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Matrices and α-Stable Bipartite Graphs[1]

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Abstract: A square $(0,1)$-matrix X of order $n \geq 1$ is called *fully indecomposable* if there exists no integer k with $1 \leq k \leq n-1$, such that X has a k by $n-k$ zero submatrix. The *reduced adjacency matrix* of a bipartite graph $G = (A, B, E)$ (having $A \cup B = \{a_1, ..., a_m\} \cup \{b_1, ..., b_n\}$ as a vertex set, and E as an edge set), is $X = [x_{ij}], 1 \leq i \leq m, 1 \leq j \leq n$, where $x_{ij} = 1$ if $a_i b_j \in E$ and $x_{ij} = 0$ otherwise. A *stable set* of a graph G is a subset of pairwise nonadjacent vertices. The *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G. A graph is called α-*stable* if its stability number remains the same upon both the deletion and the addition of any edge. We show that a connected bipartite graph has exactly two maximum stable sets that partition its vertex set if and only if its reduced adjacency matrix is fully indecomposable. We also describe a decomposition structure of α-stable bipartite graphs in terms of their reduced adjacency matrices. On the base of these findings, we obtain both new proofs for a number of well-known theorems on the structure of matrices due to Brualdi (1966), Marcus and Minc (1963), Dulmage and Mendelsohn (1958), and some generalizations of these statements. Two kinds of matrix product are also considered (namely, Boolean product and Kronecker product), and their corresponding graph operations. As a consequence, we obtain a new proof of one Lewin's theorem claiming that the product of two fully indecomposable matrices is a fully indecomposable matrix.

Key Words: fully indecomposable matrix, cover irreducible matrix, total support, Boolean product, Kronecker product, adjacency matrix, stable set, bistable bipartite graph, perfect matching, elementary graph

Category: G.2.1, G.2.2

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If U is a subset of vertices, then $G[U]$ is the subgraph of G induced

[1] C. S. Calude, G. Stefanescu, and M. Zimand (eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*

by U, i.e., the graph having U as its vertex set, and containing all the edges of G connecting vertices of U. By $G - W$ we mean either the subgraph $G[V - W]$, if $W \subset V(G)$, or the partial subgraph of G obtained by deleting the edges from W, whenever $W \subset E(G)$ (we use $G - a$, if $W = \{a\}$). If U, W are disjoint subsets of V, then (U, W) stands for the set $\{e = uw : u \in U, w \in W, e \in E\}$.

The neighborhood of a vertex $v \in V$, denoted by $N(v)$, is the set of vertices adjacent to v. For any $U \subset V(G)$, we denote $N_G(U) = \cup\{N(x) : x \in U\}$, or, if no ambiguity, $N(U)$.

By pG we mean the *disjoint union* of $p \geq 2$ copies of G. If $G_i, 1 \leq i \leq q$, are q pairwise vertex disjoint subgraphs of G, such that $V(G) = V(G_1) \cup ... \cup V(G_q)$, then we say that $G_i, 1 \leq i \leq q$, define a *decomposition* of G and we write $G = G_1 \cup ... \cup G_q$.

A subset $U \subset V(G)$ is said to be 2-dominating in G if $|N(v) \cap U| \geq 2$, for every vertex $v \in V - U$, [Gunther et al. 1993].

A stable set (i.e., a set containing pairwise nonadjacent vertices) of maximum size will be referred to as a *maximum stable set* of G. The *stability number* of G, denoted by $\alpha(G)$, is the cardinality of a maximum stable set of G.

A perfect matching is a set of non-incident edges of G covering all its vertices.

A bipartite graph is a triple $G = (A, B, E)$, where E is its edge set and $\{A, B\}$ is its bipartition; if $|A| = |B|$, then G is called *balanced bipartite*. If A, B are the only two maximum stable sets of G, then it is a *bistable bipartite* graph, [Levit and Mandrescu 1997]. Clearly, every bistable bipartite graph is also balanced, while the converse is not always true.

A graph $G = (V, E)$ is called:

(i) α^-*-stable* if $\alpha(G - e) = \alpha(G)$ for every $e \in E$, [Gunther et al. 1993];

(ii) α^+*-stable* if $\alpha(G + e) = \alpha(G)$ for each $e \notin E, e = xy$ and $x, y \in V$, [Gunther et al. 1993];

(iii) α*-stable* if it is both α^--stable and α^+-stable,
[Levit and Mandrescu 1997], [Levit and Mandrescu 2001].

