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Quotient Spaces and Coequalisers in Formal Topology[1]

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Abstract: We give a construction of coequalisers in formal topology, a predicative version of locale theory. This allows for construction of quotient spaces and identification spaces in constructive topology.

Key Words: locales, coequalisers, quotients, formal topology, predicativity, constructive topology.
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1 Introduction

Several interesting and promising approaches to the constructivisation of general topology have been suggested. Whereas the theory of metric spaces is well-developed (Bishop and Bridges [2]), and caters well for the needs of mathematical analysis, there is no general agreement of which of those approaches, to non-metric topology, will be most fruitful. This can probably only be tested against a development of a constructive theory of manifolds and algebraic topology, in which quotient spaces play a fundamental role.

In Bridges and Vîţă [4] it is argued that neighbourhood based topology does not carry enough constructive information. By their introduction of *apartness spaces* they take instead apartness of points and sets of points as a basic notion of topology. See [9] for a generalisation of apartness spaces, and the relation to neighbourhood spaces. A rather different approach is that of *point-free topology,* specifically locale theory [10, 11], which takes the lattice of open sets and a covering relation as basic notions, and then derives the notion of point and point-function. The continuous functions $\mathbb{R} \longrightarrow \mathbb{R}$ turn out to agree with those of BISH [16]. Also this approach carries around more constructive information than the neighborhood based topology does. A connection between the two approaches was given in [17], where it is shown that regular formal topologies (and so regular locales) gives rise to apartness spaces. See [1, 20] for further discussion and references.

Formal topology in the sense of Martin-Löf and Sambin [18, 19] may be considered as a predicative version of constructive locale theory [10, 11]. In order for the theory to permit the usual topological constructions, such as quotienting,

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

gluing subspaces and attaching maps, it is enough that the category of formal topologies and continuous mappings has finite limits and finite colimits. See [5, 6, 15] for surveys of earlier results, e.g. the construction of products, coproducts and equalisers. In this paper we provide the missing piece: construction of coequalisers.

In the category of locales the coequaliser of a pair of morphisms can easily be constructed as an equaliser in the dual category of frames (see e.g. [3]). The straightforward translation of this construction into the terms of formal topology is

$$\{U \in \mathcal{P}(Y) : (\forall a \in X)(a \lhd F^{-1}U \iff a \lhd G^{-1}U)\},$$

for a pair of continuous mappings $F, G : X \longrightarrow Y$ between formal topologies. From a predicative point of view the problem with this construction is the use of the full power set $\mathcal{P}(Y)$. We show that it can be replaced by a restricted set of subsets, which may indeed be constructed in, e.g., Martin-Löf type theory [12, 14].

Together with known predicative constructions of products [5] and coproducts, and equalisers [15] the above result gives that the set-presented formal topologies form a small complete and small co-complete category, just as the classical topological spaces. This indicates that the category should be adequate for constructing many of the spaces studied by methods of algebraic topology; see Section 5.

Already in the setting of neighbourhood spaces (and thus with points) surprisingly difficult predicativity problems appear when constructing quotient spaces or coequalisers [8].

2 The category of set-presented formal topologies

Following Bishop and Bridges [2], and category-theoretic practise, a subset $A = (\iota, I)$ of a given set X is an injective function $\iota : I \longrightarrow X$. An element x of X is a *member of the subset A,* if $x = \iota(a)$ for some $a \in A$. Note that this a is necessarily unique. We then write $x \in_X A$. Two subsets A and B of X are *equal* if

$$x \in_X A \iff x \in_X B.$$

From this membership definition arises easily notions of inclusion and the usual set-theoretical operations.

For any family $\mathcal{U}$ of types $T(t)$ $(t : U)$ there is a notion of $\mathcal{U}$*-set,* which is a set A which is isomorphic to a set of the form $(T(t), =_e)$ where the equivalence is

$$x =_e y \iff T(e(x, y))$$

and $e : T(t) \times T(t) \longrightarrow U$. For any set X there is then a notion of restricted power set $\mathcal{R}_{\mathcal{U}}(X)$. This is a set consisting of subsets $A = (I, \iota)$ of X where I is a $\mathcal{U}$-set. Such subsets are called *$\mathcal{U}$-subsets.* Two such are identified if they are equal as subsets. Unless the family of types have certain closure properties it will not be possible to perform the usual set operations on the restricted power set. We return to the question of what these properties might be later.

