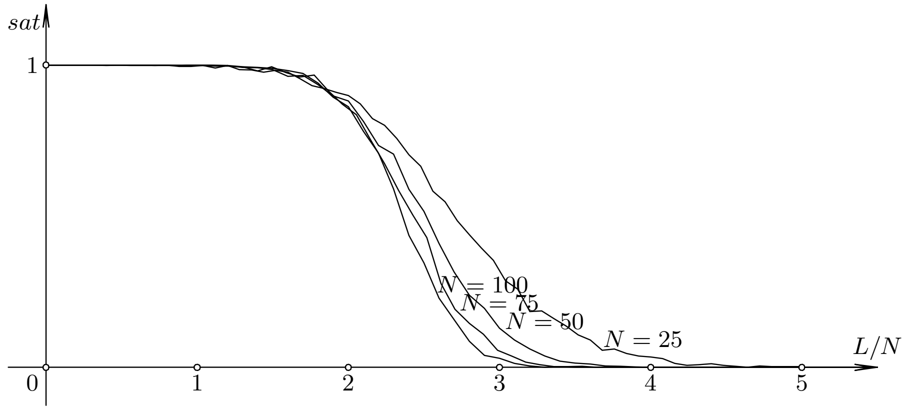

Figure 2: Satisfiability function for random k-GD-SAT, for $k = 2$, for $p = 1/2$, and for $N = 25, 50, 75, 100$

for a formula with L clauses and N variables is at most

$$2^N \left(1 - 2^{-k}\right)^L .$$

For large k, $2^N \left(1 - 2^{-k}\right)^L = 2^N \left(1 + \frac{-1}{2^k}\right)^{2^k L/2^k}$ is close to $e^{N \ln 2 - L/2^k}$ and, therefore, less than 1 for $N \ln 2 - L/2^k < 0$ i.e., for $L/N > 2^k \ln 2$. Therefore, the crossover point for k-SAT is less or equal to $2^k \ln 2$.

The same approach can be applied to k-GD-SAT. We consider random formulae of k-GD-SAT problem with parameter p, and with L clauses over N variables. Let us fix any truth assignment (out of 2^N) and observe that a random k-GD-SAT clause is satisfied by it with the following probability (sum for different clause lengths):

$$\sum_{l=0}^{\infty} \left(1 - 2^{-(k+l)}\right) p(1-p)^l .$$

This value is equal to

$$\sum_{l=0}^{\infty} p(1-p)^l - \sum_{l=0}^{\infty} 2^{-(k+l)} p(1-p)^l = p \cdot \frac{1}{1-(1-p)} - 2^{-k} p \sum_{l=0}^{\infty} \left(\frac{1-p}{2}\right)^l =$$

$$1 - 2^{-k} p \frac{1}{1 - \frac{1-p}{2}} = 1 - 2^{-k} \frac{2p}{1+p} = 1 - 2^{-\left(k - \log_2\left(\frac{2p}{1+p}\right)\right)} .$$

Hence, the expected number of satisfying truth assignments for a formula with L clauses, and over N variables, is at most

$$2^N \left(1 - 2^{-\left(k - \log_2\left(\frac{2p}{1+p}\right)\right)}\right)^L .$$

Following the same argument as above, we conclude that the crossover point $c(k, p)$ for k-GD-SAT with parameter p is less or equal to

$$2^{\left(k-\log_2\left(\frac{2p}{1+p}\right)\right)} \ln 2 = \frac{2^k}{\frac{2p}{1+p}} \ln 2 = \frac{1+p}{2p} 2^k \ln 2 \ .$$

Therefore, we have shown that the upper bound for k-GD-SAT with parameter p is

$$\left(\frac{1}{2} + \frac{1}{2p}\right) 2^k \ln 2 = 2^{k-1} \ln 2/p + 2^{k-1} \ln 2 \ . \tag{2}$$

Note that this function is linear in parameter $1/p$. Of course, for $p = 1$, the above upper bound for k-GD-SAT becomes the upper bound for crossover points for k-SAT.

It is known that the upper bound for k-SAT is asymptotically tight and can approximate crossover points [2]. On these grounds, it is possible that the above upper bound for k-GD-SAT is also asymptotically tight and can approximate crossover points for k-GD-SAT.

