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Constructive Aspects of Markov Chains[1]

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Abstract: This is a preliminary pass at examining some of the constructive issues in the theory of finite Markov chains. I trust that it is not all bad that there seem to be more questions raised than answered.

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Category: F.1

1 Introduction

Throughout, A will denote a Markov matrix with entries a_{ij} where i and j are in some set I called the set of **states**. So $a_{ij} \geq 0$ for all $i, j \in I$ and $\sum_{j \in I} a_{ij} = 1$ for each $j \in I$. Usually the number of states will be finite, but some of the results hold also when the number of states is infinite. The number a_{ij} is the probability that if our system is in state i at time n, then it will be in state j at time $n+1$. The entries in the matrix A^n are denoted by $a_{ij}^{(n)}$.

A key relation among states of a Markov chain is **reachability**. If $a_{ij}^{(n)} > 0$ for some n we say that you can **reach** j from i or you can **get to** j from i or you can **go from** i to j. If d is the number of states in the chain, then you can reach j from i if and only if the ij-th entry in the matrix

$$I + A + A^2 + \cdots + A^{d-1}$$

is different from zero. For some purposes it seems necessary (but not desirable) to require **decidable reachability**, that is, to require that each entry in the displayed matrix be either zero or nonzero. This certainly holds if each entry in the original matrix A is either zero or nonzero.

2 Transient states

A state i is said to be **transient** if there is a state j such that $a_{ij}^{(n)} > 0$ for some n and $a_{ji}^{(n)} = 0$ for all n. That is, you can reach j from i but you can't reach i from j. We will show that

Theorem 1. *If i is a state of a Markov chain with transition matrix A, then each of the following conditions implies the next:*

[1] C. S. Calude, H. Ishihara (eds.). *Constructivity, Computability, and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges.*

1. *The state i is transient,*
2. $\lim_{n\to\infty} a_{ki}^{(n)} = 0$ *for each state k,*
3. $\lim_{n\to\infty} a_{ii}^{(n)} = 0$.

Moreover, if the chain has decidable reachability, then the three conditions are equivalent to

4. $\sum_{n=1}^{\infty} a_{ii}^{(n)}$ *converges.*

The remainder of this section is devoted to proving Theorem 1.

Clearly each of (2) and (4) implies (3), even without decidable reachability. To show that (1) implies (2) we use the following lemma.

Lemma 2. *Let i, j and k be states, $m \geq 1$ and $n \geq 0$ integers. If i cannot be reached from j, then*

$$a_{ki}^{(n)} a_{ij}^{(m)} \left(1 + \left\lfloor \frac{n}{m} \right\rfloor\right) \leq 1.$$

Proof. Set $b_i^{(n)} = \sup_k a_{ki}^{(n)}$. We will show that

$$b_i^{(n)} a_{ij}^{(m)} \left(1 + \left\lfloor \frac{n}{m} \right\rfloor\right) \leq 1.$$

Note that

$$a_{ki}^{(n+1)} = \sum_s a_{ks} a_{si}^{(n)} \leq \sum_s a_{ks} b_i^{(n)} = b_i^{(n)},$$

so $b_i^{(n+1)} \leq b_i^{(n)}$. If the initial state is k, then the number $a_{ki}^{(n-tm)} a_{ij}^{(m)}$ is the probability that the chain is in state i at time $n - tm$ and in state j at time $n - tm + m$. These are disjoint events for $t = 0, \ldots, \lfloor n/m \rfloor$ because you can't get from j to i. So

$$\sum_{t=0}^{\lfloor n/m \rfloor} a_{ki}^{(n-tm)} a_{ij}^{(m)} \leq 1$$

from which it follows that

$$\sum_{t=0}^{\lfloor n/m \rfloor} b_i^{(n-tm)} a_{ij}^{(m)} \leq 1$$

and thus

$$\left(1 + \left\lfloor \frac{n}{m} \right\rfloor\right) b_i^{(n)} a_{ij}^{(m)} \leq 1$$

because $b_i^{(n)}$ decreases as n increases.