A set X is a *projective set* or a *choice-set* if the axiom of choice is valid on X. The latter means that for any set Y and for any relation R between X and Y if

$$(\forall x \in X)(\exists y \in Y)R(x,y)$$

then there is a function $f : X \longrightarrow Y$ so that

$$(\forall x \in X)R(x, f(x)).$$

As any type in Martin-Löf type theory can be equipped with an equality relation (given by an Id-type) so that it becomes a projective set $\underline{X}$, the above choice principle is sometimes referred to as *type-theoretic choice.* The principle is frequently used in Bishop-style constructivism. We thus assume that for every set X there is a projective set $\underline{X}$ and a surjective function $p_X : \underline{X} \longrightarrow X$. Then we get the following choice principle which is sometimes useful

$$(\forall x \in X)(\exists y \in Y)R(x,y) \Longrightarrow (\exists f : \underline{X} \longrightarrow Y)(\forall x \in \underline{X})R(p_X(x), f(x)). \tag{1}$$

There is a dependent version of this principle as well

$$(\forall x \in X)(\exists y \in Y_x)R(x,y) \Longrightarrow \Big(\exists f \in \prod_{u \in \underline{X}} Y_{p_X(u)}\Big)(\forall u \in \underline{X})R(p_X(u), f(u)). \tag{2}$$

Definition 1 Let S be a set, and let $\lhd$ be a relation between elements of S and subsets of S, i.e. $\lhd \subseteq S \times \mathcal{P}(S)$. Extend $\lhd$ to a relation between subsets by letting $U \lhd V$ if and only if $a \lhd V$ for all $a \in U$. For a preorder $(X, \leq)$ and a subset $U \subseteq X$, the *downwards closure* $U_{\leq}$ consists of those $x \in X$ such that $x \leq y$ for some $y \in U$. Write $a_{\leq}$ for $\{a\}_{\leq}$. When the preorder is obvious from the context we write $U \wedge V$ for $U_{\leq} \cap V_{\leq}$. A further abbreviation is $a \wedge b$ for $\{a\} \wedge \{b\}$.

Definition 2 A *formal topology* $\mathcal{S}$ is a pre-ordered set $S = (S, \leq)$ (of so-called *basic neighbourhoods*) together with a relation $\lhd \subseteq S \times \mathcal{P}(S)$, the *covering relation,* satisfying the four conditions

(R) $a \in U$ implies $a \lhd U$, (L) $a \lhd U$, $a \lhd V$ implies $a \lhd U \wedge V$,
(T) $a \lhd U$, $U \lhd V$ implies $a \lhd V$, (E) $a \leq b$ implies $a \lhd \{b\}$.

The topology is *set-presented* if there is a family of subsets $C(a,i)$ of S, where $i \in I(a)$ and $a \in S$ such that

$$a \lhd U \Longleftrightarrow (\exists i \in I(a))\, C(a,i) \subseteq U.$$

Equivalently, we may express this as: there is a family $C(w)$ $(w \in I)$ of subsets of S and a function $c : I \longrightarrow S$ so that

$$a \lhd U \iff (\exists w \in I)\, c(w) = a \;\&\; C(w) \subseteq U. \tag{3}$$

A continuous mapping between formal topologies is a certain relation between their basic neighbourhoods. To define the concept we introduce some notation. For a relation $R \subseteq S \times T$ the *inverse image of* $V \subseteq T$ *under the relation* R is, as usual,

$$R^{-1}[V] =_{\mathrm{def}} \{a \in S : (\exists b \in V)\, a\, R\, b\}$$

Notice that, in general, $R^{-1}[U] \subseteq R^{-1}[V]$ whenever $U \subseteq V$, and

$$R^{-1}[\cup_{i \in I} U_i] = \cup_{i \in I} R^{-1}[U_i]. \tag{4}$$

The relation R is naturally extended to subsets as follows. For $U \subseteq S$, let $U\, R\, b$ mean $(\forall u \in U)\, uRb$, and for $V \subseteq T$, we let $a\, R\, V$ mean $a \lhd R^{-1}[V]$.