3.2 50% Satisfiability Points

We are interested in a relationship between crossover points for different values of p (for a fixed k). For that purpose, for each p, we consider 50% satisfiability points, i.e., values of L/N for which there are 50% satisfiable formulae. For different types of the k-GD-SAT problem (for different values of k, p, and N), we experimentally approximate these 50% satisfiability points in the following way:

– we start with an interval (of values L) wide enough to cover the required point (i.e., in the left endpoint the percentage of satisfiable formulae is greater than, and in the right endpoint the percentage of satisfiable formulae is less than 50%);

– in each iteration, we generate M random k-GD-SAT formulae for the parameter p over N variables, and calculate the percentage of satisfiable formulae in the endpoints;

– we narrow the interval by binary search until the distance between the endpoints is 1; then we use linear interpolation for approximating 50% satisfiability point.

Since the phase transition region narrows as N grows, these 50% satisfiability points converge to the crossover point.[7] We consider 50% points obtained in this way for different values of N and p. First, we consider rigorous results about these points for $N = 1$.

[7] Alternatively, a crossover point (for a fixed model) can be estimated as a point at which the percentage of satisfiable formulae is approximately constant for large values of N.

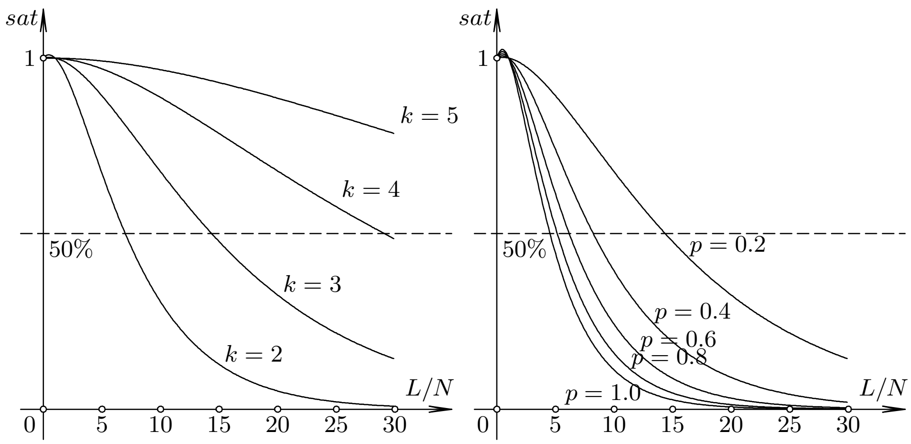

Figure 3: Satisfiability function for k-GD-SAT, for $N = 1$, $p = 0.5$, and $k = 2, 3, 4, 5$ (left) and for $N = 1$, $k = 2$, and $p = 0.2, 0.4, 0.6, 0.8, 1.0$ (right)

3.3 50% Satisfiability Points in k-GD-SAT for $N = 1$

The k-GD-SAT formula for $N = 1$ (with a single variable, a) is unsatisfiable if and only if it has a clause $a \vee a \vee \ldots \vee a$ (we will call it a-clause) and a clause $\neg a \vee \neg a \vee \ldots \vee \neg a$ (we will call it $\neg a$-clause).

Consider the k-GD-SAT with probability parameter p and one variable ($N = 1$). The probability of generating a clause of length k is equal p and the probability that that clause is an a-clause is equal to $1/2^k$. More generally, the probability of generating a clause of length l ($l \geq n$) is $p(1-p)^{l-k}$, while the probability that that clause is an a-clause is equal $1/2^l$. Hence, the total probability of generating an a-clause is equal to:

$$\sum_{l=k}^{\infty} \frac{1}{2^l} p(1-p)^{l-k} = \sum_{l=k}^{\infty} \frac{p}{2^k} \left(\frac{1-p}{2}\right)^{l-k} = \frac{p}{2^{k-1}(p+1)} .$$

The probability that a generated formula with L clauses does not have an a-clause is equal to

$$\left(1 - \frac{p}{2^{k-1}(p+1)}\right)^L .$$

The same values apply to $\neg a$-clauses.

The probability of generating a clause that is neither an a-clause nor an

$\neg a$-clause is equal to:

$$1 - 2\frac{p}{2^{k-1}(p+1)} = 1 - \frac{p}{2^{k-2}(p+1)} \ .$$

The formula is satisfiable if and only if it does not contain both an a-clause and an $\neg a$-clause. The probability of such a formula is equal to the sum of the probability of a formula with no a-clause and the probability of a formula with no $\neg a$-clause, minus the probability of a formula with neither an a-clause nor an $\neg a$-clause:

$$2\left(1 - \frac{p}{2^{k-1}(p+1)}\right)^L - \left(1 - \frac{p}{2^{k-2}(p+1)}\right)^L$$

i.e.,

$$s_k(1, L) = 2\left(1 - \frac{1}{2^{k-1}(1+1/p)}\right)^L - \left(1 - \frac{1}{2^{k-2}(1+1/p)}\right)^L \ ,$$

where $s_k(N, L)$ is the satisfiability function for the k-GD-SAT formulae (over N variables and with L clauses). Figure 3 shows the function $s_k(1, L)$, for different values of p and k.