That (1) implies (2) follows immediately from Lemma 2 upon choosing j and m so that $a_{ij}^{(m)} > 0$. This result doesn't require the number of states to be finite: we can treat the supremum $\sup_k a_{ki}^{(n)}$ in the proof of Lemma 2 as a generalized real number—a nonlocated supremum [2].

We now show that (3) implies (1) if the chain has a finite number d of states and decidable reachability.

Let S be the set of states reachable from i. Because reachability is decidable, the set S is finite and either i is transient or i is not transient. So it suffices to prove the contrapositive: not (1) implies not (3). If i is not transient, then i is reachable from any state in S. For each $j \in S$ there is $t < d$ such that $a_{ji}^{(t)} > 0$. Then

$$\sum_{t=0}^{d-1} a_{ii}^{(n+t)} = \sum_{t=0}^{d-1} \sum_{j \in S} a_{ij}^{(n)} a_{ji}^{(t)} = \sum_{j \in S} a_{ij}^{(n)} \sum_{t=0}^{d-1} a_{ji}^{(t)} \geq \inf_{j \in S} \sum_{t=0}^{d-1} a_{ji}^{(t)} > 0$$

for all n, so $a_{ii}^{(n)}$ cannot converge to zero.

It remains to show that (1) implies (4) if reachability is decidable. We state that as a separate theorem.

Theorem 3. *Let i be a transient state of a finite Markov chain with decidable reachability and transition matrix A. Then $\sum_{n=1}^{\infty} a_{ii}^{(n)}$ converges, that is, (1) implies (4).*

Proof. Suppose you can reach j from i but not vice versa. States from which i cannot be reached may be incorporated into j without affecting $a_{ii}^{(n)}$ or the fact that i is transient. Decidable reachability allows us to identify those states, so we may assume that i can be reached from every state except j, whence $a_{jj} = 1$. Then $\delta = \inf_k a_{kj}^{(d)} > 0$ for some d so

$$a_{ij}^{(nd)} \geq 1 - (1 - \delta)^n$$

for all n, whence

$$a_{ij}^{(n)} \geq 1 - (1 - \delta)^{\lfloor n/d \rfloor}$$

for all n (because $a_{ij}^{(n)}$ increases with n). Thus

$$a_{ii}^{(n)} \leq (1 - \delta)^{\lfloor n/d \rfloor}$$

for all n. Then

$$\sum_{n=md}^{\infty} a_{ii}^{(n)} \leq d \sum_{t=0}^{\infty} (1 - \delta)^{m+t} = d \frac{(1 - \delta)^m}{\delta}$$

goes to zero as m goes to infinity, so $\sum_{n=1}^{\infty} a_{ii}^{(n)}$ converges.

3 Limitations on extending Theorem 1

In Theorem 1, you can't prove that (2) implies (1) without *some* additional hypothesis on A because that would show that the real numbers were discrete. Indeed, given $0 \leq r < 1/2$, consider the Markov matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ r & 1-2r & r \\ 0 & 0 & 1 \end{pmatrix}.$$

To verify (2) for state 1 of this matrix, we need to show that, for each $\varepsilon > 0$, there exists n such that $a_{i1}^{(n)} < \varepsilon$ for all $n \geq N$. As $a_{i1}^{(n)} \leq r$ for all $n \geq 1$, if $r < \varepsilon$, we can choose $N = 1$. If, on the other hand, $r > 0$, then state 1 is transient so Theorem 1 provides the N. Thus (2) holds for state 1. However, if state 1 is transient, then you can get from 1 to some state j but not back. If $j = 2$, then $r = 0$ while if $j = 3$, then $r > 0$.

Note that this example does not satisfy (4). Clearly $a_{11}^{(n)} \geq r(1-2r)^{n-2}$, so $\sum a_{11}^{(n)} \geq r/(2r) = 1/2$ if $r > 0$, and $\sum a_{11}^{(n)} = 0$ if $r = 0$.