Let $G = (A, B, E)$ be a bipartite graph, where $A = \{a_1, a_2, ..., a_m\}$ and also $B = \{b_1, b_2, ..., b_n\}$. Then G can be characterized by its *adjacency matrix*, which is a square $(0, 1)$-matrix of order $m + n$

$$\begin{bmatrix} O & X \\ X^t & O \end{bmatrix},$$

where

$$X = [x_{ij}], 1 \leq i \leq m, 1 \leq j \leq n,$$

with $x_{ij} = 1$ if $a_i b_j \in E$ and $x_{ij} = 0$ otherwise. X is called the *reduced adjacency matrix* of the bipartite graph G. Any $(0, 1)$-matrix of size m by n is the reduced adjacency matrix of a bipartite graph. If G is balanced bipartite, then its reduced adjacency matrix is a square $(0, 1)$-matrix of order $n = |A| = |B|$.

The *term rank* $\rho = \rho(X)$ of a $(0,1)$-matrix X of size m by n is the maximal number of 1's of X with no two of 1's on a line (i.e., on a row or on a column).

A collection of n elements of a square $(0,1)$-matrix X of order n is called a *diagonal* of X provided no two elements belong to the same row or column of X. A *nonzero diagonal* of X is a diagonal containing no 0's.

A square $(0,1)$-matrix X of order n is called *partly decomposable* if $n = 1$ and its unique entry is zero, or $n > 1$ and there is an integer k with $1 \leq k \leq n-1$, such that X has a k by $n-k$ zero submatrix. A square matrix is *fully indecomposable* provided it is not partly decomposable, [Marcus and Minc 1963]. By permuting the lines of X, the partly decomposable matrix X can be written in the form

$$X = \begin{bmatrix} X_1 \; O \\ X_2 \; X_3 \end{bmatrix},$$

where O is a zero matrix of size k by $n-k$, while X_1 and X_3 are square matrices of orders k and $n-k$, respectively.

Decomposition structures of α^+-stable and α-stable bipartite graphs were first established in [Levit and Mandrescu 2001]. On the base of these findings we obtain both new proofs for several well-known theorems on the structure of matrices that may be found in [Brualdi 1966a], [Brualdi 1966b], [Brualdi 1966c], [Brualdi 1967], [Marcus and Minc 1963], [Dulmage and Mendelsohn 1958], and also some generalizations of these statements. Some new results on reduced adjacency matrices of α-stable bipartite graphs are presented, as well. For example, we show that a connected bipartite graph has exactly two maximum stable sets that partition its vertex set if and only if its reduced adjacency matrix is fully indecomposable.

The paper is organized as follows: for the sake of self-consistency, section 2 contains a series of results referring to the structure of bistable, α^+-stable, and α-stable bipartite graphs. We use these findings in section 3, in order to prove some assertions on reduced adjacency matrices associated with bipartite graphs. Sections 4 and 5 are dealing with two different kinds of matrix product, (namely, Boolean and Kronecker), and their corresponding graph operations.

2 α-Stable bipartite graphs

In this section we recall some results concerning the structure of α-stable and α^+-stable bipartite graphs in terms of bistable bipartite graphs.

The following theorem describes some stability properties of general graphs.

Theorem 1. *[Haynes et al. 1990] A graph G is:*

(i) *α^--stable if and only if each of its maximum stable sets is a 2-dominating set in G;*

(ii) α^+*-stable if and only if no pair of vertices is contained in all its maximum stable sets.*

As an immediate consequence, one can deduce that a disconnected graph G, with components $H_1, ..., H_p$, is α-stable if and only if the following assertions are valid:
(i) each $H_i, 1 \leq i \leq p$, is α-stable;
(ii) at most one of $H_i, 1 \leq i \leq p$, has

$$|\cap\{S_i : S_i \text{ is a stability system in } H_i\}| = 1.$$

The following theorem is a generalization of a similar result for trees shown in [Gunther et al. 1993]).