If we consider the value of L such that the above probability $s_k(1, L)$ is equal to 50%, then we get the equation: $s_k(1, L) = 0.5$. This relation gives an implicit function L on the parameter $1/p$. The numerical solutions for this equation, for different values of p and k, are shown in Figure 4 (left). These results suggest that the 50% satisfiability curves for $N = 1$ are asymptotically linear in parameter $1/p$.

3.4 50% Satisfiability Points in *k*-GD-SAT for $N > 1$

We performed a series of experiments and obtained experimental approximation for 50% satisfiability points for different values of k, p, and N.

Figure 5 shows 50% satisfiability points for k-GD-SAT, for $k = 2$, $N = 50$, and for $1/p$ ranging from 1 to 101 by step 1. The points are determined as described in §3.2, with $M = 1000$. We determined the line that is the least square fit (i.e., the line for which the sum of squares of residuals is minimized) and we measured residuals for all points and for the fit given by this line. This line is given by $y = 0.987281x + 0.558788$ (i.e., $L/N = 0.987281/p + 0.558788$) and the corresponding residuals are shown in Figure 6 (left). Although there is a noise in these results due to the relatively small sets of formulae used in the experiments, all residuals are much less then 1 (on interval from $1/p = 1$ to $1/p = 101$). Similar results were obtained for other values of k and N. These results provide further evidence that 50% satisfiability curves for the k-GD-SAT model are (asymptotically) linear (in parameter $1/p$).

Figure 6 (right) gives a closer look on the same curve: it shows 50% satisfiability points for $k = 2$, $N = 50$, $M = 1000$, $1/p$ ranging from 1 to 5 by step 0.01.

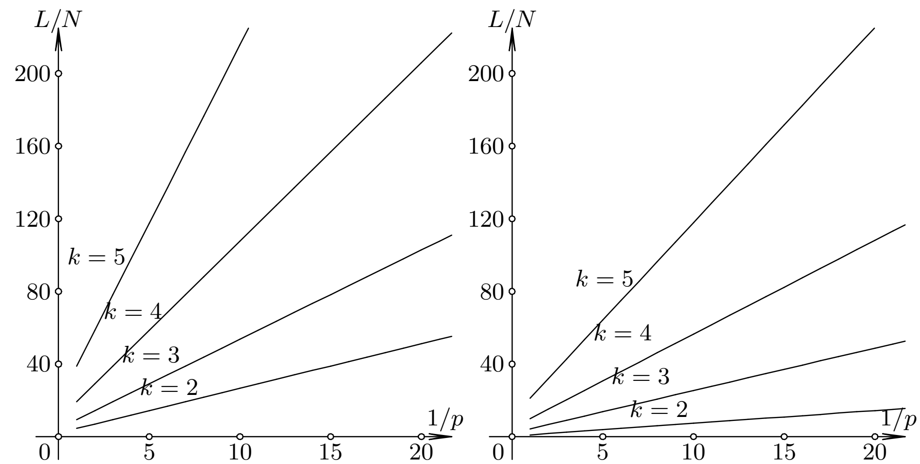

Figure 4: 50% points for k-GD-SAT, for $N = 1$, $k = 2, 3, 4, 5$ and for $1/p$ ranging from 1 to 20 (left), and crossover points for $k = 2, 3, 4, 5$ based on Gent/Walsh conjecture (right)

It can be observed that for values $1/p$ close to 1, the 50% satisfiability points are below the line $y = 0.987281x + 0.558788$. Similar interesting behavior was observed for $2 + p$-SAT: up to value 0.4 for probability parameter p the crossover curve behaves as $1/(1-p)$ (and the problem behaves as in the class P), and then it changes its behavior (this change is called second-order phase transition) [1].

We call a *crossover curve* for the k-GD-SAT model a curve determined by the points $(1/p, c(k,p))$. As said, for a fixed k and probability parameter p, the sequence of 50% satisfiability points converges to the crossover point for those parameters. Consequently, curves determined by 50% satisfiability points (in parameter $1/p$) approach crossover curves, when N grows. If, as suggested by the above results, 50% satisfiability points belong to lines, the crossover curve for k-GD-SAT for each k is linear, hence we will call it the *crossover line*.