Theorem 3, that (1) implies (4), is classically true for infinite chains also. Suppose you can get from state i to state j but not from j to i, and let θ be one minus the positive probability of some path from i to j that doesn't return to i on the way. Then $0 \leq \theta < 1$ and the probability of returning m times to state i is at most θ^m. So the expected number of returns, $\sum_{n=1}^{\infty} a_{ii}^{(n)}$, is bounded by

$$\sum m\theta^m = \frac{\theta}{(1-\theta)^2}.$$

However $\sum_{n=1}^{\infty} a_{ii}^{(n)}$ need not converge constructively; indeed we need not be able to show that $\lim_{n\to\infty} a_{11}^{(n)} = 0$. Let the state space be the natural numbers $\{1, 2, 3, \ldots\}$ and define the Markov matrix by

$$\begin{aligned} a_{21} = a_{23} &= 1/2, \\ a_{11} &= 1, \\ a_{i,i+1} &= 1 \text{ if } s_i = 0, \text{ otherwise } a_{i1} = 1, \end{aligned}$$

where s_i is a sequence of zeros and ones. State 2 is certainly transient, but if $\lim_{n\to\infty} a_{22}^{(n)} = 0$, then there exists i_0 such that if $s_i = 0$ for all $i \leq i_0$, then $s_i = 0$ for all i.

You need some sort of condition on A, in the finite case, to prove Theorem 3, that (1) implies (4). With no condition, the implication from (1) to (4) allows us to show that the real numbers are discrete. Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & r & 1-r \end{pmatrix}.$$

Clearly state 1 is absorbing and state 2 is transient. We will show that if $\sum a_{22}^{(n)}$ converges, then $r = 0$ or $r > 0$. The recursion for $a_{22}^{(n)}$ and $a_{23}^{(n)}$ is given by $a_{22}^{(n)} = a_{23}^{(n-1)} r$ and

$$a_{23}^{(n)} = (1-r)a_{23}^{(n-1)} + a_{22}^{(n-1)}/2 = (1-r)a_{23}^{(n-1)} + a_{23}^{(n-2)} r/2$$

with initial conditions $a_{23}^{(0)} = 0$ and $a_{23}^{(1)} = 1/2$. The characteristic equation of the recursion for $a_{23}^{(n)}$ is $X^2 = (1-r)X + r/2$ with roots

$$\lambda_+ = \frac{1-r}{2} + \frac{1}{2}\sqrt{1+r^2} \text{ and}$$
$$\lambda_- = \frac{1-r}{2} - \frac{1}{2}\sqrt{1+r^2}$$

from which we get the formula

$$a_{23}^{(n)} = \frac{\lambda_+^n - \lambda_-^n}{2\sqrt{1+r^2}}$$

which works for $n = 0$ and $n = 1$. Note that if $r > 0$, then $0 \le \lambda_+ < 1$ and $|\lambda_-| \le \lambda_+$. So, for $r > 0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_{22}^{(n)} &= \frac{r}{2\sqrt{1+r^2}} \left(\frac{1}{1-\lambda_+} - \frac{1}{1-\lambda_-} \right) \\ &= \frac{r}{2\sqrt{1+r^2}} \left(\frac{\lambda_+ - \lambda_-}{(1-\lambda_+)(1-\lambda_-)} \right) \\ &= \frac{r}{2\sqrt{1+r^2}} \left(\frac{\sqrt{1+r^2}}{r/2} \right) = 1.\end{aligned}$$

Thus the expected number of returns to state 2 is 1. That makes sense because whenever the state is 2, there is a probability of 1/2 of getting absorbed into state 1 on the next step, and otherwise the system will eventually return to 2 (if $r > 0$). Of course if $r = 0$, then $\sum_{n=1}^{\infty} a_{22}^{(n)} = 0$. So if $\sum_{n=1}^{\infty} a_{22}^{(n)}$ converges to s, then $s < 1$ implies $r = 0$, and $s > 0$ implies that some $a_{22}^{(n)} > 0$, so $r > 0$ there being no way to get from state 2 to state 2 without transiting from state 3 to state 2.

Other questions remain: Does (3) imply (2) without assuming decidable reachability? As $a_{ii}^{(m+n)} \ge a_{ik}^{(m)} a_{ki}^{(n)}$ it does if k can be reached from i. We can prove that (3) implies (2) when the number of states is (at most) three.

Proposition 4. *In any three-state Markov chain, if* $a_{11}^{(n)} \to 0$*, then* $a_{21}^{(n)} \to 0$.