Definition 3 Let $\mathcal{S} = (S, \leq, \lhd)$ and $\mathcal{T} = (T, \leq', \lhd')$ be formal topologies. A relation $R \subseteq S \times T$ is a *continuous mapping,* or *continuous morphism,* from $\mathcal{S}$ to $\mathcal{T}$ (and we write $R : \mathcal{S} \longrightarrow \mathcal{T}$) if

(A1) $a\, R\, b$, $b \lhd' V$ implies $a\, R\, V$,

(A2) $a \lhd U$, $U\, R\, b$, implies $a\, R\, b$,

(A3) $a\, R\, T$, for all $a \in S$,

(A4) $a\, R\, V$, $a\, R\, W$ implies $a\, R\, (V_{\leq'} \cap W_{\leq'})$.

Remark. Note that by $b \lhd' \{b\}$, (A1) and (A2)

$$\{a\}\, R\, b \iff a\, R\, b \iff a \lhd R^{-1}\{b\} \iff a\, R\, \{b\}.$$

Moreover (A4) may be replaced by the condition

(A4') $a\, R\, b, a\, R\, c \Longrightarrow a\, R\, (b_{\leq'} \cap c_{\leq'})$.

The next properties are useful for checking closure under composition. Denote by $\tilde{U} = \{a : a \lhd U\}$ — the saturation of U in the topology.

Proposition 1 *Let $R : \mathcal{S} \longrightarrow \mathcal{T}$ be a continuous mapping. Then:*

(i) $U \lhd V$ *implies* $R^{-1}[U] \lhd R^{-1}[V]$,

(ii) $b\, R\, U$ *iff* $b\, R\tilde{U}$,

(iii) $R^{-1}[U]^{\sim} = R^{-1}[\tilde{U}]^{\sim}$. $\square$

Let $\mathbf{FTop}_s$ be the following category of set-presented formal topologies and continuous mappings. For a formal topology $\mathcal{S} = (S, \leq, \lhd)$ we define a continuous mapping $I : \mathcal{S} \longrightarrow \mathcal{S}$ (the identity) by

$$aIb \Longleftrightarrow a \lhd \{b\}.$$

For continuous mappings, $R_1 : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ and $R_2 : \mathcal{S}_2 \longrightarrow \mathcal{S}_3$, between formal spaces, define the composition

$$a(R_2 \circ R_1)b \Longleftrightarrow a \lhd R_1^{-1}[R_2^{-1}\{b\}].$$

This is a continuous mapping $(R_2 \circ R_1) : \mathcal{S}_1 \longrightarrow \mathcal{S}_3$.

3 Construction of coequalisers

Let F and G be continuous mappings $\mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathbf{FTop}_s$. A set $\mathcal{R}(Y)$of subsets of Y is said to be *adequate for F and G* if (H1) – (H3) below are satisfied.

(H1) $Y \in \mathcal{R}(Y)$.

(H2) $U, V \in \mathcal{R}(Y)$ implies $U \wedge V \in \mathcal{R}(Y)$. Here $\wedge$ is taken with respect to the preorder of $\mathcal{Y}$.

(H3) For any subset U of Y with $b \in U$ such that U satisfies the equivalence

$$(\forall a \in X)(a \lhd_{\mathcal{X}} F^{-1}U \Longleftrightarrow a \lhd_{\mathcal{X}} G^{-1}U)$$

there is already some $V \in \mathcal{R}(Y)$ with $b \in V \subseteq U$ satisfying the equivalence.

Lemma 1. *Let $F, G : \mathcal{X} \longrightarrow \mathcal{Y}$ be a pair of continuous morphisms in $\mathbf{FTop}_s$. If $\mathcal{R}(Y)$ is adequate for the pair F and G, then the following defines a coequaliser of the pair: the formal topology $\mathcal{Q} = (Q, \leq_{\mathcal{Q}}, \lhd_{\mathcal{Q}})$ where*

$$Q = \{U \in \mathcal{R}(Y) : (\forall a \in X)(a \lhd_{\mathcal{X}} F^{-1}U \Longleftrightarrow a \lhd_{\mathcal{X}} G^{-1}U)\}$$

and $U \leq_{\mathcal{Q}} V$ iff $U \lhd_{\mathcal{Y}} V$, and where $U \lhd_{\mathcal{Q}} \mathcal{U}$ iff $U \lhd_{\mathcal{Y}} \cup \mathcal{U}$ for $\mathcal{U} \subseteq \mathcal{R}(Y)$. Moreover the coequalising morphism $P : \mathcal{Y} \longrightarrow \mathcal{Q}$ is given by: $a\,P\,U$ iff $a \lhd_{\mathcal{Y}} U$.