Theorem 2. *[Levit and Mandrescu 1997] If G is a connected bipartite graph, then the following assertions are equivalent:*
(i) *G is α^+-stable;*
(ii) *G has a perfect matching;*
(iii) *G possesses two maximum stable sets that partition its vertex set.*

Figure 1 illustrates some basic differences between α^+-stable and α^--stable graphs. Namely, both are bipartite, but G_1 is α^+-stable and non-α^--stable (since it has a perfect matching and a non-2-dominating maximum stable set), while G_2 is α^--stable and non-α^+-stable (because its unique maximum stable set is 2-dominating and it has no perfect matching).

Figure 1: α^+-stable and α^--stable bipartite graphs: G_1 and G_2, respectively.

A bipartite graph $G = (A, B, E)$ is said to be *cover-irreducible* if it is balanced and A, B are its only minimum vertex covers [Dulmage and Mendelsohn 1967].

In [Lovász and Plummer 1977], [Lovász and Plummer 1986] a graph H is defined as *elementary* if the union of all its perfect matchings forms a connected subgraph of H. The next theorem extends the characterization of elementary bipartite graphs [Hetyei 1964]), [Lovász and Plummer 1977]).

Theorem 3. *[Levit and Mandrescu 2001] If $G = (A, B, E)$ is a bipartite graph with at least* 4 *vertices, then the following assertions are equivalent:*
(i) *G is cover-irreducible;*

(ii) *G is bistable;*

(iii) *for any proper subset U of A or of B, $|N(U)| > |U|$ holds;*

(iv) *G is balanced and for any proper subset U of A, $|N(U)| > |U|$ holds;*

(v) *$G - a - b$ is α^+-stable, for any $a \in A$ and $b \in B$;*

(vi) *$G - a - b$ has a perfect matching, for every $a \in A$ and $b \in B$;*

(vii) *G is connected and each of its edges lies in some perfect matching;*

(viii) *G is elementary;*

(ix) *G can be written in the form $G = G_0 \cup H_1 \cup H_2 \cup ... \cup H_k$, where G_0 consists of two vertices and an edge joining them, and H_i is an even path which joins two points of $G_0 \cup H_1 \cup ... \cup H_{i-1}$ in different color classes and has no other point in common with $G_0 \cup H_1 \cup ... \cup H_{i-1}$.*

The α-stable connected bipartite graphs are completely characterized, as follows.

Theorem 4. *[Levit and Mandrescu 2001] If G is a connected bipartite graph, then the following assertions are equivalent:*

(i) *G is α-stable;*

(ii) *$G = G_1 \cup ... \cup G_k, k \geq 1$, where each $G_i, 1 \leq i \leq k$, is bistable bipartite and has at least 4 vertices;*

(iii) *G has perfect matchings and $\cap\{M : M$ is a perfect matching of $G\} = \emptyset$;*

(iv) *for every vertex of G there exist at least two edges incident to this vertex and contained in some perfect matchings;*

(v) *each vertex of G belongs to some alternating cycle.*

Finally, let us recall the following result.

Proposition 5. *[Levit and Mandrescu 2001] A connected bipartite graph G is α^+-stable if and only if it admits a decomposition as $G = G_1 \cup ... \cup G_k$, where all G_i are bistable bipartite.*

Figure 2 offers an example of decomposition of an α^+-stable bipartite graph into vertex-disjoint and bistable bipartite components: $G = G_1 \cup G_2 \cup G_3$.

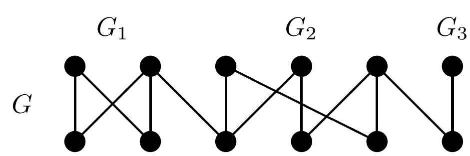


Figure 2: A decomposition of G into bistable components: $G = G_1 \cup G_2 \cup G_3$.

3 Matrices and bipartite graphs

It is not difficult to see that the unity matrix $I_n, n \geq 1$, is the reduced adjacency matrix of nK_2, i.e., of the graph consisting of n disjoint copies of K_2. A simple generalization of this observation is as follows.

Lemma 6. *A bipartite graph G is disconnected if and only if its adjacency matrix X can be written as*

$$X = \begin{bmatrix} X_1 & O & O & . & . & O \\ O & X_2 & O & . & . & O \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ O & O & . & & . & . X_k \end{bmatrix}, \tag{1}$$

where the blocks $X_1, X_2, ..., X_k$ are the adjacency matrices corresponding to the $k \geq 2$ connected components of G, respectively.

The following findings emphasize an impact of a stable set on the structure of the reduced adjacency matrix.