Proof. First we show that $a_{32}a_{21} = a_{23}a_{31} = a_{21}a_{31} = 0$. If $a_{32}a_{21} > 0$, then

$$a_{12}^{(n)} \le \frac{a_{11}^{(n+1)}}{a_{21}}$$
$$a_{13}^{(n)} \le \frac{a_{11}^{(n+2)}}{a_{32}a_{21}}$$

so

$$1 = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)} \leq a_{11}^{(n)} + \frac{a_{11}^{(n+1)}}{a_{21}} + \frac{a_{11}^{(n+2)}}{a_{32}a_{21}}$$

which contradicts $a_{11}^{(n)} \to 0$. By symmetry, $a_{23}a_{31} = 0$ also.

If $a_{21}a_{31} > 0$, then

$$1 = a_{11}^{(n)} + a_{12}^{(n)} + a_{13}^{(n)} \leq a_{11}^{(n)} + \frac{a_{11}^{(n+1)}}{a_{21}} + \frac{a_{11}^{(n+1)}}{a_{31}}$$

which contradicts $a_{11}^{(n)} \to 0$. Thus $a_{21}a_{31} = 0$.

As $a_{11}^n \leq a_{11}^{(n)} \to 0$, we have $a_{11} < 1$ so either $a_{12} > 0$ or $a_{13} > 0$. If $a_{12} > 0$, then from $a_{11}^{(1+n)} \geq a_{12}a_{21}^{(n)}$, we can conclude that $a_{21}^{(n)} \to 0$ as desired. So we may assume that $a_{13} > 0$. If a_{21} and a_{31} are both small, then

$$\begin{aligned} a_{21}^{(n)} &= a_{21}^{(n-1)}a_{11} + a_{22}^{(n-1)}a_{21} + a_{23}^{(n-1)}a_{31} \\ &\leq a_{21}^{(n-1)}a_{11} + a_{21} + a_{31} \end{aligned}$$

for all n, so, using this as a recursion and noting that $a_{21}^{(0)} = 0$, we have

$$a_{21}^{(n)} \leq \frac{a_{21} + a_{31}}{1 - a_{11}}$$

is small for all n. So we may assume that either $a_{21} > 0$ or $a_{31} > 0$. If $a_{31} > 0$, then $a_{23} = a_{21} = 0$ from the equations of the first paragraph, so $a_{22} = 1$ whence $a_{21}^{(n)} = 0$ for all n. There remains the case $a_{21} > 0$. Then $a_{32} = 0$ and $a_{31} = 0$, from the equations of the first paragraph, so $a_{33} = 1$. Recall that we may assume that $a_{13} > 0$, so 1 is a transient state whence $a_{21}^{(n)} \to 0$.

For four states, it looks like one will have to dig deeper.

I have been unable to settle whether (4) implies (2), or whether (4) implies (1), without decidable reachability.

Condition 3, that $a_{ii}^{(n)} \to 0$, is some sort of weak transience. Does it admit a good characterization? Let P_{ij} be the set of simple paths (no repeated vertices) from i to j. For $\xi \in P_{ij}$, define

$$a_\xi = \prod_{(s,t)\in\xi} a_{st}$$

and set

$$L_{ij} = \sum_{\xi \in P_{ij}} a_\xi .$$

The letter "L" is for "leads to", and L_{ij} is the probability of traveling along a simple path to j if you start at i. So j can be reached from i if and only if $L_{ij} > 0$ if and only if $a_\xi > 0$ for some $\xi \in P_{ij}$. If $a_{11}^{(n)} \to 0$, then $a_{11} < 1$ and

$\prod L_{i1} = 0$. This is a pretty weak implication, even classically. If we are going to draw serious consequences from $a_{11}^{(n)} \to 0$, then we need to do better than that.

Condition 1, that state i is (strongly) transient, can be phrased as there exist j so that $L_{ji} = 0$ and $L_{ij} > 0$. A weaker version is

– For all $\varepsilon > 0$, there exists j such that $L_{ji} < \varepsilon$ and $L_{ij} > 0$.