Proof. By (H2) it follows that Q is closed under $\wedge$. Using this it is straightforward to check that $\mathcal{Q}$ is a formal topology. It is as well set-presented since, if $C(a, i)$ $(i \in I(a))$ is the set-presentation of $\mathcal{Y}$, then we get a set-presentation (D, J) for $\mathcal{Q}$ by letting for $U \in \mathcal{R}(Y)$

$$J(U) =_{\mathrm{def}} \{(f, g) : f \in \Phi,\ g \in \prod_{x \in \underline{U}} \prod_{y \in \underline{C}(p_U(x), f(x))} Q_{p'_x(y)}\}$$

where $\Phi = (\Pi x \in \underline{U})I(p_U(x))$, $Q_u = \{U \in Q : u \in U\}$, and $p'_x(y) = p_{C(p_U(x),f(x))}(y)$. Moreover, let

$$D(U,(f,g)) =_{\mathrm{def}} \{g(x)(y) : x \in \underline{U}, y \in \underline{C}(p_U(x), f(x))\}.$$

Here, as before, $p_S : \underline{S} \longrightarrow S$ is the surjection associated with the choice-set of S. Set-presentability now follows. Indeed, using the choice principle (2) we get the equivalences

$$\begin{aligned}
U \lhd_{\mathcal{Q}} \mathcal{U} &\iff (\forall a \in U)\; a \lhd \cup\mathcal{U} \\
&\iff (\forall a \in U)(\exists i \in I(a))\; C(a,i) \subseteq \cup\mathcal{U} \\
&\iff (\exists f \in \Phi)(\forall a \in \underline{U})\; C(p_U(a), f(a)) \subseteq \cup\mathcal{U} \\
&\iff (\exists f \in \Phi)(\forall a \in \underline{U})\Big(\forall b \in \underline{C}(p_U(a), f(a))\Big)\; p'_a(b) \in \cup\mathcal{U}
\end{aligned}$$

Note that $p \in \cup\mathcal{U}$ is equivalent to $(\exists V \in Q_p)\, V \in \mathcal{U}$. Now abbreviating $\underline{C}(p_U(a), f(a))$ as $E(a,f)$ and then using type-theoretic choice twice, we obtain further equivalences:

$$\begin{aligned}
&\iff (\exists f \in \Phi)(\forall a \in \underline{U})(\forall b \in E(a,f))(\exists V \in Q_{p'_a(b)})\; V \in \mathcal{U} \\
&\iff \Big(\exists f \in \Phi\Big)\Big(\exists g \in (\Pi x \in \underline{U})(\Pi y \in E(x,f))Q_{p'_x(y)}\Big) \\
&\qquad\qquad \Big(\forall a \in \underline{U}\Big)\Big(\forall b \in E(a,f)\Big)\; g(a)(b) \in \mathcal{U}. \\
&\iff (\exists (f,g) \in J(U))\; D(U,(f,g)) \subseteq \mathcal{U}.
\end{aligned}$$

This proves set-presentability.

Next, to check that P is a continuous morphism is easy. For instance, to verify condition (A3): Trivially, for any $a \in Y$ we have $a \lhd_{\mathcal{Y}} Y$. By (H1), $Y \in \mathcal{R}(Y)$ and then by (A3) for F and G we have $Y \in Q$. It follows that $a \lhd_{\mathcal{Q}} P^{-1}[Q]$. Thus condition (A3) for P is verified.

The equation $P \circ F = P \circ G$ is clear by the definition of Q.

To verify the universal property of P, suppose that $H : \mathcal{Y} \longrightarrow \mathcal{Z}$ is a continuous morphism such that $H \circ F = H \circ G$. Thus we have for all $a \in X$ and $b \in Z$

$$a \lhd_{\mathcal{X}} F^{-1}[H^{-1}b] \iff a \lhd_{\mathcal{X}} G^{-1}[H^{-1}b]. \tag{5}$$

Now define $K : \mathcal{Q} \longrightarrow \mathcal{Z}$ by

$$U\, K\, c \iff_{\mathrm{def}} (\forall a \in U)\, a\, H\, c. \tag{6}$$

We first prove that K is a morphism.

