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Deriving Consensus for Hierarchical Incomplete Ordered Partitions and Coverings

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Abstract: A method for determining consensus of hierarchical incomplete ordered partitions and coverings of sets is presented in this chapter. Incomplete ordered partitions and coverings are often used in expert information analysis. These structures should be useful when an expert has to classify elements of a set into given classes, but referring to several elements he does not know to which classes they should belong. The hierarchical ordered partition is a more general structure than incomplete ordered partition. In this chapter we present definitions of the notion of hierarchical incomplete ordered partitions and coverings of sets. The distance functions between hierarchical incomplete ordered partitions and coverings are defined. We present also algorithms of consensus determining for a finite set of hierarchical incomplete ordered partitions and coverings.

Keywords: hierarchical incomplete ordered partition and covering, consensus methods
Categories: E.1, H.2.1, I.2.4, I.2.11

1 Introduction

An ordered partition of a set *X* is a sequence of some non-empty subsets of *X*, which are disjoint with each other and whose sum is equal *X*. An ordered covering of a set *X* is a sequence of some subsets of *X*, whose sum is equal *X*. These two structures have been proved to be useful for experts to represent their opinions in a classification task [Danilowicz, 92]. Incomplete ordered partition is more general structure than ordered partition, where the sum of the subsets occurring in the sequence must not be equal set *X*. In work [Hernes, 04] we have presented in detail the notion of incomplete ordered partition of a set. It turned out that this tool might be very useful in case if an expert has uncertainty and incomplete knowledge in a classification task.

In this paper we present some other generalized structures of incomplete ordered partitions and coverings of sets. They are hierarchical incomplete ordered partitions and coverings of sets. These structures should allow an expert to realize a classification task in different levels. Owing to this his opinion can be more precise. Of course the structure enables reflecting the incompleteness and uncertainty of the expert.

Consensus problem is most often formulated as follows: *For given a set of elements representing solutions of experts of some problem one should determine an element which best represent these elements*. The determined element is called a consensus of the set. Consensus determination task is dependent on the following elements: First, the structure of the given elements; second, the distance functions between these elements and third, the criterion for consensus determining. Many structures have been investigated in detail. For example, semillatices [Barthelemy, 91], n-tree [Day, 87], linear relations [Arrow, 63], [Kemeny, 59], multi-value tuples [Nguyen, 02]. Different criteria for consensus choice tasks have been analyzed in work [Nguyen, 01].

In this paper we deal with the consensus choice task for sets of hierarchical incomplete ordered partitions and coverings. For this aim we will define these structures and the distance function between them. Next we work out a method for determining a consensus for a set of hierarchical incomplete ordered partitions and a set of hierarchical incomplete ordered coverings.

In Sec. 2 we mention some notions referring incomplete ordered partitions and coverings. In Section 3 we present the notions of hierarchical incomplete ordered partitions and coverings and the distance functions between them and the method for consensus determining. Some conclusions and directions for future works are included in Section 4.

2 Incomplete Ordered Partitions and Coverings

In this section we present the notions of incomplete ordered partitions and coverings of a set and the distance function between them. These notions regarding incomplete partitions have been presented and analyzed in work [Hernes, 04].

Definition 2.1.
By a K-class incomplete ordered partition of finite set $X = \{x_1, \ldots, x_N\}$ *we call any sequence* $P = \langle P_1, \ldots, P_K \rangle$ *where* $P_i \cap P_j = \emptyset$ *(for* $i \neq j$*;* $i,j = 1, \ldots, K$*) and* $\bigcup_{i=1,2,\ldots,K} P_i \subseteq X$.

Example 2.1.
Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. The examples of incomplete ordered partitions of set X are:

$P = \langle \{x_1, x_2\}, \{x_5, x_3, x_6\} \rangle$ (2-class partition),

$P = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{\emptyset\} \rangle$ (3-class partition),

$P = \langle \{x_7\}, \{x_3, x_4, x_2\}, \{x_1\} \rangle$ (3-class partition),

$P = \langle \{x_1\}, \{x_2\}, \{x_3, x_7\}, \{x_4\}, \{x_5, x_6\} \rangle$ (5-class partition).