The strong and weak versions are equivalent under decidable reachability. The weak version is implied by (3). If $\sup_{t \le d} a_{ii}^{(n+t)} < \varepsilon/d$, choose j so that $a_{ij}^{(n)} > 1/d$. Then $L_{ij} > 0$ and $\sup_{t \le d} a_{ii}^{(n+t)} \ge a_{ij}^{(n)} L_{ji} \ge L_{ji} d$, so $L_{ji} < \varepsilon$. For the converse, we would want to get a bound on $a_{ii}^{(n)}$ from $L_{ij} > \delta$ and $L_{ji} < \varepsilon$.

4 First passages

Let $f_{ki}^{(m)}$ be the probability of landing on i for the first time at step m starting from k. (*Land* is a key word here—that is, $f_{ii}^{(1)} = a_{ii}$.) Then

$$\sum_m f_{ki}^{(m)} \le 1.$$

We would like to show that this sum, which represents the probability of getting from k to i, actually converges, but it need not. If

$$A = \begin{pmatrix} 1-\varepsilon & \varepsilon \\ 0 & 1 \end{pmatrix},$$

then $f_{12}^{(m)} = (1-\varepsilon)^{m-1}\varepsilon$ is 0, if $\varepsilon = 0$, and sums to 1 if $\varepsilon > 0$. With decidable reachability we get convergence.

Theorem 5. *If reachability is decidable, then $\sum_m f_{ki}^{(m)}$ converges for every k and i.*

Proof. Suppose that reachability is decidable. Let S be the set of states from which i is reachable, excluding i itself. Restricting A to S gives rise to a sub-markov chain with matrix A_S. We will show that the probability of surviving in this chain for n steps (or more) goes to zero as n goes to infinity.

If $\sum_{s \in S} a_{ks} < 1$ for each $k \in S$, then take

$$\theta = \max_{k \in S} \sum_{s \in S} a_{ks} < 1.$$

The probability of surviving n steps in this chain is clearly at most θ^n. Apply this argument to the matrix A_s^d, where d is the number of states in S. We get θ so that the probability of surviving dn steps of A_S is at most θ^n. So the probability of surviving n steps of A_S is at most $\theta^{[n/d]}$ which goes to zero.

It remains to observe that $\sum_{m>n} f_{ki}^{(m)}$ is bounded by the probability of surviving n steps of A_S starting at k.

This theorem gives us a direct proof that (3) implies (2) in Theorem 1 if reachability is decidable.

Corollary 6. *If reachability is decidable, and* $a_{ii}^{(n)} \to 0$, *then* $a_{ki}^{(n)} \to 0$ *for each* k.

Proof. We have

$$a_{ki}^{(n)} = \sum_{m \le n} f_{ki}^{(m)} a_{ii}^{(n-m)} \le \sum_{n/2 \le m \le n} f_{ki}^{(m)} + \max_{m < n/2} a_{ii}^{(n-m)}.$$

The second term on the right goes to zero by hypothesis. The first term on the right goes to zero by Theorem 5.

We can't show that $\sum_m f_{ki}^{(m)}$ converges in general, but do we need that in order to prove Corollary 6? We have seen in Proposition 4 that Corollary 6 holds if the number of states is at most three. The equality in

$$a_{ki}^{(n)} = \sum_{m \le n} f_{ki}^{(m)} a_{ii}^{(n-m)} \le \sum_{n/2 \le m \le n} f_{ki}^{(m)} + \max_{m < n/2} a_{ii}^{(n-m)}$$

seems to be the right thing to look at.

We can show that $f_{ki}^{(n)} \to 0$.

Theorem 7. *For any states* i *and* k, *we have* $\lim_{n\to\infty} f_{ki}^{(n)} = 0$.

Proof. Let S be the set of states other than i. Given $\varepsilon > 0$, write S as the disjoint union $R \cup T$ so that if $j \in R$, then $a_{ji} < \varepsilon$, and if $j \in T$, then $a_{ji} > 0$. Let B be A with the i-th row changed to $b_{ij} = \delta_{ij}$, so we have made i an absorbing state. Then, for $k \in S$,

$$f_{ki}^{(n)} = \sum_{j \in S} b_{kj}^{(n-1)} a_{ji} \le \varepsilon + \sum_{j \in T} b_{kj}^{(n-1)}.$$

If $j \in T$, then j is a transient state of B, so $b_{kj}^{(n-1)}$ goes to zero by Theorem 1. Thus the theorem is true for $k \in S$. If $k = i$, then for $n > 1$,

$$f_{ki}^{(n)} = \sum_{j \in S} a_{ij} f_{ji}^{(n-1)} \to 0.$$

5 Recurrent states

A state i is **recurrent** if whenever j can be reached from i, then i can be reached from j. This may or may not be a good definition, but it does parallel the definition of a transient state. Note that if reachability is decidable, then each state is either transient or recurrent (and not both).