Lemma 7. *Let S be a proper subset of the vertex set of $G = (A, B, E)$, with $p + q$ vertices, where $p = |S \cap A| \geq 1$ and $q = |S \cap B| \geq 1$. Then S is stable in G if and only if its reduced adjacency matrix X can be written as*

$$X = \begin{bmatrix} X_1 & O \\ X_2 & X_3 \end{bmatrix},$$

where O is a p by q zero matrix.

Proof. By using an appropriate indexing for A and for B, we may suppose that

$$S \cap A = \{a_1, ..., a_p\} \text{ and } S \cap B = \{b_{n-q+1}, ..., b_n\}.$$

Therefore, S is stable in G if and only if $x_{ij} = 0$ for every $i \in \{1, ..., p\}$ and $j \in \{n - q + 1, ..., n\}$, i.e., X has exactly the form announced above.

Proposition 8. *Let $G = (A, B, E)$ be a connected balanced bipartite graph with $2n$ vertices and X be its reduced adjacency matrix. Then G has a stable set of n vertices that meets both A and B if and only if X is partly decomposable.*

Proof. If $p = |S \cap A|$, then $q = |S \cap B| = n - p$, and by Lemma 7, we obtain X in the form

$$X = \begin{bmatrix} X_1 & O \\ X_2 & X_3 \end{bmatrix},$$

where O is a p by $n - p$ zero matrix, $1 \leq p \leq n - 1$, i.e., the reduced adjacency matrix X is partly decomposable.

Proposition 9. *A balanced bipartite graph is bistable if and only if its reduced adjacency matrix is fully indecomposable.*

Proof. Since a bistable bipartite graph $G = (A, B, E)$ is connected and has only A and B as maximum stable sets, Proposition 8 ensures that its reduced adjacency matrix can not be partly decomposable. The converse is clear.

Following the terminology from [Dulmage and Mendelsohn 1967], let us recall that for a balanced bipartite graph $G = (A, B, E)$, a *cover* is a pair of subsets $\{A_0, B_0\}$ of A and B respectively, such that for every edge $ab \in E$, either $a \in A_0$ or $b \in B_0$. G is *cover irreducible* if its only minimum covers are $\{A, \emptyset\}$ and $\{\emptyset, B\}$. The reduced adjacency matrix of a cover irreducible bipartite graph is a *cover irreducible matrix*.

It was shown in [Lovász and Plummer 1986] that elementary bipartite graphs and the cover irreducible bipartite graphs are the same. It turns out that bistable bipartite graphs are exactly cover irreducible bipartite graphs, and fully indecomposable matrices coincide with cover irreducible matrices. Our approach is based, in principal, on the *bistable property*. Combining Theorem 3 and Proposition 9, we get the following result from [Brualdi et al. 1980].

Corollary 10. *Let G be a balanced bipartite graph with $2n$ vertices and X be its reduced adjacency matrix. Then X is fully indecomposable if and only if G is connected and every edge belongs to a perfect matching.*

We also obtain a simple proof for the following characterization of fully indecomposable matrices from [Marcus and Minc 1963], [Brualdi 1966a].

Theorem 11. *A $(0,1)$-matrix X of order $n \geq 2$ is fully indecomposable if and only if every 1 of X belongs to a nonzero diagonal and every 0 of X belongs to a diagonal whose other elements equal 1.*

Proof. Let $G = (A, B, E)$ be a balanced bipartite graph with $|A| = |B| = n$, having X as its reduced adjacency matrix. Then, according to Proposition 9 and Theorem 2, X is fully indecomposable if and only if $G - a - b$ is α^+-stable for every $a \in A$ and $b \in B$, i.e., for any $i, j \in \{1, ..., n\}$, the submatrix Y, obtained by deleting the row i and the column j of X, has a nonzero diagonal, and this completes the proof.

Corollary 12. *[Marcus and Minc 1963] A fully indecomposable $(0,1)$-matrix X of order n contains at most $n(n-2)$ zero entries.*

Proof. Let $G = (A, B, E)$ be a balanced bipartite graph with X as its reduced adjacency matrix. By Proposition 9, G is bistable and according to Theorem 3*(iii)*, $|N(v)| \geq 2$ holds for any vertex v of G. Consequently, any row of X

cannot have more than $n-2$ zeros, and hence X cannot contain more than $n(n-2)$ zero entries. On the other hand, $C_{2n}, n \geq 2$, is bistable and its reduced adjacency matrix has exactly $n(n-2)$ zero entries.