3.5 Gent/Walsh Conjecture

In k-GD-SAT it holds that $\phi(l) = p(1-p)^{l-k}$ $(l = 2, 3, \ldots)$, so thanks to the equation (1) (from §2.4), we can approximate crossover points $c(k,p)$ (for k-GD-SAT, for parameter p): using the estimates $c_2 = 1$, $c_3 \approx 4.267$, $c_4 \approx 9.931$, $c_5 \approx 21.117$, $c_6 \approx 43.37$, $c_7 \approx 87.79$ [20] and $c_k \approx 2^k \ln 2$ for $k > 7$ (these are upper bounds from [2], the lower bounds give almost identical results), we obtain an estimation for $c(k,p)$ shown in Figure 4 in parameter $1/p$.

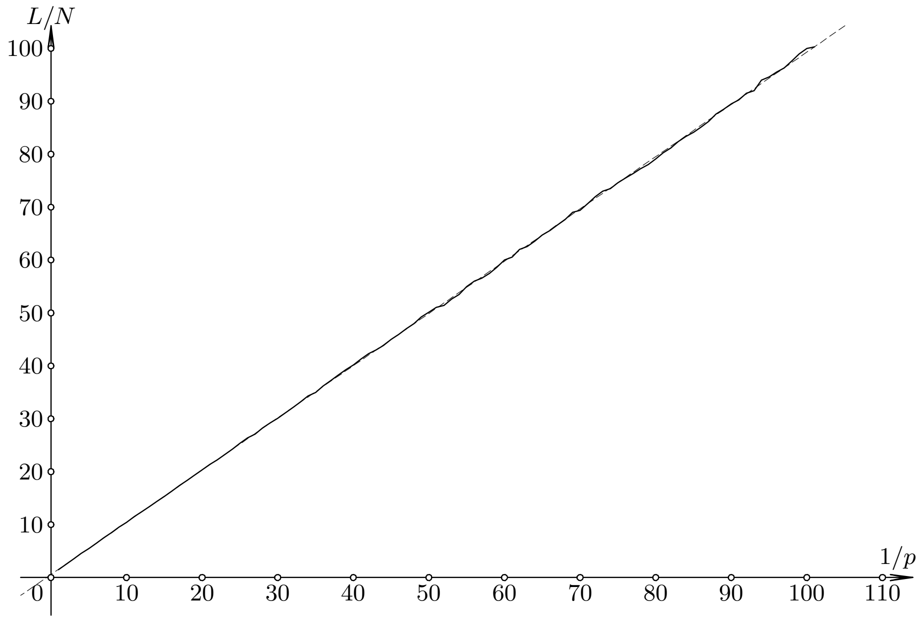

Figure 5: 50% satisfiability points for k-GD-SAT, for $N = 50$, $k = 2$, $1/p$ ranging from 1 to 101 (left) and the line $y = 0.987281x + 0.558788$

Using the equation (1) and $c_k \approx 2^k \ln 2$, we can also estimate the asymptotic behavior of crossover points $c(k, p)$ in the following way:

$$\frac{1}{c(k,p)} \approx \sum_{l=k}^{\infty} \frac{p(1-p)^{l-k}}{2^l \ln 2} = \frac{p}{2^k \ln 2} \sum_{i=0}^{\infty} \left(\frac{1-p}{2}\right)^i = \frac{p}{2^{k-1} \ln 2(1+p)}$$

which yields (the estimate is the same as the equality (2)):

$$c(k,p) \approx \left(\frac{1}{2} + \frac{1}{2p}\right) 2^k \ln 2 = 2^{k-1} \ln 2/p + 2^{k-1} \ln 2 \; . \qquad (3)$$

This result also suggests that the crossover curves for k-GD-SAT are linear (in parameter $1/p$).

4 Linking Crossover Points in k-SAT and k-GD-SAT

In this section we consider the relationship between crossover points for k-SAT and k-GD-SAT. The first approach is based on the conjecture by Gent and Walsh from [10] and the second one is based on our experimental results.

