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On the Meaning of Positivity Relations for Regular Formal Spaces[1]

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Abstract: A careful analysis of the original definition of formal topology led to the introduction of a new primitive, namely a positivity relation between elements and subsets. This is, in other terms, a direct intuitionistic treatment of the notion of closed subset in formal topology. However, since formal open subsets do not determine formal closed subsets uniquely, the new concept of positivity relation is not yet completely clear. Here we begin to illustrate the general idea that positivity relations can be regarded as a further, powerful tool to describe properties of the associated formal space. Our main result is that, keeping the formal cover fixed, by suitably redefining the positivity relation of a regular formal topology one can obtain any given set-indexed family of points as the corresponding formal space.

Key Words: Formal topology, positivity relation, formal spaces, formal reals, regular formal topologies
Category: F.1

The original definition of formal topology [4] included among primitive data a unary positivity predicate Pos. Its interpretation was that a formal basic neighbourhood a is positive, that is $Pos(a)$ is true, if it is inhabited by some formal point. However, the presence of Pos was not sufficiently well motivated and some scholars openly expressed their doubts about its introduction. Over ten years later, rather than abandoning the predicate of positivity, a more careful analysis of the primitive concepts led to its strenghtening, namely to the introduction of a new positivity relation between elements and subsets [5]. The main outcome of the new definition is a direct intuitionistic treatment of the notion of closed set. In particular, it brings to the new notion of basic topology, which contains a formal cover and a positivity relation compatible with it.

An obstacle to a satisfactory understanding of this new concept arises from the fact that in general, for any fixed covering relation, there exist multitudes of

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positivity relations compatible with it. In other words, the formal open subsets do not determine the formal closed subsets, at least if one tries to achieve this by means of compatibility alone. So we are naturally led to the question of what kind of information is carried by positivity relations.

Here we begin to illustrate the general idea that positivity relations can be regarded as a natural tool to describe properties of the associated formal space, such as capturing subspaces for instance. We are going to show this in a somewhat general setting, also paving the way for future developments. In fact, we show that keeping the formal cover fixed, by suitably redefining the positivity relation of a regular formal topology one can obtain any given set-indexed family of points as the corresponding formal space. We look at the simple results given here as the starting point for a more accurate investigation into the meaning of positivity relations and more generally into the possibilities allowed by the introduction of existential statements in formal topology (cf. [5]). We refer the reader to [5] for all the notions involved here.

1 Minimal Closed Subsets

Let $\mathcal{S} = (S, \lhd, \ltimes)$ be a basic topology. We shall say that $F_i \subseteq S$ $(i \in I)$ is a family of $\ltimes$-independent subsets if

1. for every $i \in I$, F_i is $\ltimes$-closed and inhabited;

2. for every $i, j \in I$, $F_i \subseteq F_j$ implies $F_i = F_j$.

We also say that a subset $F \subseteq S$ is $\ltimes$-*minimal* if it is $\ltimes$-closed, inhabited and if $G \subseteq F$ implies $G = F$ for all $\ltimes$-closed inhabited subsets G of S.

Proposition 1. *Let $\mathcal{S} = (S, \lhd, \ltimes)$ be a basic topology. Let $F_i \subseteq S$ $(i \in I)$ be a family of $\ltimes$-independent subsets. Define $\ltimes_I$ by putting*

$$a \ltimes_I F \;\equiv\; (\exists i \in I)(a \,\epsilon\, F_i \;\wedge\; F_i \subseteq F).$$

Then the following hold:

1. $\ltimes_I$ is a positivity relation compatible with $\lhd$, that is $(S, \lhd, \ltimes_I)$ is a basic topology;

2. $\ltimes_I$ is weaker than $\ltimes$, that is $a \ltimes_I F$ implies $a \ltimes F$;

3. the $\ltimes_I$-minimal subsets are precisely the F_i's, that is F is a $\ltimes_I$-minimal subset if and only if $F = F_i$ for some $i \in I$.