(A1): Let $U \in Q$ and suppose $U\, K\, c$ and $c \lhd_{\mathcal{Z}} W$. The property to be shown is $U \lhd_{\mathcal{Q}} K^{-1}W$, i.e. $U \lhd_{\mathcal{Y}} \cup K^{-1}W$. Let $a \in U$ be arbitrary. Then, since H is a morphism, $a \lhd_{\mathcal{Y}} H^{-1}W$. Consider any $c \in H^{-1}W$. Thus there is $b \in W$ with $c \in H^{-1}b$. Applying (5) and the adequacy condition (H3), we obtain $V \in Q$ with $c \in V \subseteq H^{-1}b$. Thus by definition (6) we have $V\,K\,b$, so $V \in K^{-1}W$ and $c \lhd_{\mathcal{Y}} \cup K^{-1}W$. Since c was arbitrary, $H^{-1}W \lhd_{\mathcal{Y}} \cup K^{-1}W$, and thereby $a \lhd_{\mathcal{Y}} \cup K^{-1}W$.

(A2): immediate.

(A3): this follows since $Y \in Q$ by (H1).

(A4'): Suppose $U\,K\,c$ and $U\,K\,d$. Thus $U \lhd_{\mathcal{Y}} H^{-1}c$ and $U \lhd_{\mathcal{Y}} H^{-1}d$. From this follows, using (A4') for H, that $U \lhd_{\mathcal{Y}} H^{-1}[c \wedge d]$. Let $y \in H^{-1}[c \wedge d]$, i.e. assume that there is $e \leq c$ and $e \leq d$ with $y\,H\,e$. By (H3) and (5) (with $b = e$) we find $W \in Q$ satisfying $W \subseteq H^{-1}e$ and $y \in W$. Thus $W\,K\,e$ and so $W \in K^{-1}[c \wedge d]$. Since $y \in W$, we get $y \lhd_{\mathcal{Q}} K^{-1}[c \wedge d]$. Thus $H^{-1}[c \wedge d] \lhd_{\mathcal{Q}} K^{-1}[c \wedge d]$, since y was arbitrary. Now $U \lhd_{\mathcal{Q}} K^{-1}[c \wedge d]$ follows.

We finally need to prove that K is the unique morphism such that $K \circ P = H$, that is

$$a \lhd_{\mathcal{Y}} P^{-1}[K^{-1}c] \Longleftrightarrow a \lhd_{\mathcal{Y}} H^{-1}c. \tag{7}$$

The direction $\Rightarrow$ is clear by the definition of P and K. To prove $\Leftarrow$, assume $a \lhd_{\mathcal{Y}} H^{-1}c$. Thus $a \in H^{-1}c$. By (H3) and (5) there is $U \in Q$ with $a \in U \subseteq H^{-1}c$. Hence $U \lhd_{\mathcal{Y}} H^{-1}c$, i.e. $U\,K\,c$, so in particular we have $a \lhd_{\mathcal{Y}} \cup K^{-1}c$. Thereby $a \lhd_{\mathcal{Y}} P^{-1}K^{-1}c$.

The uniqueness of K is proved as follows. Suppose $K_2 : \mathcal{Q} \longrightarrow \mathcal{Z}$ is another morphism satisfying (7). For $U \in Q$ and $c \in Z$ we have

$$\begin{aligned} U\,K_2\,c &\Leftrightarrow U \lhd_{\mathcal{Q}} K_2^{-1}c \\ &\Leftrightarrow (\forall a \in U) a \lhd_{\mathcal{Y}} \cup K_2^{-1}c \\ &\Leftrightarrow (\forall a \in U) a \lhd_{\mathcal{Y}} P^{-1}K_2^{-1}c \\ &\Leftrightarrow (\forall a \in U) a \lhd_{\mathcal{Y}} H^{-1}c \\ &\Leftrightarrow U\,K\,c. \end{aligned}$$

Thus $K_2 = K$. $\square$

4 Existence of adequate, restricted power sets

Let $F, G : \mathcal{X} \longrightarrow \mathcal{Y}$ be continuous morphisms between set-presentable formal topologies. Let $C(a,i)$ ($i \in I(a)$, $a \in X$) be a set-presentation of $\mathcal{X}$. Consider its equivalent form $C(w)$ ($w \in I$) with $c : I \longrightarrow X$ as in (3). Now let $\mathcal{U}$ be a type-theoretic universe $\mathsf{T}(t)$ ($t : \mathsf{U}$), closed under Σ-constructions, and which is such that X, Y, I are $\mathcal{U}$-sets, the relation $\leq_{\mathcal{X}}$ is a $\mathcal{U}$-subset of $X \times X$, and the relations F and G are $\mathcal{U}$-subsets of $X \times Y$. Moreover $C(w)$ is an $\mathcal{U}$-subset of X

for each $w \in I$. Further, we assume the universe to contain the positive natural numbers $\mathbb{Z}_+$.