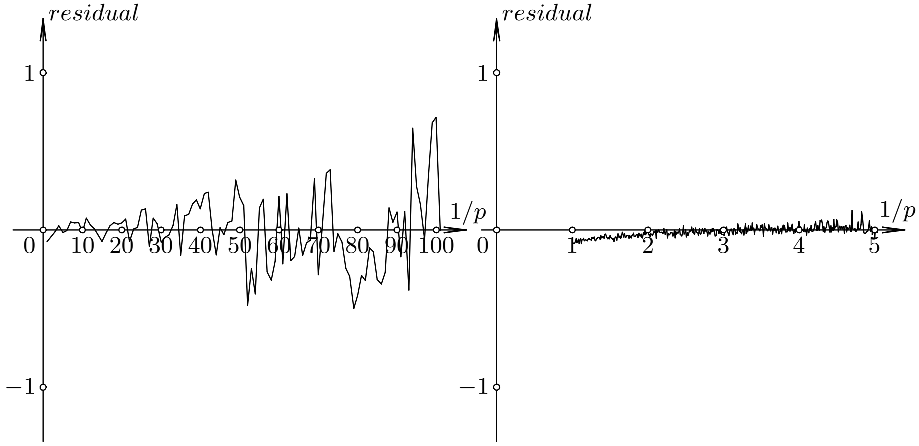

Figure 6: Residuals from the line $L/N = 0.987281/p + 0.558788$ of the 50% satisfiability points for $k = 2$, $N = 50$, $M = 1000$, $1/p$ ranging from 1 to 101 by step 1 (left) and of 50% satisfiability points for $k = 2$, $N = 50$, $M = 1000$, $1/p$ ranging from 1 to 5 by step 0.01 (right)

4.1 Approach Based on Gent/Walsh Conjecture

The experiments suggest that the crossover curve for k-GD-SAT (for a fixed k) behaves asymptotically as a line (in parameter $1/p$). This line can be approximated by experiments or via the crossover points c_k for k-SAT by conjecture given by the equation (1). We are also interested in the opposite direction: given a crossover line for k-GD-SAT (for a fixed k), can we compute the crossover points c_k for k-SAT?

Following the equation (1), it holds that:

$$\sum_{l=k}^{\infty} \frac{\phi(l)}{c_l} = \frac{1}{c_\phi}$$

where $\phi(l)$ is the probability of generating a clause of length l. For the k-GD-SAT model with probability parameter p, it holds that:

$$\sum_{l=k}^{\infty} \frac{p(1-p)^{l-k}}{c_l} = \frac{1}{c(k,p)}$$

i.e.,

$$\frac{1}{c_k}(1-p)^0 + \frac{1}{c_{k+1}}(1-p)^1 + \frac{1}{c_{k+2}}(1-p)^2 + \ldots + \frac{1}{c_{n+1}}(1-p)^{n-1} + \ldots = \frac{1}{p \cdot c(k,p)} \ .$$

For a fixed k, for different values of p (p_i), this equality yields systems (in unknown variables x_i) of the form:

$$\begin{array}{c} x_1 + a_1 x_2 + \ldots + a_1^{n-1} x_n = b_1 \\ x_1 + a_2 x_2 + \ldots + a_2^{n-1} x_n = b_2 \\ \ldots \\ x_1 + a_n x_2 + \ldots + a_n^{n-1} x_n = b_n \end{array} \tag{4}$$

where a_1, a_2, …, a_n are distinct real numbers: $a_i = 1 - p_i$, $b_i = 1/(p_i c(k, p_i))$, and $x_i = 1/(c_{k+i-1})$. This system has the following (unique) solution [23]:

$$x_j = (-1)^{n+j} \sum_{i=1}^{n} \frac{b_i f_{ij}}{(a_i - a_1) \ldots (a_i - a_{i-1})(a_i - a_{i+1}) \ldots (a_i - a_n)}$$

where f_{ij} is the sum of all possible products of $n - j$ out of $n - 1$ numbers a_1, a_2, …, a_{i-1}, a_{i+1}, …, a_n.

We observed that the crossover curve behaves asymptotically as a line for large values of $1/p$, so we should consider large values of this parameter: let $1/p$ take n values between $n + 1$ and $2n$ and let $b_i = 1/(p_i(\lambda/p_i + \mu))$, for values λ and μ that determine the crossover line $\lambda/p_i + \mu$. If the equation (1) is valid, then the solutions x_i would converge to the values c_k (crossover points for k-SAT) as n grows. However, solving this system[8] gives solutions such that x_j converges to $\alpha\beta^j$. Indeed, taking $x_j = 1/c_j$ to be $\alpha\beta^j$, the equation (1) becomes:

$$\sum_{i=k}^{\infty} \frac{p(1-p)^{i-k}}{\alpha\beta^i} = \frac{1}{c(k,p)}$$

which gives the following linear relationship:

$$c(k, p) = \alpha\beta^{k-1}(\beta - 1)\frac{1}{p} + \alpha\beta^{k-1} \ .$$

Since the system (4) has a unique solution, it is of the form $x_j = \alpha\beta^j$. Hence, this approach cannot give exact values for c_k for small k (since the c_k do not behave as $\alpha\beta^k$ for small k, although $c_{k+1}/c_k \to 2$, for $k \to \infty$). Further, this suggests that the equation (1) gives good asymptotic estimates, but is not an exact equality. It gives only an approximate relationship between crossover points for k-SAT and k-GD-SAT.