A $(0,1)$-matrix of order $n \geq 2$ is said to be with *total support* provided each of its 1's belongs to a nonzero diagonal.

Proposition 13. *[Brualdi and Ryser 1991] Let X be a $(0,1)$-matrix of order $n \geq 2$ with total support, and let G be the bipartite graph whose reduced adjacency matrix is X. Then G is connected if and only if X is fully indecomposable.*

Proof. Clearly, X is with total support if and only if each edge of G is contained in a perfect matching of G. Therefore, taking into account Theorem 3*(vii)* and Proposition 9, we get that:

$$G \ \ is\ connected \ \Leftrightarrow G\ is\ bistable \ \Leftrightarrow X\ is\ fully\ indecomposable,$$

and this completes the proof.

We can now characterize the bipartite graphs whose reduced adjacency matrices are with total support.

Proposition 14. *The reduced adjacency matrix X of a bipartite graph G has total support if and only if each connected component of G is bistable bipartite.*

Proof. If G is connected, then according to Proposition 9, X has total support if and only if G is bistable. If G is disconnected, Lemma 6 implies that X can be written in the form (1), and then X has total support if and only if all the blocks $X_1, ..., X_k$ have total support, i.e., according to Propositions 9 and 13, all connected components of G are bistable bipartite.

Proposition 15. *Let G be a balanced bipartite graph with $2n$ vertices and X be its reduced adjacency matrix. Then the following assertions are equivalent:*
(i) *G is α^+-stable;*
(ii) *X has a nonzero diagonal;*
(iii) *$\rho(X) = n$;*
(iv) *$per(X) > 0$.*

Proof. According to Theorem 2, G is α^+-stable if and only if it has a perfect matching, i.e., its reduced adjacency matrix X has a nonzero diagonal, and this clearly is equivalent to both *(iii)* and *(iv)*.

Corollary 16. *[Minc 1969] A $(0,1)$-matrix X of order $n \geq 2$ is fully indecomposable if and only if every $(n-1)$-square submatrix Y of X has $per(Y) > 0$.*

Proof. Suppose X is the reduced adjacency matrix of the balanced bipartite graph $G = (A, B, E)$. According to Proposition 9, X is fully indecomposable if and only if G is bistable bipartite, and by Theorem 3, this happens if and only if $G - a - b$ is α^+-stable, for every $a \in A$ and $b \in B$, i.e., by virtue of the Proposition 15, if and only if $per(Y) > 0$ holds for each $(n-1)$-square submatrix Y of X.

Corollary 17. *[Brualdi 1966b] Let X be a square $(0,1)$-matrix of order n and let X_{ij} denote the matrix obtained from X by striking the i-th row and the j-th column. Then X is fully indecomposable if and only if $per(X_{ij}) > 0$.*

Proof. Let $G = (A, B, E)$ be a bipartite graph whose reduced adjacency matrix is X. By Proposition 9, X is fully indecomposable if and only if G is bistable, i.e., according to Theorem 3*(vi)*, $G-a-b$ has a perfect matching for every $a \in A$ and $b \in B$, that is, by Proposition 15, the matrix X_{ab} has a positive permanent.

Theorem 18. *Let G be a balanced bipartite graph with $2n$ vertices and X be its reduced adjacency matrix. Then G is α-stable if and only if X can be written as*

$$X = \begin{bmatrix} X_1 & X_{12} & X_{13} & .. & X_{1k} \\ O & X_2 & X_{23} & .. & X_{2k} \\ . & . & . & \cdots & \\ . & . & . & \cdots & \\ . & . & . & \cdots & \\ O & O & O & .. & X_k \end{bmatrix}, \tag{2}$$

where $X_1, ..., X_k$ are fully indecomposable matrices of order at least 2.

Proof. According to Theorem 4*(ii)*, G is α-stable if and only if it admits a decomposition as $G = G_1 \cup ... \cup G_k$, where all $G_i, 1 \leq i \leq k$, are simultaneously α-stable, bistable bipartite, and pairwise vertex-disjoint. Hence, using an appropriate indexing for the vertices of G, X can be written in the form (2), with $X_1, ..., X_k$ as reduced adjacency matrices corresponding to $G_1, ..., G_k$. By Proposition 9, each X_i is fully indecomposable. In addition, every X_i is of order at least two, since it corresponds to G_i, which has at least 4 vertices, because it is a bistable bipartite and α-stable graph.