Then form the restricted power set $\mathcal{R}_{\mathcal{U}}(Y)$ with respect to the family $\mathcal{U}$. The following is a set-theoretic collection principle.

Lemma 2. *Suppose $\mathcal{U}$ is as above. Let H be a $\mathcal{U}$-subset of $X \times Y$, and let A be a $\mathcal{U}$-subset of X. Then for any subset B of Y with*

$$A \subseteq H^{-1}B,$$

there is a $\mathcal{U}$-subset Z of Y with $A \subseteq H^{-1}Z$ and $Z \subseteq B$.

Proof. Suppose $A \subseteq H^{-1}B$. This is equivalent to

$$(\forall x \in A)(\exists y \in B)\, x\, H\, y.$$

Thus by principle (1) there is some $f : \underline{A} \longrightarrow B$ with $(\forall x \in \underline{A})\; p_A(x)\, H\, f(x)$. The image of this function $Z = \{y \in B : (\exists x \in \underline{A})\; f(x) = y\}$ is a $\mathcal{U}$-subset of Y since $\mathcal{U}$ is closed under Σ. It is clear that $A \subseteq H^{-1}Z$, by the surjectivity of p_A, and $Z \subseteq B$. □

Lemma 3. *Suppose $\mathcal{R}(Y) = \mathcal{R}_{\mathcal{U}}(Y)$ is as above. Take a subset U of Y such that*

$$(\forall a \in X)(a \lhd F^{-1}U \Rightarrow a \lhd G^{-1}U). \tag{8}$$

Then for any $V \in \mathcal{R}(Y)$ with $V \subseteq U$, there is $W \in \mathcal{R}(Y)$ with $W \subseteq U$ and

$$(\forall a \in X)(a \lhd F^{-1}V \Rightarrow a \lhd G^{-1}W). \tag{9}$$

Moreover, the above holds with F and G interchanged in both (8) and (9).

Proof. First note that by reflexivity (R) and transitivity of covers (T) we may replace the left hand $a \lhd \cdots$ by $a \in \cdots$ in both (8) and (9). This modified statement is what we shall prove.

Let $V \in \mathcal{R}(Y)$ with $V \subseteq U$. Then by (8) and transitivity of covers we get

$$(\forall a \in F^{-1}V)\, a \lhd G^{-1}U.$$

Using the set-presentation of the cover this may be rephrased as

$$(\forall a \in F^{-1}V)(\exists w \in I)\; (c(w) = a \;\&\; C(w) \subseteq G^{-1}U).$$

By Lemma 2, and since $C(w) \in \mathcal{R}(X)$, the statement $C(w) \subseteq G^{-1}U$ is equivalent to $(\exists Z \in \mathcal{R}(Y))\,(C(w) \subseteq G^{-1}(Z) \;\&\; Z \subseteq U)$. Thus we have

$$(\forall a \in F^{-1}V)(\exists Z \in \mathcal{R}(Y))(a \lhd G^{-1}Z \;\&\; Z \subseteq U). \tag{10}$$

Let $S = F^{-1}V$. By the principle (1), we get $H : \underline{S} \longrightarrow \mathcal{R}(Y)$ so that

$$(\forall s \in \underline{S})(p_S(s) \lhd G^{-1}H(s) \;\&\; H(s) \subseteq U). \tag{11}$$

Let

$$W = \bigcup_{s\in\underline{S}} H(s).$$

Now $W \in \mathcal{R}(X)$ and by transitivity

$$(\forall s \in \underline{S})(p_S(s) \lhd G^{-1}W \;\&\; W \subseteq U).$$

Since p_S is surjective this means that

$$(\forall a \in F^{-1}V)\; a \lhd G^{-1}W,$$

proving (9) by transitivity of covers. □

Lemma 4. *The restricted power set $\mathcal{R}(Y) = \mathcal{R}_{\mathcal{U}}(Y)$ as constructed above is adequate for F and G.*

Proof. Condition (H1) is trivial since the subset $Y = \{x \in Y : x = x\}$ belongs to $\mathcal{R}(Y)$. Condition (H2) follows since the relation $\leq_{\mathcal{Y}}$ is a $\mathcal{U}$-subset of $Y \times Y$ and $\mathcal{U}$ is closed under Σ.