4.2 Approach Based on Experimental Results

We performed a series of experiments aimed at estimating 50% crossover points for k-GD-SAT, for different values of k and p. Since the arguments presented in §3

[8] We have developed a special-purpose C++ library for dealing with large numbers (more precisely, fractions with numerators and denominators that are large numbers) and solving the system (4) with total precision.

N	$k=2$	$k=3$	$k=4$
25	$1.0325/p+0.6933$	$2.4304/p+2.1505$	$5.2235/p+4.8269$
50	$0.9860/p+0.5727$	$2.4085/p+1.9873$	$5.2051/p+4.8086$
75	$0.9630/p+0.6205$	$2.4027/p+2.0287$	$5.0798/p+4.5017$
100	$0.9586/p+0.5243$	$2.4026/p+2.0192$	n/a

N	$k=5$	$k=6$
25	$10.886/p+10.180$	$21.884/p+24.697$
50	$10.859/p+10.105$	n/a
75	n/a	n/a
100	n/a	n/a

Table 2: Experimentally approximated 50% satisfiability lines (in the form $y = a(1/p) + b$) based on values in $1/p = 10$ and $1/p = 50$, for $k = 2, 3, 4, 5, 6$, and for $N = 25, 50, 75, 100$.

suggest that there is a linear relationship between 50% crossover points for each k and N, in order to approximate a line consisting of 50% crossover points, it suffices to determine two of its points. Because of the considerations (from 3.4) concerning values $1/p$ close to 1 for 2-GD-SAT, we determine 50% crossover points (in a way described in §3.2) for $1/p = 10$ and $1/p = 50$, for $k = 2, 3, 4, 5, 6$, and for $N = 25, 50, 75, 100$. We used $M = 5000$ for $k \leq 3$ or $N = 25$, while because of high computational cost we used smaller values for M (1000 or 100) for other combinations of k and N. For some combinations of k and N we did not perform full experiments.[9]

Experimentally estimated 50% satisfiability lines are shown in Figure 7 and their coefficients (in the form $y = a(1/p) + b$) are shown in Table 2.

These experimental results suggest that all crossover curves intersect in one point (or approach a single point) — let it be (x_c, y_c). On the other hand, since for $p = 1$, k-GD-SAT becomes k-SAT, the crossover line for k-GD-SAT passes through the point $(1, c_k)$ (for simplicity, we omit the effect of slightly increased residuals for $k = 2$ and $1/p$ close to 1). Therefore, a crossover line for k-GD-SAT is determined by:

$$y = \frac{c_k - y_c}{1 - x_c} x + \frac{-c_k x_c + y_c}{1 - x_c} \ .$$

i.e.,

$$c(k,p) = \frac{c_k - y_c}{1 - x_c} \cdot \frac{1}{p} + \frac{-c_k x_c + y_c}{1 - x_c} \ .$$

[9] For instance, for $k = 4$ and $N = 100$, processing a single formula for $p = 0.1$ takes more than one hour of CPU time on a PC computer working on 1.7 GHz, so it is very difficult to produce relevant experimental results.