Theorem 19. *Let G be a balanced bipartite graph with $2n$ vertices and X be its reduced adjacency matrix. Then G is α^+-stable if and only if X can be written in the form (2), where all $X_1, ..., X_k$ are fully indecomposable matrices.*

Proof. By Proposition 5, G is α^+-stable if and only if it admits a decomposition as $G = G_1 \cup ... \cup G_k$, where all $G_i, 1 \leq i \leq k$, are bistable balanced bipartite and pairwise vertex-disjoint. Hence, using an appropriate indexing for the vertices of G, the matrix X can be written in the form (2), with $X_1, ..., X_k$ as reduced adjacency matrices corresponding to $G_1, ..., G_k$, and therefore being fully indecomposable, by Proposition 9.

As an outcome, we obtain the following well-known result.

Theorem 20. *[Dulmage and Mendelsohn 1958], [Brualdi 1966a] Let X be a $(0,1)$-matrix of order n with term rank $\rho(X)$ equal to n. Then there exist permutation matrices P and Q of order n and an integer $k \geq 1$ such that $P \times A \times Q$ has the form (2), where all $X_1, ..., X_k$ are square fully indecomposable matrices.*

Proof. Let G be a balanced bipartite graph, whose reduced adjacency matrix is X. By Proposition 15*(i)*, G is α^+-stable, and according to Proposition 5 it admits a decomposition as $G = G_1 \cup ... \cup G_k$, where all the graphs G_i are bistable bipartite and pairwise vertex-disjoint. Hence, using an appropriate indexing for the vertices of G, the matrix X can be written according to the form *(*2*)*, with $X_1, ..., X_k$ as reduced adjacency matrices corresponding to $G_1, ..., G_k$. Proposition 9 ensures that all $X_1, ..., X_k$ are fully indecomposable.

Combining Theorem 20 and Theorem 4*(ii),(iii)* we deduce the following.

Corollary 21. *Let X be a $(0,1)$-matrix of order n with $\rho(X) = n$. Then the number of 1 by 1 blocks X_i in the matrix (2) is equal to the number of common elements of all nonzero diagonals of X.*

We end this section with a characterization of the reduced adjacency matrix corresponding to an α-stable bipartite graph.

Proposition 22. *Let G be a balanced bipartite graph and X be its reduced adjacency matrix. Then G is α-stable if and only if for every non-zero entry x_{ij} of X there exists a non-zero diagonal of X that does not contain it.*

Proof. According to Theorem 4*(iii)*, $G = (A, B, E)$ is α-stable if and only if it has perfect matchings and

$$\cap\{M : M \text{ } is \text{ } a \text{ } perfect \text{ } matching \text{ } of \text{ } G\} = \emptyset,$$

that is, G has perfect matchings, and for each edge $e \in E$ there is a perfect matching M such that $e \notin M$. In other words, G is α-stable if and only if for every non-zero entry x_{ij} of X, there is a non-zero diagonal of X that does not contain it.

4 Boolean product of matrices

Let $G = (A, B, E)$ and $H = (B, C, F)$ be two balanced bipartite graphs on $2n$ vertices. We define the *bipartite join* of G with H as the bipartite graph

$$G * H = (A, C, W),$$

where $ac \in W$ if and only if there is $b \in B$, such that $ab \in E$ and $bc \in F$.

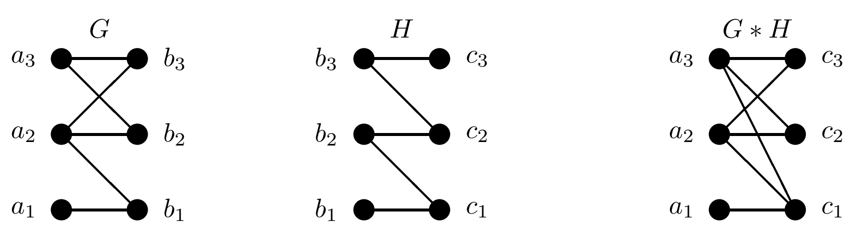
The *Boolean matrix product* of two $(0, 1)$-matrices X, Y is a $(0, 1)$-matrix denoted by $X \bullet Y$ and having the same zero and non-zero entries as the usual matrix product $X \times Y$. Hence $X \bullet Y$ is fully indecomposable if and only if $X \times Y$ is fully indecomposable. The term *Boolean* refers to the property of the *Boolean addition operation* saying that: $1 + 1 = 1$. For an example, see Figure 3. Using this notation we have the following result.