By $NU_K(X)$ we denote the set of all K-class incomplete ordered partitions of set X. A K-class incomplete ordered partition can be represented by a characteristic vector:

$$p = \langle p_1, p_2, ..., p_N \rangle$$

where p_i is the index of the class to which element x_i belongs for i = 1,..., N . If element x_i does not belong to any class then $p_i = 0$.

Example 2.2.
Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $P = \langle \{x_4\}, \{x_3, x_7, x_2\}, \{x_1\} \rangle$. Then vector $p = \langle 3,2,2,1,0,0,2 \rangle$ is the characteristic vector of incomplete ordered partition P.

Incomplete ordered partitions have the following properties:

- The elements of set X can not repeat;
- An incomplete ordered partition has not to include all of elements of set X:
- The sequence of classes is important, the sequence of elements in classes is not important;
- An incomplete ordered partition may consist of any number of classes.

Definition 2.2.
By a K-class incomplete ordered covering of finite set $X = \{x_1, \ldots, x_N\}$ *we call any sequence* $C = \langle C_1, \ldots, C_K \rangle$ *where* $C_i \subseteq X$ *for* $i = 1,\ldots,K$.

Definition 2.3.
a) Let $P = \langle P_1, \ldots, P_l, \ldots, P_K \rangle \in NU_K(X)$ *and* $x_n \notin P_l$. *By operation*

$$A_n^l(P) = \langle P_1, \ldots, P_l \cup \{x_n\}, \ldots, P_K \rangle$$

we call an addition of element x_n *to class* P_l.
b) Let $P = \langle P_1, \ldots, P_k, \ldots, P_K \rangle \in NU_K(X)$ *and* $x_n \in P_k$. *By operation*

$$E_n^k(P) = \langle P_1, \ldots, P_k \setminus \{x_n\}, \ldots, P_K \rangle$$

we call an elimination of element x_n *from class* P_k.

Notice that an addition operation $A_n^l(P)$ transforms an incomplete partition to an incomplete partition. An elimination operation $E_n^k(P)$ also transforms an incomplete partition into an incomplete partition.

Example 2.3.
Let

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\},$$
$$P = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{\varnothing\} \rangle \text{ and}$$
$$Q = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{x_2\} \rangle.$$

If we want to transform incomplete ordered partition P into incomplete ordered partition Q, we have to add element x_2 to the third class of P, so $Q = A_2^3(P) = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{\varnothing\} \cup \{x_2\} \rangle$.

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$,

$P = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{x_2\} \rangle$ and

$Q = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{\varnothing\} \rangle$.

If we want to transform partition P into partition Q then we have to eliminate element x_2 from the third class, so

$$Q = E_2^3(P) = \langle \{x_5\}, \{x_1, x_4, x_6\}, \{x_2\} \setminus \{x_2\} \rangle .$$

Definition 2.4.

Let $C = \langle C_1, \ldots, C_l, \ldots, C_K \rangle \in V_K(X)$ *. En elimination of element* x_n *from class* C_l *we called operation* $E_n^l : V_K(X) \to V_k(X)$ *where*

$$E_n^l(C) = \langle C_1, \ldots, C_l \setminus \{x_n\}, \ldots, C_K \rangle$$

and an addition of element x_m *to class* C_l *we called operation* A_m^l*, where*

$$A_m^l(C) = \langle C_1, \ldots, C_l \cup \{x_m\}, \ldots, C_K \rangle .$$

Elimination E_n^l causes removal of element x_n from class C_l when $x_n \in C_l$. If $x_n \notin C_l$ then $E_n^l(C) = C$. On the other hand operation A_m^l causes addition of element x_m to class C_l if $x_m \notin C_l$ and if $x_m \in C_l$ then $A_m^l(C) = C$.

Definition 2.5.

Distance $\omega(P,Q)$ *between incomplete ordered partitions P and Q is equal the minimal number of additions and eliminations needed to transform the incomplete ordered partition P to the incomplete ordered partition Q.*

Example 2.4

Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$,

$P = \langle \{x_5\}, \{x_1, x_4\}, \{x_2\} \rangle$, and

$Q = \langle \{x_3\}, \{x_2, x_4, x_6\}, \{x_1\} \rangle$.