To prove (H3) suppose that $U \subseteq Y$ satisfies the equivalence

$$(\forall a \in X)(a \lhd F^{-1}U \iff a \lhd G^{-1}U).$$

Let $b \in U$. We construct V_n $(n \in \mathbb{Z}_+)$ with $V_n \in \mathcal{R}(Y)$ and $V_n \subseteq U$ as follows.

(base) $V_1 = \{b\}$

(suc0) $(\forall a \in X)[a \lhd F^{-1}V_n \Rightarrow a \lhd G^{-1}V_{2n}]$

(suc1) $(\forall a \in X)[a \lhd G^{-1}V_n \Rightarrow a \lhd F^{-1}V_{2n+1}]$

By Lemma 3 we can find such V_{2n} and V_{2n+1} as in (suc0) and (suc1). Now put

$$V_\infty = \bigcup_{n\in\mathbb{Z}_+} V_n.$$

Then $V_\infty \in \mathcal{R}(Y)$ since $\mathcal{U}$ is contains $\mathbb{Z}_+$. Clearly, $V_\infty \subseteq U$.

Suppose now that $a \in F^{-1}[V_\infty]$. Thus by (4) we have $a \in F^{-1}V_n$ for some $n \in \mathbb{Z}_+$. Hence by (suc0) $a \lhd G^{-1}V_{2n}$, and hence $a \lhd G^{-1}V_\infty$. We have shown

$$(\forall a \in X)(a \in F^{-1}V_\infty \Rightarrow a \lhd G^{-1}V_\infty),$$

and by transitivity of covers, in fact,

$$(\forall a \in X)(a \lhd F^{-1}V_\infty \Rightarrow a \lhd G^{-1}V_\infty).$$

The proof of the converse implication is the same, exchanging F and G, and using (suc1) instead of (suc0). This establishes the lemma with V_∞ as the set satisfying the required equivalence. □

A *universe forming operator* is a type construction which over any family of types builds a type universe closed under Π, Σ, W, I and $+$-constructions [14].

Theorem 5. *Under the assumption of universe forming operators, coequalisers exists in the category* $\mathbf{FTop}_s$.

Proof. The result follows by Lemma 1 and Lemma 4, noting that with the help of the universe operator we can build the required universe $\mathcal{U}$ for given maps F and G. □

We note that the closure of the universe under Π, W, $+$ and I-types is not actually used.

5 Examples

By reformulating standard topological constructions in terms of limits and colimits, we may now perform them in the category of formal topologies.

1. The two-dimensional $\mathbb{T}^2$ torus may be constructed as the coequaliser of the followings maps $\mathbb{R}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{R}^2$

$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x},$$
$$(\mathbf{x}, \mathbf{n}) \mapsto \mathbf{x} + \mathbf{n}.$$

2. The n-dimensional real projective space $\mathbb{R}P^n$ may be constructed as a coequaliser of two maps $\mathbb{R}^{n+1} \times \mathbb{R}_{\neq 0} \longrightarrow \mathbb{R}^{n+1}$

$$(\mathbf{x}, \lambda) \mapsto \mathbf{x},$$
$$(\mathbf{x}, \lambda) \mapsto \lambda\mathbf{x}.$$

3. The Möbius band may be constructed as the coequaliser of the maps $\ell, r : [0,1] \longrightarrow [0,1]^2$ where $\ell(x) = (0, x)$ and $r(x) = (1, 1-x)$.

4. Pushouts may be constructed using sums and coequalisers. Various glued spaces may be constructed using pushouts. For $A \hookrightarrow X$ and $f : A \longrightarrow Y$, the pushout gives the attaching map construction:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, dotted] \\ X \arrow[r, dotted] & Y \cup_f X \end{tikzcd}$$

5. The special case of 3, where $Y = 1$ is the one-point space, gives the space X/A where A in X is collapsed to a point.

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