Lemma 23. *If X, Y are the reduced adjacency matrices of the balanced bipartite graphs G and H, respectively, then the Boolean matrix product $X \bullet Y$ is the reduced adjacency matrix of the graph $G * H$.*

Proof. If $X = (x_{ij}), Y = (y_{ij})$ and $X \bullet Y = (z_{ij})$, then clearly we have:

$$\begin{aligned} z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj} \neq 0 \Leftrightarrow \\ \textit{there exists } k \in \{1, ..., n\}, \textit{ such that } x_{ik} = y_{kj} = 1 \Leftrightarrow \\ \textit{there is some } b_k \in B, \textit{ so that } a_i b_k \in E \textit{ and } b_k c_j \in F \Leftrightarrow \\ a_i c_j \in W, \end{aligned}$$

which validates the assertion.



$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad Z = X \bullet Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3: $G * H$ and its corresponding reduced adjacency matrix $Z = X \bullet Y$.

Corollary 24. *Every balanced bipartite graph G on $2n, n \geq 1$, vertices is isomorphic to $G * nK_2$.*

Proposition 25. *Let $G = (A, B, E)$ and $H = (B, C, F)$ be balanced bipartite graphs. Then the following assertions are true:*

(i) *if G and H are α^+-stable, then $G * H$ is α^+-stable;*

(ii) *if one of G, H is α^+-stable and the other is bistable bipartite, then $G * H$ is bistable bipartite;*

(iii) *if both G and H are bistable bipartite, then $G * H$ is bistable bipartite, as well.*

Proof. *(i)* Taking into account the definition of $*$-operation, it is clear that $G*H$ has a perfect matching, whenever both G and H have a perfect matching. Hence, Theorem 2 implies that $G*H$ is α^+-stable whenever G and H are both α^+-stable.

(ii) Suppose that G is α^+-stable and H is bistable bipartite. If D is an arbitrary proper subset of A or of C, then according to Theorem 3 and Hall's marriage theorem we get:

$$|D| < |N_G(D)| \leq |N_H(N_G(D))| = |N_{G*H}(D)|,$$

i.e., $G * H$ is bistable, by virtue of the same Theorem 3.

The assertion *(iii)* is a consequence of *(ii)*.

Corollary 26. *Let X, Y be $(0, 1)$-matrices of order n. If $per(X) > 0$ and Y is fully indecomposable, then $X \times Y$ is fully indecomposable.*

Proof. Let $G = (A, B, E)$ be a bipartite graph with a perfect matching and $H = (B, C, F)$ be a balanced bipartite graph. Let X and Y be the reduced adjacency matrices of G and H, respectively. Lemma 23 implies that $X \bullet Y$ is the reduced adjacency matrix of the graph $G * H$.

By Theorem 2 G is α^+-stable, while by Proposition 9, H is bistable bipartite. According to Proposition 25*(ii)*, $G * H$ is also bistable bipartite. Hence, Proposition 9 ensures that $X \bullet Y$ is fully indecomposable. Therefore, $X \times Y$ is fully indecomposable, as well.

Actually, Corollary 26 is a strengthening of Theorem 27.

Theorem 27. *[Lewin 1971] The product of any finite number of fully indecomposable matrices is a fully indecomposable matrix.*

Corollary 28. *[Marcus and Minc 1963] If X is a fully indecomposable $(0, 1)$-matrix, then $X \times X^t$ is fully indecomposable.*

5 Kronecker product of matrices

Let $G = (A, B, E)$ and $H = (C, D, F)$ be two balanced bipartite graphs on $2n$ vertices. The *Kronecker product* of graphs G and H is the graph

$$G \otimes H = (A \times C, B \times D, U),$$

where $(a, c)(b, d) \in U$ if and only if $ab \in E$ and $cd \in F$. In these notations we have the following result.