If we want to transform the incomplete ordered partition P in the incomplete ordered partition Q, we have to perform the following operations:

$$P = \langle \{x_5\}, \{x_1, x_4\}, \{x_2\} \rangle \xrightarrow{E_5^1} \langle \{\varnothing\}, \{x_1, x_4\}, \{x_2\} \rangle$$

$$\xrightarrow{A_3^1} \langle \{x_3\}, \{x_1, x_4\}, \{x_2\} \rangle \xrightarrow{E_1^2} \langle \{x_3\}, \{x_4\}, \{x_2\} \rangle$$

$$\xrightarrow{E_2^3} \langle \{x_3\}, \{x_4\}, \{\varnothing\} \rangle \xrightarrow{A_2^2} \langle \{x_3\}, \{x_4, x_2\}, \{\varnothing\} \rangle$$

$$\xrightarrow{A_6^2} \langle \{x_3\}, \{x_4, x_2, x_6\}, \{\varnothing\} \rangle \xrightarrow{A_1^3} \langle \{x_3\}, \{x_4, x_2, x_6\}, \{x_1\} \rangle = Q$$

Thus partition Q can be obtained by performing 7 operations and this is the minimal number of operations which is required for transforming partition P into partition Q. Then we have $\omega(P,Q) = 7$.

However, this method requires a lot of operations for calculating distances, below we present a more simple method using characteristic vectors.

Theorem 2.1.
Let P, $Q \in NU_K(X)$ and p, q be the characteristic vectors of incomplete ordered partitions P, Q, respectively. Then:

$$\omega(P,Q) = \sum_{i=1}^{N} [p_i \oplus q_i]$$

where N is the cardinality of set X and

$$[p_i \oplus q_i] = 0 \quad \textit{iff} \quad p_i = q_i$$

$$[p_i \oplus q_i] = 1 \quad \textit{iff} \quad (p_i \neq q_i) \wedge (p_i \cdot q_i = 0)$$

$$[p_i \oplus q_i] = 2 \quad \textit{iff} \quad (p_i \neq q_i) \wedge (p_i \cdot q_i \neq 0).$$

The proof of this theorem is given in [Hernes, 04].

Example 2.5
Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$,
$P = \langle \{x_5\}, \{x_1, x_4\}, \{x_2\} \rangle$, and
$Q = \langle \{x_3\}, \{x_2, x_4, x_6\}, \{x_1\} \rangle$.

Characteristic vectors of them are: p = <2,3,0,2,1,0,0> and q = <3,2,1,2,0,2,0>.
We calculate the distance between P and Q:

$p_1 \oplus q_1 = 2$,
$p_2 \oplus q_2 = 2$,
$p_3 \oplus q_3 = 1$,
$p_4 \oplus q_4 = 0$,
$p_5 \oplus q_5 = 1$,
$p_6 \oplus q_6 = 1$,
$p_7 \oplus q_7 = 0$.

Thus the distance $\omega(P,Q)$ = 2+2+1+0+1+1+0 = 7.

This method requires a smaller number of operations than the method from Definition 2.4.

Definition 2.5.
Distance $\mu(C,D)$ ($C,D \in V_K(X)$) is equal the minimal number of additions and eliminations needed to transform ordered covering C to ordered covering D.

From this definition it follows the following theorem:

Theorem 2.2.

$$\mu(C,D) = \sum_{i=1}^{K} \sum_{j=1}^{N} |c_{ij} - d_{i,j}|$$

where c_{ij} and d_{ij} are elements of matrixes representing coverings C and D, respectively.

3 Hierarchical Incomplete Ordered Partitions and Coverings

Incomplete ordered partition should be used when expert knowledge is incomplete and referring to some object he does not know to which class it should belong. A more general form of incomplete ordered partition is hierarchical incomplete ordered partition. For example, set of all the animals can be classified to different types, next animals of a type can be classified to sub-types, and so on. Thus there arises a hierarchy. However, an expert may not know to which type an animal should belong, thus this whole classification is an incomplete hierarchical ordered partition of the set of animals. Besides, if the expert classifies an animal to different types then we have to deal with an incomplete hierarchical ordered covering of the set of all animals.

3.1. Definition of Hierarchical Incomplete Ordered Partition

Definition 3.1.
By a hierarchical incomplete ordered partition of finite set $X = \{x_1, \ldots, x_N\}$, *whose dendrite is tree T we call the following function*:

$$P : W \cup \{r\} \rightarrow 2^X$$

which meets conditions:

a) $P(r) = X$,

b) $P(b) \subset P(a)$ *if node b is a son of node a*,

c) $P(b) \cap P(c) = \emptyset$ *if nodes b and c are sons of node a*,

d) $\bigcup_{a \in L} P(a) \subseteq X$,

where:
T – directed tree, r – root, W – set of nodes in T which are not root, L – set of leaves of tree T.