Proof. 1. The relation is coreflexive since if $a \ltimes_I F$ then $a \in F_i \subseteq F$ for some i. It is cotransitive since if $\{b : b \ltimes_I F\} \subseteq G$, viz.

$$\forall b \in S \, \forall j \in I \; (\, b \,\epsilon\, F_j \subseteq F \;\Rightarrow\; b \,\epsilon\, G \,),$$

then it follows immediately that $F_j \subseteq F$ gives $F_j \subseteq G$ for all j, and therefore $\{b : b \ltimes_I F\} \subseteq \{b : b \ltimes_I G\}$. It is easy to see that it is also compatible with $\lhd$, since the F_i's are assumed to be closed subsets in $\mathcal{S}$.

2. Assume that $a \ltimes_I F$. By definition, there exists i such that $a \,\epsilon\, F_i \subseteq F$. Since every F_i is a $\ltimes$-closed subset, we get $a \ltimes F_i$ which together with $F_i \subseteq F$ yields $a \ltimes F$.

3. First we show that each F_i is minimal closed for $\ltimes_I$. Clearly each F_i is closed for $\ltimes_I$, since $a \,\epsilon\, F_i$ implies $a \ltimes_I F_i$ by definition. That every F_i is inhabited is among our assumptions on the family of subsets F_i $(i \in I)$. Suppose that we have $G \subseteq F_i$ where G is inhabited and closed for $\ltimes_I$, and let $a \,\epsilon\, G$ for some $a \in S$. Then also $a \ltimes_I G$, and by definition of $\ltimes_I$ this implies that we can find j such that $F_j \subseteq G$, hence $F_j \subseteq F_i$, and by our assumptions on the family, $F_j = F_i$, which yields $F_i = G$.

Next suppose that $F \subseteq S$ is minimal closed for $\ltimes_I$. Then $a \,\epsilon\, F$ for some $a \in S$ because F is inhabited, and hence $a \ltimes_I F$ because F is $\ltimes_I$-closed. By definition, this says that $a \,\epsilon\, F_i \subseteq F$ for some $i \in I$; as observed previously, F_i is closed for $\ltimes_I$, thus by $\ltimes_I$-minimality of F we get $F = F_i$.

An alternative indirect proof of the fact that $(S, \lhd, \ltimes_I)$ is a basic topology is interesting enough to be given here. Given the family $F_i \subseteq S$ $(i \in I)$, since I is a set the structure $(I, \Vdash, S)$, where $i \Vdash a \equiv a \,\epsilon\, F_i$, is a basic pair. One can see that the positivity relation induced on S (as described in [5]) is exactly $\ltimes_I$ as defined above. The corresponding cover is given by $a \lhd_I U \equiv \forall i \in I(a \,\epsilon\, F_i \Rightarrow F_i \between U)$, and so $a \lhd U$ implies $a \lhd_I U$ because each F_i is closed. Then, since $\ltimes_I$ is compatible with $\lhd_I$, it is a fortiori compatible also with $\lhd$.

2 Regular Formal Topologies

Recall that $\mathcal{S} = (S, \lhd, \ltimes)$ is regular if $\forall a \in S \; a \lhd wc(a)$, where

$$wc(a) = \{c : S \lhd \{a\} \cup c^*\},$$

c^* denoting $\{b : b \downarrow c \lhd \varnothing\}$. *Unless otherwise stated, throughout the present section $\mathcal{S}$ is a fixed regular formal topology.* We shall be using the following generalization of a well-known property of regular formal topologies (compare with [5]) which says that if $\alpha \subseteq \beta$ are any formal points then $\alpha = \beta$; it holds in particular for the formal continuum:

Lemma 2. *Let α be a formal point of $\mathcal{S}$. Then α is minimal closed for $\ltimes$.*

Proof. Clearly, α is inhabited and $\ltimes$-closed by definition, since it is a formal point. Suppose $G \subseteq \alpha$ for some inhabited $\ltimes$-closed subset G of $\mathcal{S}$, and let $a \,\epsilon\, \alpha$. Then since α is a point there is some $c \,\epsilon\, \alpha \cap wc(a)$. Since $S \lhd \{a\} \cup c^*$, choosing $d \,\epsilon\, G$ (which is possible since G is inhabited) we get $d \lhd \{a\} \cup c^*$, which yields $G \between \{a\} \cup c^*$ by closedness of G and by compatibility. Now, assume $G \between c^*$: then we find some $b \,\epsilon\, \alpha \cap c^*$ (because $G \subseteq \alpha$), but this gives $b \downarrow c \lhd \varnothing$ by definition of c^*, which cannot hold since $b, c \,\epsilon\, \alpha$ and α is a formal point. Therefore we must have $G \between \{a\}$, that is $a \,\epsilon\, G$. This shows that $\alpha \subseteq G$, and concludes the proof.