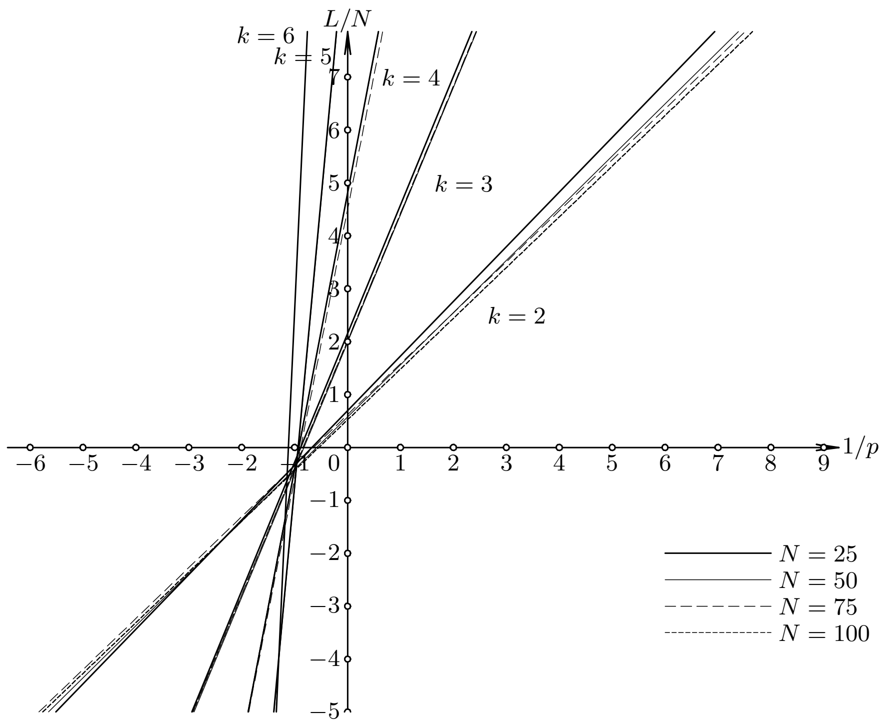

Figure 7: Experimentally approximated 50% satisfiability points for $N = 25, 50, 75, 100$, based on values for $1/p = 10$ and $1/p = 50$, and for $k = 2, 3, 4, 5, 6$.

If the above hypothesis holds, then we could also use the obtained equation the other way round: for computing the values c_k, on the basis of points on crossover curves for k-GD-SAT:

$$c_k = \frac{p(1 - x_c) \cdot c(k, p) - y_c p + y_c}{1 + p} .$$

For $x_c = -1$ and $y_c = 0$ (values roughly suggested by the experimental results), we have:

$$c(k, p) = \left(\frac{1}{2} + \frac{1}{2p}\right) c_k$$

and

$$c_k = \frac{2p}{p + 1} \cdot c(k, p) .$$

N	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
25	2.0033	4.8100	10.3748	21.6436	44.3031
50	1.8969	4.7404	10.3382	21.5818	n/a
75	1.8638	4.7374	10.0545	n/a	n/a
100	1.8382	4.7355	n/a	n/a	n/a

Table 3: Estimates of values c_k based on the points on crossover curves for k-GD-SAT for $1/p = 10$.

If we replace $c(k, p)$ by the approximation given by 50% satisfiability point for k-GD-SAT (for some fixed N), we will get an approximation for c_k. Table 3 shows that these approximations for c_k approach their known values. Notice that the values from Table 3 can also serve as approximations for 50% satisfiability points for k-SAT for given values of N (they are close to the results that can be obtained experimentally). Of course, the estimate for the common intersection point (x_c, y_c) is the subject of further refinement.

Finally, for large values of k, we can approximate c_k by $2^k \ln 2$ [2], and we have the following estimate for $c(k, p)$:

$$c(k,p) = \left(\frac{1}{2} + \frac{1}{2p}\right) 2^k \ln 2 \ ,$$

the estimate same to the one given by the equations (2) and (3) from §3.1 and §3.5. This shows that the hypothesis that crossover lines intersect in one point (or approach a single point) is consistent with the available theoretical and experimental results.

The above equations (dependent on the given hypotheses) give a simple relationship between all crossover points in k-SAT and k-GD-SAT. This interesting relationship is important because knowing crossover points from one model, enable us to estimate the crossover point from the other and vice versa. A deep understanding of one model would give us a deep understanding of the other, and vice versa. Also, this relationship provides a new link between P and NP-complete problems, leading possibly to a deeper understanding of the relationship between these complexity classes.

5 Conclusions and Future Work

In this paper, we presented a new random SAT model — k-GD-SAT, based on probability parameter p that controls geometrical distribution on clause lengths. We provided experimental evidence about the phase transition for this model.

The experimental results and theoretical analysis also suggest that for each k, there is a linear relationship between crossover points (in parameter $1/p$). Further, our results and analysis suggest that all these crossover lines intersect in one point (or approach a single point). This leads to an intriguing property, a relationship that links all crossover points in k-SAT and k-GD-SAT. Knowing crossover points from one model enables us to estimate the crossover point from the other and vice versa.

In our future work, we are planning to perform further, more extensive experiments, to look for further experimental confirmation of the hypotheses presented in this paper, and to obtain finer approximations of relevant parameters. We will look, on the basis of the presented results, for ways of improving the conjecture given by the equation (1). We will also look for theoretical explanations of our experimental results, following successful approaches for estimating crossover points in other classes of SAT problems. In particular, we will try to apply the approach from [2] in order to estimate lower bound for crossover points for k-GD-SAT problem. In this paper, we considered the relationship between crossover points for different models and model parameters. In further work, we will also more deeply consider the relationship between 50% crossover points for different values of N. We are planning to use the finite-scaling method [19] which provides a transformation by which, for one SAT model, satisfiability functions for all N scale to a single one. This way, with appropriate parameters of this scaling (that will be a subject of our research) — all 50% satisfiability curves would be mapped to a single line. This would provide new information on the k-GD-SAT model and provide simple estimates for the 50% satisfiability points for any particular values of k, p, and N. We are also planning to look at computational complexity issues and performance of SAT solvers on k-GD-SAT problem, in particular following approaches from [7] and [6].