By $NU_T(X)$ we denote the set of all hierarchical incomplete ordered partitions of set *X* whose dendrite is tree *T*.

Example 3.1.
Let X={ x_1, x_2, x_3, x_4, x_5, x_6} and tree T is:

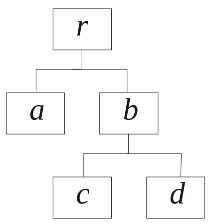

Figure 1: Dendrite T

An example of hierarchical incomplete ordered partition for which T is a dendrite is presented as follows:

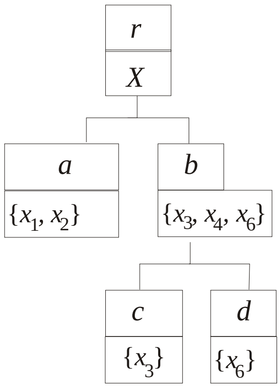

Figure 2: Example of hierarchical incomplete ordered partitions for tree T

Definition 3.2.
Let $P,P' \in NU_T(X)$, we say that partition P' arises from partition P in the result of moving an element $x \in X$ from leaf a to leaf b, if $x \in P(a)$ and $x \in P'(b)$ and the position of the remaining elements in this partitions is the same.

We can determine the weight of the moving operation:

By Z we denote set of nodes of tree T which belong to the shortest way joining leaves a and b. By the weight of a node we understand value $1/l$ where l is the level of this node (the root has level equal 0). Then the weight of a moving operation of an element from node a to node b is equal the sum of weights of the nodes belonging to Z.

Definition 3.3.
By the distance $\gamma(P,Q)$ between hierarchical incomplete ordered partitions $P,Q \in NU_T(X)$ we understand the minimal sum of the weights of moving operations needed to transform partition P to partition Q.

Such defined distance function γ is a metric.

3.2. Definition of Hierarchical Incomplete Ordered Coverings

Definition 3.4.
By a hierarchical ordered covering of finite set $X = \{x_1, \ldots, x_N\}$, whose dendrite is tree T we call function:

$$C : W \cup \{r\} \rightarrow 2^X$$

which satisfies conditions:

$a)$ $C(r) \subseteq X$,

b) $C(b) \subseteq P(a)$ *if node b is a son of node a.*

By $V_T(X)$ we denote the set of all *K*-class hierarchical ordered coverings of set X whose dendrite is tree T.

Example 3.2.
Let $X = \{ x_1, x_2, x_3, x_4, x_5, x_6\}$ and tree *T* be:

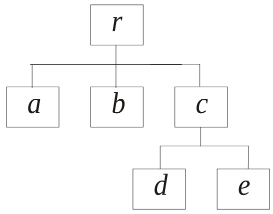

Figure 3: Directed tree T

Two examples of hierarchical ordered coverings of set *X* and dendrite *T* are given in Figures 4a and 4b. Notice that in a hierarchical ordered covering for a node *a* which is not a leaf we have $\bigcup_{b\in S(a)} C(b) \subseteq C(a)$ where *S*(*a*) is the set of sons of *a*.