We want to consider two other related conditions:

1. $\sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$,
2. $\sum_{n=0}^{\infty} a_{ii}^{(n)} = \infty$. (This is the "potential-matrix criterion." It says that the expected number of visits to i is infinite.)

To see that these two conditions are equivalent, consider the generating functions

$$f_{ii}(t) = \sum_{n=1}^{\infty} f_{ii}^{(n)} t^n \text{ and}$$
$$a_{ii}(t) = \sum_{n=0}^{\infty} a_{ii}^{(n)} t^n.$$

Then

$$a_{ii} = 1 + a_{ii} f_{ii}$$

that is

$$(1 - f_{ii}) a_{ii} = 1.$$

Now $a_{ii}(r)$ and $f_{ii}(r)$ converge if $|r| < 1$ because their coefficients are probabilities. Note that $f_{ii}(r) < 1$ if $|r| < 1$, and that

$$\lim_{r \to 1} f_{ii}(r) = \sum_{n=1}^{\infty} f_{ii}^{(n)}$$
$$\lim_{r \to 1} a_{ii}(r) = \sum_{n=0}^{\infty} a_{ii}^{(n)}$$

in the obvious sense (both sides of each equation are sups). So the two conditions are equivalent. How do they relate to the recurrency of i?

Suppose that i is recurrent. We will show that Condition 2 holds. That is, given any B, we will to find N so that $\sum_{n=0}^{N} a_{ii}^{(n)} > B$. Let

$$\theta_j = \sup_{t \le d} a_{ji}^{(t)}$$

and let S be a finite set of states such that $\theta_j > 0$ for all $j \notin S$. We may take S to be all states initially. We induct on the cardinality of S. Choose $N > d$ so that $N/d > 1 + B/\theta_j$ for each state $j \notin S$. We have

$$\sum_j \sum_{n=0}^{N-d} a_{ij}^{(n)} = N - d + 1$$

so $\sum_{n=0}^{N-d} a_{ij}^{(n)} > N/d - 1$ for some state j. Because i is recurrent, $\theta_j > 0$. If $j \in S$, then we may replace S by $S \setminus \{j\}$ and we are done by induction. If $j \notin S$, then

$$\sum_{n=0}^{N-d} a_{ij}^{(n)} > B/\theta_j$$

so

$$\sum_{n=0}^{N} a_{ii}^{(n)} \geq \sum_{n=0}^{N-d} a_{ij}^{(n)} \theta_j > B.$$

which proves that if i is recurrent, then Condition 2 holds. I don't know about the converse in general, but if reachability is decidable, and Condition 2 holds, then i cannot be transient (because of Theorem 1 part 4), so i must be recurrent.

Let's return to the notion of transience in relation to the two sums $x = \sum_{n=1}^{\infty} f_{ii}^{(n)}$, and $y = \sum_{n=1}^{\infty} a_{ii}^{(n)}$. Because

$$\lim_{r \to 1} f_{ii}(r) = \sum_{n=1}^{\infty} f_{ii}^{(n)}$$

$$\lim_{r \to 1} a_{ii}(r) = \sum_{n=0}^{\infty} a_{ii}^{(n)}$$

and

$$(1 - f_{ii}) a_{ii} = 1,$$

the sum x, which is a supremum, is located if and only if the sum y is located (including ∞). So $\sum_{n=1}^{\infty} f_{ii}^{(n)}$ converges to a number less than 1 if and only if $\sum_{n=1}^{\infty} a_{ii}^{(n)}$ converges. The latter condition is (4) of Theorem 1, so we have another definition of "transient" that is equivalent to (4).

References

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