We make an observation before we come to our main result. Recall that $\mathcal{P}t(\mathcal{S})$ is the collection of formal points, that is the formal space associated with $\mathcal{S}$.

Lemma 3. *Let $\mathcal{S} = (S, \lhd, \ltimes)$ be an arbitrary formal topology. Let $\ltimes'$ be a positivity predicate weaker than $\ltimes$ (that is $a \ltimes' F$ implies $a \ltimes F$) and compatible with $\lhd$; let $\mathcal{S}'$ be the formal topology $(S, \lhd, \ltimes')$. Then $\mathcal{P}t(\mathcal{S}') \subseteq \mathcal{P}t(\mathcal{S})$. Moreover, if the $\ltimes'$-closed subset $F \subseteq S$ is $\ltimes$-minimal, then it is also $\ltimes'$-minimal.*

Proposition 4. *Suppose F_i, $i \in I$ is a $\ltimes$-independent family of closed subsets of $\mathcal{S}$. Then if α is a point of the formal topology*

$$\mathcal{S}_I = (S, \lhd, \ltimes_I),$$

there exists some $i \in I$ such that $\alpha = F_i$.

Proof. Let α be a point for $\ltimes_I$; then α is also a point for $\ltimes$ (Lemma 3), hence a minimal closed subset for $\ltimes$ (by Lemma 2), hence *a fortiori* a minimal closed subset for $\ltimes_I$ (again by Lemma 3): thus $\alpha = F_i$ for some i, by Proposition 1.

Corollary 5. *Suppose α_i, $i \in I$ is a set-indexed family of formal points of $\mathcal{S}$. Then the α_i's are precisely the formal points of a formal topology $\mathcal{S}_I = (S, \lhd, \ltimes_I)$, with $\ltimes_I$ weaker than $\ltimes$.*

Proof. We apply Proposition 1 to the family α_i, $i \in I$, which is immediately seen to be a $\ltimes$-independent family, by observing that formal points are inhabited $\ltimes$-closed and by regularity of $\mathcal{S}$ (Lemma 2). So consider the positivity relation $\ltimes_I$ on $\mathcal{S}$ associated to this family as in Proposition 1. It is clear that all the α_i's are formal points for $\ltimes_I$, since they are both formal points of $\mathcal{S}$ and closed subsets for $\ltimes_I$ (as before, by Proposition 1). The converse is an immediate consequence of Proposition 4.

In this corollary, the assumption that $\mathcal{S}$ is regular is used only to prove that $\alpha_i \subseteq \alpha_j$ implies $\alpha_i = \alpha_j$, for $i, j \in I$. So the corollary applies to formal topologies $\mathcal{S}$ in which the ordering of $\mathcal{P}t(\mathcal{S})$ is discrete, or also to every formal topology $\mathcal{S}$, by restricting to families α_i, $i \in I$, in which each α_i is minimal.

Before we come to the next corollary, a few observations are in order. Recall that if $f : \mathcal{S} \to \mathcal{T}$ is a morphism of formal topologies and $F \subseteq S$ is a closed subset then the image $f(F)$ is closed. Moreover, if $f = g : \mathcal{S} \to \mathcal{T}$ are *equal* morphisms (equality of morphisms is defined as $f^{-}b \lhd g^{-}b$ *and* $g^{-}b \lhd f^{-}b$ *for every* $b \in T$) then $f(F) = g(F)$ for every closed subset $F \subseteq S$ (see [6] for a detailed proof). Therefore if f is an isomorphism, the rule

$$F \mapsto f(F)$$

gives a bijection between the collection of closed subsets of $\mathcal{S}$ and those of $\mathcal{T}$, in particular a bijection between the minimal closed subsets. It follows that the cardinality of the collection of minimal closed subsets of a formal topology is invariant under isomorphism.