Lemma 29. *If X and Y are the reduced adjacency matrices of the balanced bipartite graphs G and H, respectively, then the Kronecker matrix product $X \otimes Y$ is the reduced adjacency matrix of the graph $G \otimes H$.*

Proof. If $X = (x_{ij}), Y = (y_{ij})$ and $X \otimes Y = (z_{ij})$, then we have:

$$\begin{aligned} z_{ij} &= z_{(k-1)m+p,(r-1)m+q} = x_{kr} y_{pq} = 1 \Leftrightarrow \\ x_{kr} &= 1 \textit{ and } y_{pq} = 1 \Leftrightarrow \\ a_k b_r &\in E \textit{ and } c_p d_q \in F \Leftrightarrow (a_k, c_p)(b_r, d_q) \in U, \end{aligned}$$

i.e., $X \otimes Y$ is the reduced adjacency matrix of $G \otimes H$.

Proposition 30. *If $G = (A, B, E)$ and $H = (C, D, F)$ are α^+-stable, then their Kronecker product $G \otimes H$ is also α^+-stable.*

Proof. Let $\{(a_i, b_i) : 1 \le i \le n\}$ and $\{(c_j, b_j) : 1 \le j \le m\}$ be perfect matchings in G, H respectively, which exist by virtue of Theorem 2. Hence, according to the same theorem, $G \otimes H$ is also α^+-stable, since

$$\{(a_i, c_j)(b_i, d_j) : 1 \le i \le n, 1 \le j \le m\}$$

is a perfect matching in $G \otimes H$.

Corollary 31. *Let X, Y be two $(0, 1)$-matrices of order n, m, respectively. Then*

$$\begin{aligned} &\rho(X \otimes Y) \ge \rho(X)\rho(Y), \\ &\textit{and if } \rho(X) = n, \rho(Y) = m, \textit{ then } \rho(X \otimes Y) = \rho(X)\rho(Y). \end{aligned}$$

Proof. Let $G = (A, B, E)$ and $H = (C, D, F)$ be bipartite graphs having X, Y as reduced adjacency matrices, respectively. If the edge sets

$$\{a_i b_i : 1 \le i \le \rho(X)\} \textit{ and } \{c_j b_j : 1 \le j \le \rho(Y)\}$$

are maximum matchings in G and H, respectively, then

$$M = \{(a_i, c_j)(b_i, d_j) : 1 \le i \le \rho(X), 1 \le j \le \rho(Y)\}$$

is a matching in $G \otimes H$, and consequently, we have

$$\rho(X \otimes Y) \geq |M| \geq \rho(X)\rho(Y).$$

If $\rho(X) = n$ and $\rho(Y) = m$, i.e., both G and H have perfect matchings, then M is a perfect matching in $G \otimes H$, and this ensures that $\rho(X \otimes Y) = \rho(X)\rho(Y)$.

Proposition 32. *If $G = (A, B, E)$ is α-stable and $H = (C, D, F)$ is α^+-stable, then their Kronecker product $G \otimes H$ is α-stable.*

Proof. Let X, Y, Z be the corresponding reduced adjacency matrices of G, H and K. By Proposition 22, for every non-zero entry

$$z_{ij} = z_{(k-1)m+p,(r-1)m+q} = x_{kr}y_{pq}$$

of Z, there is a non-zero diagonal $\{x_{1i_1}, x_{2i_2}, ..., x_{ni_n}\}$ of X that does not contain x_{kr}, and clearly, the blocks

$$\{x_{1i_1}Y, x_{2i_2}Y, ..., x_{ni_n}Y\}$$

contain one non-zero diagonal of Z, since Y has at least one non-zero diagonal. According to Proposition 22, $G \otimes H$ is α-stable.

Corollary 33. *The Kronecker product of any two α-stable bipartite graphs is α-stable.*

Theorem 34. *[Brualdi 1967] The Kronecker product of two fully indecomposable matrices is a fully indecomposable matrix.*

As a consequence, we get the following result.

Corollary 35. *The Kronecker product of two bistable bipartite graphs is a bistable bipartite graph.*

6 Conclusions

In this paper we have investigated the intimate relationship between the structures of α-stable bipartite graphs and their corresponding reduced matrices.

The mutual transfer of the results was done via the following bridge:

$$bistable\ bipartite\ graphs\ vis-a-vis\ fully\ indecomposable\ matrices.$$

On the base of this correspondence, we have obtained new proofs and extensions of several well-known theorems on matrices.

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