References

1. Achlioptas, D., Kirousis, L. M., Kranakis, E., Krizanc, D.: Rigorous Results for Random $(2+p)$-SAT, Proceedings of RALCOM' 97, 1997.
2. Achlioptas, D., Naor, A., Peres, Y.: Rigorous location of phase transitions in hard optimization problems, *Nature*, **435**, 2005, 759–764.
3. Cook, S. A.: The complexity of theorem-proving procedures, *STOC '71: Proceedings of the third annual ACM symposium on Theory of computing*, ACM Press, 1971.
4. Cook, S. A., Mitchell, D. G.: Finding Hard Instances of the Satisfiability Problem: A Survey, *DIMACS Series in Descrete Mathematics and Theoretical Computer Science*, **35**, 1997, 1–17.
5. Davis, M., Logemann, G., Loveland, D.: A machine program for theorem-proving, *Communications of the ACM*, **5**(7), 1962, 394–397, ISSN 0001-0782.
6. Franco, J., Paull, M.: Probabilistic Analysis of the David Putnam procedure for solving the satisfiability problem, *Discrete Applied Mathematics*, **5**, 1983, 77–87.

7. Franco, J., Swamanithan, R.: Average Case Results for Satisfiability Algorithms Under the Random Clause Model, *Anals of Mathematics and Artificial Intelligence*, **20**, 1997, 357–391.
8. Friedgut, E.: Sharp threshold for graph properties and the k-sat problem, *Journal of the American Mathematical Society*, **12**, 1999, 1017–1054.
9. Gent, I., Macintyre., E., Prosser, P., Walsh, T.: The Constraidness of Search, *Proceedings of AAAI-96*, Menlo Park, AAAI Press/MIT Press., 1996.
10. Gent, I. P., Walsh, T.: The SAT phase transition, *Proceedings of ECAI-94*, 1994.
11. Goerdt, A.: A treshold for unsatisfiability, *Proceedings of the 17th International Symposium on Mathematical Foundations of Computer Science*, 1992.
12. Goldberg, A.: *On the complexity of the satisfiability problem*, Technical Report Courant Computer Science Report No. 16, New York University, 1979.
13. Gu, J., Purdom, P. W., Franco, J., Wah, B.: Algorithms for the Satisfiability (SAT) Problem: A Survey, *DIMACS Series in Descrete Mathematics and Theoretical Computer Science*, **35**, 1997, 19–151.
14. Hooker, J., Fedjki, C.: Branch-and-cut soultion of inference problems in propositional logic, *Ann. Math. Artif. Intell.*, **1**, 1990, 123–139.
15. Janičić, P.: GD-SAT model and crossover line, Journal of Experimental and Theoretical Artificial Intelligence, **13**(3), 2001, 181–198.
16. Janičić, P., Dedić, N., Terzić, G.: On Different Models for Generating Random SAT Problems, Computing and Informatics *(former* Computers and Artificial Intelligence*)*, **20**(5), 2001, 451–469.
17. Jovanović, D., Janičić, P.: Logical Analysis of Hash Functions, *Frontiers of Combining Systems (FroCoS)* (B. Gramlich, Ed.), 3717, Springer-Verlag, 2005.
18. Kautz, H., Selman, B.: The State of SAT, *Discrete and Applied Mathematics*, To appear.
19. Kirkpatrick, S., Selman, B.: Critical behaviour in the satisfiability of random boolean expressions, *Science*, **264**, 1994, 1297–1301.
20. Mertens, S., Mézard, M., Zecchina, R.: *Threshold Values of Random K-SAT from the Cavity Method*, Technical Report CoRR cs.CC/0309020, 2003.
21. Mitchell, G. D., Selman, B., Levesque, J. H.: Hard and easy distributions of SAT problems, *Proceedings AAAI-92*, AAAI Press/The MIT Press, San Jose, CA, 1992.
22. Moskewicz, M. W., Madigan, C. F., Zhao, Y., Zhang, L., Malik, S.: Chaff: engineering an efficient SAT solver, *DAC '01: Proceedings of the 38th conference on Design automation*, ACM Press, 2001, ISBN 1-58113-297-2.
23. Проскуряков И. В.: Сборник задач по линеиноu алгебре, Наука, Москва, 1966.
24. Zhang, L., Malik, S.: The Quest for Efficient Boolean Satisfiability Solvers, *CAV '02*, Springer-Verlag, 2002, ISBN 3-540-43997-8.