One more observation concerning regular topologies. For any points α, β let $\alpha \# \beta$ abbreviate that *there exist* $a \epsilon \alpha$, $b \epsilon \beta$ *such that* $a \downarrow b \lhd \emptyset$.

Proposition 6. *The relation # is an intuitionistic apartness relation, i.e. it enjoys the following properties:*

1. $\neg \alpha \# \beta \Leftrightarrow \alpha = \beta$;
2. $\alpha \# \beta \Rightarrow \beta \# \alpha$;
3. $\alpha \# \beta \Rightarrow \alpha \# \gamma \vee \beta \# \gamma$.

Proof. If α, β are any points of a regular formal topology $\mathcal{S}$ and $a \epsilon \alpha$, then there is $c \epsilon \alpha \cap wc(a)$ such that $\beta \between \{a\} \cup c^{*}$ (see the proof of lemma 2).

To prove the first property (the non-trivial implication $\Rightarrow$), assume $a \epsilon \alpha$: since $c^{*} \between \beta$ cannot hold under the assumption $\neg \alpha \# \beta$, we must have $\beta \between \{a\}$, that is $a \epsilon \beta$; this shows $\alpha \subseteq \beta$, hence $\alpha = \beta$. The second property is trivial.

To prove the third property, let $a \downarrow b \lhd \emptyset$ for some $a \epsilon \alpha$ and $b \epsilon \beta$: then there are $c \epsilon \alpha \cap wc(a)$ and $d \epsilon \beta \cap wc(b)$ such that $\gamma \between \{a\} \cup c^{*}$ and $\gamma \between \{b\} \cup d^{*}$; but since $c^{*} \between \gamma \Rightarrow \alpha \# \gamma$, $d^{*} \between \gamma \Rightarrow \beta \# \gamma$ and $a, b \in \gamma$ cannot hold, in any case we conclude $\alpha \# \gamma$ or $\beta \# \gamma$.

In particular # is classically equivalent to $\neq$, which amounts to the well-known fact that the space $\mathcal{P}t(\mathcal{S})$ is Hausdorff when $\mathcal{S}$ is regular.

Corollary 7. *By varying only the positivity relation, it is possible to derive several non-isomorphic formal topologies from the given formal topology $\mathcal{S}$ (unless $\mathcal{S}$ is too small), none of them having points.*

Proof. The idea is clear: choose any $\ltimes$-independent family of closed subsets of $\mathcal{S}$ such that no point of $\mathcal{S}$ occurs among the F_i's, and apply proposition 4. In practice, such a family can be obtained from a nice doubly-indexed family of points

$$\alpha_j : \mathcal{P}t(\mathcal{S}) \qquad i \in I, j \in J(i)$$

such that $\forall i \in I\ \exists j, j' \in J(i)\ \alpha_j \# \alpha'_j$, by setting, for every $i \in I$,

$$F_i = \bigcup_{j \in J(i)} \alpha_j.$$

This is a closed inhabited subset, but not convergent, and hence not a point.

For the topology of real numbers, every ($\ltimes$-independent) family of closed subsets comes from a (nice) doubly-indexed family of formal points by the method explained in the proof of the corollary, cf. Lemma 9.

3 Subspaces of the Formal Continuum

In what follows, we let $\mathcal{R}$ denote the formal topology $(\mathbb{Q} \times \mathbb{Q}, \lhd, \ltimes)$ of *real numbers* (also referred to as the *real line*, or *continuum* [5]); here $\ltimes$ is the standard co-inductively generated positivity relation [3].

Substantially, the following fact already appeared in [2]; here we state it in the general language of formal topology.

Lemma 8. *Let (p_n, q_n), for n natural number, be a sequence of pairs of rational numbers such that for all n we have $p_n < p_{n+1} < q_{n+1} < q_n$, and assume that the length $q_n - p_n$ goes to zero as n tends to infinity. Let*

$$\alpha = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} : \ p < p_n < q_n < q \text{ for some } n\}.$$

Then α is a formal point.