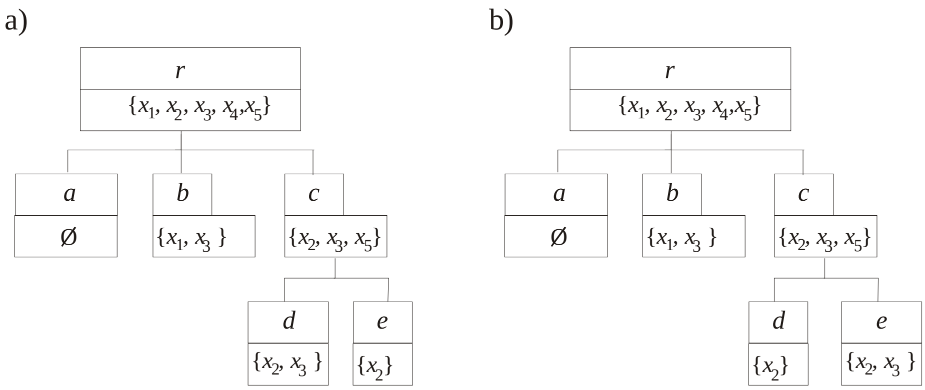

Figure 4: Examples of hierarchical ordered coverings for tree T

Definition 3.5.
Let $C, C', C'' \in V_T(X)$. *We say that covering* C' *arises from partition C in the result of elimination of an element* $x \in X$ *from node a, if* $C'(a) = C(a) \setminus \{x\}$ *and for every node c being not a descendant of node a there is* $C(c) = C'(c)$, *and for every node c being a descendant of node a there is* $C'(c) = C(c) \setminus \{x\}$. *We say that covering* C'' *arises from partition C in the result of addition of an element* $x \in X$ *to node b if* $C''(b) = C(b) \cup \{x\}$ *and for every node c being not an ancestor of node a there is*

$C(c) = C''(c)$ *and for every node c being an ancestor of node a there is* $C''(c) = C(c) \cup \{x\}$.

Definition 3.6.
By the distance $\zeta(C,D)$ *between hierarchical ordered coverings* $C,D \in V_T(X)$ *we call the minimal number of eliminations and additions needed to transform covering C to covering D.*

Such defined distance function ζ is also a metric.

3.3. Criteria for Consensus Choice

In this sub-section we present a brief overview of consensus choice problem. The wide description of this problem may be found in [Nguyen, 01]. We assume that the subject of interests is a finite universe U of objects. Let $\Pi(U)$ denote the set of subsets of U. By $\hat{\prod}_k(U)$ we denote the set of k-element subsets (with repetitions) of set U for $k \in N$, and let $\hat{\prod}(U) = \bigcup_{k>0} \hat{\prod}_k(U)$. Each element of set $\hat{\prod}(U)$ is called a *profile*. The structure of this universe is a distance function

$$d: U \times U \to \Re^+,$$

which satisfies the following conditions:

Nonnegative: $(\forall x,y \in U)[\delta(x,y) \geq 0]$,

Reflexive: $(\forall x,y \in U)[\delta(x,y) = 0 \text{ iff } x=y]$,

Symmetrical: $(\forall x,y \in U)[\delta(x,y) = \delta(y,x)]$.

Let us notice that the above conditions are for half metrics. A space (U,d) defined in this way does not need to be a metric space. Therefore it is called a *distance space*.

By a consensus choice function in space (U,d) we mean a function

$$c: \hat{\prod}(U) \to \Pi(U).$$

For $A \in \hat{\prod}(U)$, the set $c(A)$ is called the *representation* of the profile A, where an element of $c(A)$ is called a *consensus* (or a *representative*) of the profile A.

The most popular criteria for consensus choice are the following two conditions:

$$C_1(A) = \left\{x \in U: \sum_{y \in A} d(x,y) = \min_{z \in U} \sum_{y \in A} d(z,y)\right\},$$

and

$$C_2(A) = \left\{x \in U: \sum_{y \in A} (d(x,y))^2 = \min_{z \in U} \sum_{y \in A} (d(z,y))^2\right\}.$$

These two functions have been proved to be very useful [Nguyen, 01]. However, for many structures of the objects of universe U the problem of determining values of function C_2 is NP-hard. Therefore, most often function C_1 is used.

3.4. Determining Consensus for Hierarchical Incomplete Ordered Partitions

For determining a consensus of a set of hierarchical incomplete ordered partitions we will use function C_1 defined above.

Let A be a profile of hierarchical incomplete ordered partitions of a set X. We denote $\alpha(a,x)$ as the number of occurrences of element x in node a of tree T w in partitions belonging to A. We prove the following:

Theorem 3.1.
For the profile A an element $x \in X$ appears in node a of tree T in a consensus $c \in C_1(A)$, that is $x \in c(a)$ if the following inequality is true:

$$\alpha(a,x) \geq M/2$$

where M jest cardinality of set X.