Proof. Of course α is inhabited and upward closed. It is also convergent, since if $(p, q), (r, s) \epsilon \alpha$ where $p < p_n < q_n < q$ and $r < p_m < q_m < s$, then clearly

$$\max\{p, r\} < p_{\max\{n,m\}} < q_{\max\{n,m\}} < \min\{q, s\}.$$

Assume that we are given a covering U of $(p, q) \epsilon \alpha$: we want to prove that α *splits* U, i.e. that $\alpha \between U$. It is no loss of generality to assume that U is either $wc(p, q) = \{(p', q') : p < p' < q' < q\}$ or $\{(p, s), (r, q)\}$ for some $p < r < s < q$. In the first case, by definition of α there is n such that $p < p_n < q_n < q$, hence for this choice of n we shall have $(p_n, q_n) \epsilon \alpha \cap U$ and therefore $\alpha \between U$. If on the other hand $U = \{(p, s), (r, q)\}$, by selecting n large enough we can assume that $q_n - p_n < s - r$ and $p < p_n < q_n < q$; the middle point of (p_n, q_n) will be either not greater or not smaller than the middle point of (r, s): if, say, it is not greater, then it must be $p < p_n < q_n < s$ and therefore $(p, s) \epsilon U \cap \alpha$ by definition of α.

It is not necessary to prove that α *enters* $\ltimes$, since this property is redundant when $\ltimes$ is coinductively generated.

The proof of the next lemma shows that the converse is true: given a point of $\mathcal{R}$, it is the point associated to some converging sequence of intervals as above. This says that the collection $\mathcal{P}t(\mathcal{R})$ is set-indexed. The lemma is also meant for purposes not directly related to the present exposition (see [6]).

Lemma 9. *Let F be a formal closed subset of $\mathcal{R}$, and let $(p_1, q_1) \,\epsilon\, F$. Then there exists a formal point $\alpha \subseteq F$ lying in (p_1, q_1).*

Proof. First of all, we note that for any $(p, q) \,\epsilon\, F$ it is possible to find $(p', q') \,\epsilon\, F$ such that $p < p' < q' < q$ and $(q' - p') < \frac{2}{3}(q - p)$; indeed, if $(p, q) \,\epsilon\, F$, then by closedness of F either $(p, q - \frac{1}{3}(q - p)) \,\epsilon\, F$ or $(p + \frac{1}{3}(q - p), q) \,\epsilon\, F$: by closedness of F again, in the first case there is a $(p', q') \,\epsilon\, F$ such that $p < p' < q' < q - \frac{1}{3}(q - p)$, while in the second case there exists some $(p', q') \,\epsilon\, F$ such that $p + \frac{1}{3}(q - p) < p' < q' < q$.

Next, we start from our $(p_1, q_1) \,\epsilon\, F$; by the above observation, there exists $(p_2, q_2) \,\epsilon\, F$ such that $p_1 < p_2 < q_2 < q_1$ and $q_2 - p_2 < \frac{2}{3}(q_1 - p_1)$; by means of the type-theoretic choice principle, one constructs inductively a sequence $(p_n, q_n) \,\epsilon\, F$ such that $p_n < p_{n+1} < q_{n+1} < q_n$ and $q_{n+1} - p_{n+1} < (\frac{2}{3})^n (q_1 - p_1)$ for every n.

We now apply Lemma 1 to this sequence in order to produce the required point α (clearly $\alpha \subseteq F$ since $(p_n, q_n) \,\epsilon\, F$ for all n and F is upward closed, being closed).

By the above considerations, every subcollection of $\mathcal{P}t(\mathcal{R})$, ie every predicate $\mathcal{P}(\alpha)$ prop $\alpha : \mathcal{P}t(\mathcal{R})$, does in fact correspond to a set-indexed family of points α_i, $i \in I$. Therefore for formal reals we can assert the following stronger version of Corollary 5:

Proposition 10. *Every subcollection of $\mathcal{P}t(\mathcal{R})$ coincides with $\mathcal{P}t(\mathcal{R}')$ for some formal topology*

$$\mathcal{R}' = (\mathbb{Q} \times \mathbb{Q}, \lhd, \ltimes').$$

The fact that the collection of points of the real line is set-indexed in Type Theory is already well-known [1], but we think it is nonetheless worthwhile to state the above elementary observations carefully.

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