Proof
Let $c \in C_1(A)$. For each element $x \in X$ where $\alpha(a,x) > M/2$ if $x \in c$ then its share in the sum of distanced from consensus c to the elements of profile A is equal $M - \alpha(a,x) < M/2$. If $x \notin c$ then its share in the sum of distances from consensus c to the elements of profile A is equal $\alpha(a,x) > M/2$. Thus for this case the sum of distances from consensus c to the elements of profile A should be minimal if $x \in c$. Similarly we can state that if $\alpha(a,x) < M/2$ then there should be $x \notin c$. In the case of $\alpha(a,x) = M/2$ independently of $x \notin c$ or $x \in c$ its share is always equal $M/2$. So the sum of distances from consensus c to the elements of profile A is minimal independently of $x \notin c$ or $x \in c$.

Owing to Theorem 3.1 we have an effective algorithm for determining a consensus for a profile of hierarchical incomplete ordered partitions.

3.5. Determining Consensus for Hierarchical Incomplete Ordered Coverings

For determining a consensus of a set of hierarchical incomplete ordered coverings we will also use function C_1 defined above. Let there be given a profile A of M hierarchical incomplete ordered partitions of set X:

$$A = \{C^{(1)}, C^{(2)}, \ldots, C^{(M)}\}.$$

We prove the following theorem:

Theorem 3.2.
Let $C \in C_1(A)$, then for each element $x \in X$:

- $x \in C(a)$ *if* $\alpha(a,x) > M/2$,
- $x \in C(a)$ *or* $x \notin c(a)$ *if* $\alpha(a,x) = M/2$,
- $x \notin C(a)$ *if* $\alpha(a,x) < M/2$.

Proof
Let $S(D) = \sum_{i=1}^{M} \zeta(D, C^{(i)})$ for $D \in V_T(X)$. Then

$$S(C) = \sum_{i=1}^{M} \zeta(C, C^{(i)}) = \min_{D \in V_T(X)} S(D) = \min_{D \in V_T(X)} \sum_{i=1}^{M} \zeta(C, C^{(i)}).$$

From the property of function ζ it follows that:

$$S(C) = \sum_{i=1}^{M} \sum_{\alpha \in W} card[D(a) - C^{(i)}(a)] / l(a) = \sum_{i=1}^{M} \sum_{\alpha \in W} \sum_{x \in X} g_i(x,a)] / l(a),$$

where

$$g_i(x,a) = \begin{cases} 1, \text{ when } x \in D(a) - C^{(i)}(a) \\ 0 \text{ otherwise} \end{cases} .$$

Hence

$$S(D) = \sum_{x \in X} \sum_{\alpha \in W} [\sum_{i=1}^{M} g_i(x,a)] / l(a).$$

Notice that $S(D)$ has minimal value when every component of sum $\sum_{i=1}^{M} g_i(x,a)$ takes minimal value. If $x \in D(a)$ then there should be

$$\sum_{i=1}^{M} g_i(x,a) = M - \gamma(a,x) \ .$$

On the other hand if $x \notin D(a)$ then

$$\sum_{i=1}^{M} g_i(x,a) = \gamma(a,x) \ .$$

Let's consider the following three cases:

(1) $\gamma(a,x) > M/2$. Then $\sum_{i=1}^{M} g_i(x,a)$ takes the minimal value if and only then if $x \in D(a)$.

(2) $\gamma(a,x) = M/2$. Then $\sum_{i=1}^{M} g_i(x,a)$ takes the minimal value independently of $x \in D(a)$ or $x \notin D(a)$.

(3) $\gamma(a,x) < M/2$. Then $\sum_{i=1}^{M} g_i(x,a)$ takes the minimal value if and only then if $x \notin D(a)$.

If node *a* is an ancestor of node *b* then we have $\gamma(a,x) \geq \gamma(b,x)$ and according on (1)-(3) we always can build a hierarchical covering *C* which is a consensus of coverings $C^{(1)}, \ldots, C^{(M)}$.

Theorem 3.2 allows in an effective algorithm to determine a consensus for a profile of hierarchical incomplete ordered coverings.

4 Conclusions

In this paper two structures for representing expert knowledge in classification tasks are presented. Both of them enable an expert to express the incompleteness and uncertainty of his knowledge. For these structures we have defined the distance functions and shown how to calculate their values. Next we have dealt with the consensus choice for profiles consisting of hierarchical incomplete ordered partitions and coverings. We have proved that there is a possibility to construct effective algorithms for determining the consensus as the value of consensus function C_1. The future work should concern working out algorithm for determining consensus for according to criterion of minimal sum of squared distances for these structures, i.e. function C_2.

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